Complex Analytic and Differential Geometry

Jean-Pierre Demailly

Université de Grenoble I
Institut Fourier, UMR 5582 du CNRS
38402 Saint-Martin d'Hères, France

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Complex Differential Calculus and Pseudoconvexity

This introductory chapter is mainly a review of the basic tools and concepts which will be employed in the rest of the book: differential forms, currents, holomorphic and plurisubharmonic functions, holomorphic convexity and pseudoconvexity. Our study of holomorphic convexity is principally concentrated here on the case of domains in $\mathbb{C}^n$. The more powerful machinery needed for the study of general complex varieties (sheaves, positive currents, hermitian differential geometry) will be introduced in Chapters II to V. Although our exposition pretends to be almost self-contained, the reader is assumed to have at least a vague familiarity with a few basic topics, such as differential calculus, measure theory and distributions, holomorphic functions of one complex variable, ... Most of the necessary background can be found in the books of (Rudin, 1966) and (Warner, 1971); the basics of distribution theory can be found in Chapter I of (Hörmander 1963). On the other hand, the reader who has already some knowledge of complex analysis in several variables should probably bypass this chapter.

§1. Differential Calculus on Manifolds

§1.A. Differentiable Manifolds

The notion of manifold is a natural extension of the notion of submanifold defined by a set of equations in $\mathbb{R}^n$. However, as already observed by Riemann during the 19th century, it is important to define the notion of a manifold in a flexible way, without necessarily requiring that the underlying topological space is embedded in an affine space. The precise formal definition was first introduced by H. Weyl in (Weyl, 1913).

Let $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$. We denote by C^k the class of functions which are k-times differentiable with continuous derivatives if $k \neq \omega$, and by C^ω the class of real analytic functions. A *differentiable manifold* M of real dimension m and of class C^k is a topological space (which we shall always assume Hausdorff and separable, i.e. possessing a countable basis of the topology), equipped with an atlas of class C^k with values in $\mathbb{R}^m$. An *atlas* of class C^k is a collection of homeomorphisms $\tau_\alpha : U_\alpha \longrightarrow V_\alpha$, $\alpha \in I$, called *differentiable charts*, such that $(U_\alpha)_{\alpha \in I}$ is an open covering of M and V_α an open subset of $\mathbb{R}^m$, and such that for all $\alpha, \beta \in I$ the *transition map*

$$(1.1) \quad \tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1} : \tau_\beta(U_\alpha \cap U_\beta) \longrightarrow \tau_\alpha(U_\alpha \cap U_\beta)$$

Fig. I-1 Charts and transition maps

is a C^k diffeomorphism from an open subset of V_β onto an open subset of V_α (see Fig. 1). Then the components $\tau_\alpha(x) = (x_1^\alpha, \ldots, x_m^\alpha)$ are called the *local coordinates* on U_α defined by the chart τ_α ; they are related by the transition relation $x^\alpha = \tau_{\alpha\beta}(x^\beta)$.

If $\Omega \subset M$ is open and $s \in \mathbb{N} \cup \{\infty, \omega\}$, $0 \le s \le k$, we denote by $C^s(\Omega, \mathbb{R})$ the set of functions f of class C^s on Ω, i.e. such that $f \circ \tau_\alpha^{-1}$ is of class C^s on $\tau_\alpha(U_\alpha \cap \Omega)$ for each α ; if Ω is not open, $C^s(\Omega, \mathbb{R})$ is the set of functions which have a C^s extension to some neighborhood of Ω.

A *tangent vector* ξ at a point $a \in M$ is by definition a differential operator acting on functions, of the type

$$C^1(\Omega, \mathbb{R}) \ni f \longmapsto \xi \cdot f = \sum_{1 \le j \le m} \xi_j \frac{\partial f}{\partial x_j}(a)$$

in any local coordinate system $(x_1, \ldots, x_m)$ on an open set $\Omega \ni a$. We then simply write $\xi = \sum \xi_j \, \partial/\partial x_j$. For every $a \in \Omega$, the n-tuple $(\partial/\partial x_j)_{1 \le j \le m}$ is therefore a basis of the *tangent space* to M at a, which we denote by $T_{M,a}$. The *differential* of a function f at a is the linear form on $T_{M,a}$ defined by

$$df_a(\xi) = \xi \cdot f = \sum \xi_j \, \partial f/\partial x_j(a), \qquad \forall \xi \in T_{M,a}.$$

In particular $dx_j(\xi) = \xi_j$ and we may write $df = \sum (\partial f/\partial x_j) dx_j$. Therefore $(dx_1, \ldots, dx_m)$ is the dual basis of $(\partial/\partial x_1, \ldots, \partial/\partial x_m)$ in the cotangent space $T^\star_{M,a}$. The disjoint unions $T_M = \bigcup_{x \in M} T_{M,x}$ and $T^\star_M = \bigcup_{x \in M} T^\star_{M,x}$ are called the *tangent* and *cotangent bundles* of M.

If ξ is a vector field of class C^s over Ω, that is, a map $x \mapsto \xi(x) \in T_{M,x}$ such that $\xi(x) = \sum \xi_j(x) \, \partial/\partial x_j$ has C^s coefficients, and if η is another vector field of class C^s with $s \ge 1$, the *Lie bracket* $[\xi, \eta]$ is the vector field such that

(1.2) $[\xi, \eta] \cdot f = \xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f)$.

In coordinates, it is easy to check that

$$(1.3)\quad [\xi, \eta] = \sum_{1 \le j,k \le m} \Big(\xi_j \frac{\partial \eta_k}{\partial x_j} - \eta_j \frac{\partial \xi_k}{\partial x_j}\Big) \frac{\partial}{\partial x_k}.$$

§1.B. Differential Forms

A differential form u of degree p, or briefly a p-form over M, is a map u on M with values $u(x) \in \Lambda^p T^\star_{M,x}$. In a coordinate open set $\Omega \subset M$, a differential p-form can be written

$$u(x) = \sum_{|I|=p} u_I(x)\, dx_I,$$

where $I = (i_1, \ldots, i_p)$ is a multi-index with integer components, $i_1 < \ldots < i_p$ and $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. The notation $|I|$ stands for the number of components of I, and is read *length* of I. For all integers $p = 0, 1, \ldots, m$ and $s \in \mathbb{N} \cup \{\infty\}$, $s \le k$, we denote by $C^s(M, \Lambda^p T^\star_M)$ the space of differential p-forms of class C^s, i.e. with C^s coefficients u_I. Several natural operations on differential forms can be defined.

§1.B.1. Wedge Product. If $v(x) = \sum v_J(x)\, dx_J$ is a q-form, the *wedge product* of u and v is the form of degree $(p+q)$ defined by

$$(1.4)\quad u \wedge v(x) = \sum_{|I|=p, |J|=q} u_I(x) v_J(x)\, dx_I \wedge dx_J.$$

§1.B.2. Contraction by a tangent vector. A p-form u can be viewed as an antisymmetric p-linear form on T_M. If $\xi = \sum \xi_j\, \partial/\partial x_j$ is a tangent vector, we define the *contraction* $\xi \,\lrcorner\, u$ to be the differential form of degree $p-1$ such that

$$(1.5)\quad (\xi \,\lrcorner\, u)(\eta_1, \ldots, \eta_{p-1}) = u(\xi, \eta_1, \ldots, \eta_{p-1})$$

for all tangent vectors η_j. Then $(\xi, u) \longmapsto \xi \,\lrcorner\, u$ is bilinear and we find easily

$$\frac{\partial}{\partial x_j} \,\lrcorner\, dx_I = \begin{cases} 0 & \text{if } j \notin I, \\ (-1)^{l-1} dx_{I \smallsetminus \{j\}} & \text{if } j = i_l \in I. \end{cases}$$

A simple computation based on the above formula shows that contraction by a tangent vector is a *derivation*, i.e.

$$(1.6)\quad \xi \,\lrcorner\, (u \wedge v) = (\xi \,\lrcorner\, u) \wedge v + (-1)^{\deg u} u \wedge (\xi \,\lrcorner\, v).$$

§1.B.3. Exterior derivative. This is the differential operator

$$d : C^s(M, \Lambda^p T_M^\star) \longrightarrow C^{s-1}(M, \Lambda^{p+1} T_M^\star)$$

defined in local coordinates by the formula

$$(1.7) \quad du = \sum_{|I|=p,\, 1\le k\le m} \frac{\partial u_I}{\partial x_k}\, dx_k \wedge dx_I.$$

Alternatively, one can define du by its action on arbitrary vector fields $\xi_0, \ldots, \xi_p$ on M. The formula is as follows

$$\begin{aligned} du(\xi_0, \ldots, \xi_p) &= \sum_{0\le j\le p} (-1)^j \xi_j \cdot u(\xi_0, \ldots, \widehat{\xi_j}, \ldots, \xi_p) \\ &\quad + \sum_{0\le j<k\le p} (-1)^{j+k} u([\xi_j, \xi_k], \xi_0, \ldots, \widehat{\xi_j}, \ldots, \widehat{\xi_k}, \ldots, \xi_p). \end{aligned} \tag{1.7$'$}$$

The reader will easily check that (1.7) actually implies (1.7′). The advantage of (1.7′) is that it does not depend on the choice of coordinates, thus du is intrinsically defined. The two basic properties of the exterior derivative (again left to the reader) are:

$$(1.8) \quad d(u \wedge v) = du \wedge v + (-1)^{\deg u} u \wedge dv, \qquad (\textit{Leibnitz' rule})$$

$$(1.9) \quad d^2 = 0.$$

A form u is said to be *closed* if $du = 0$ and *exact* if u can be written $u = dv$ for some form v.

§1.B.4. De Rham Cohomology Groups. Recall that a cohomological complex $K^\bullet = \bigoplus_{p\in\mathbb{Z}}$ is a collection of modules K^p over some ring, equipped with differentials, i.e., linear maps $d^p : K^p \to K^{p+1}$ such that $d^{p+1} \circ d^p = 0$. The *cocycle, coboundary* and *cohomology modules* $Z^p(K^\bullet)$, $B^p(K^\bullet)$ and $H^p(K^\bullet)$ are defined respectively by

$$(1.10) \quad \begin{cases} Z^p(K^\bullet) = \operatorname{Ker} d^p : K^p \to K^{p+1}, & Z^p(K^\bullet) \subset K^p, \\ B^p(K^\bullet) = \operatorname{Im} d^{p-1} : K^{p-1} \to K^p, & B^p(K^\bullet) \subset Z^p(K^\bullet) \subset K^p, \\ H^p(K^\bullet) = Z^p(K^\bullet)/B^p(K^\bullet). & \end{cases}$$

Now, let M be a differentiable manifold, say of class C^∞ for simplicity. The *De Rham complex* of M is defined to be the complex $K^p = C^\infty(M, \Lambda^p T_M^\star)$ of smooth differential forms, together with the exterior derivative $d^p = d$ as differential, and $K^p = \{0\}$, $d^p = 0$ for $p < 0$. We denote by $Z^p(M, \mathbb{R})$ the cocycles (closed p-forms) and by $B^p(M, \mathbb{R})$ the coboundaries (exact p-forms). By convention $B^0(M, \mathbb{R}) = \{0\}$. The *De Rham cohomology group* of M in degree p is

$$(1.11) \quad H^p_{\mathrm{DR}}(M, \mathbb{R}) = Z^p(M, \mathbb{R})/B^p(M, \mathbb{R}).$$

When no confusion with other types of cohomology groups may occur, we sometimes denote these groups simply by $H^p(M,\mathbb{R})$. The symbol $\mathbb{R}$ is used here to stress that we are considering real valued p-forms; of course one can introduce a similar group $H^p_{\mathrm{DR}}(M,\mathbb{C})$ for complex valued forms, i.e. forms with values in $\mathbb{C}\otimes\Lambda^pT^\star_M$. Then $H^p_{\mathrm{DR}}(M,\mathbb{C})=\mathbb{C}\otimes H^p_{\mathrm{DR}}(M,\mathbb{R})$ is the complexification of the real De Rham cohomology group. It is clear that $H^0_{\mathrm{DR}}(M,\mathbb{R})$ can be identified with the space of locally constant functions on M, thus

$$H^0_{\mathrm{DR}}(M,\mathbb{R})=\mathbb{R}^{\pi_0(X)},$$

where $\pi_0(X)$ denotes the set of connected components of M.

Similarly, we introduce the De Rham cohomology groups with compact support

$$(1.12)\quad H^p_{\mathrm{DR},c}(M,\mathbb{R})=Z^p_c(M,\mathbb{R})/B^p_c(M,\mathbb{R}),$$

associated with the De Rham complex $K^p=C^\infty_c(M,\Lambda^pT^\star_M)$ of smooth differential forms with compact support.

§1.B.5. Pull-Back. If $F:M\longrightarrow M'$ is a differentiable map to another manifold M', $\dim_{\mathbb{R}}M'=m'$, and if $v(y)=\sum v_J(y)\,dy_J$ is a differential p-form on M', the pull-back $F^\star v$ is the differential p-form on M obtained after making the substitution $y=F(x)$ in v, i.e.

$$(1.13)\quad F^\star v(x)=\sum v_I\big(F(x)\big)\,dF_{i_1}\wedge\ldots\wedge dF_{i_p}.$$

If we have a second map $G:M'\longrightarrow M''$ and if w is a differential form on M'', then $F^\star(G^\star w)$ is obtained by means of the substitutions $z=G(y)$, $y=F(x)$, thus

$$(1.14)\quad F^\star(G^\star w)=(G\circ F)^\star w.$$

Moreover, we always have $d(F^\star v)=F^\star(dv)$. It follows that the pull-back $F^\star$ is closed if v is closed and exact if v is exact. Therefore $F^\star$ induces a morphism on the quotient spaces

$$(1.15)\quad F^\star:H^p_{\mathrm{DR}}(M',\mathbb{R})\longrightarrow H^p_{\mathrm{DR}}(M,\mathbb{R}).$$

§1.C. Integration of Differential Forms

A manifold M is *orientable* if and only if there exists an atlas (τ_α) such that all transition maps $\tau_{\alpha\beta}$ preserve the orientation, i.e. have positive jacobian determinants. Suppose that M is oriented, that is, equipped with such an atlas. If $u(x)=f(x_1,\ldots,x_m)\,dx_1\wedge\ldots\wedge dx_m$ is a continuous form of maximum degree $m=\dim_{\mathbb{R}}M$, with compact support in a coordinate open set Ω, we set

(1.16) $$\int_M u = \int_{\mathbb{R}^m} f(x_1, \ldots, x_m)\, dx_1 \ldots dx_m.$$

By the change of variable formula, the result is independent of the choice of coordinates, provided we consider only coordinates corresponding to the given orientation. When u is an arbitrary form with compact support, the definition of $\int_M u$ is easily extended by means of a partition of unity with respect to coordinate open sets covering $\operatorname{Supp} u$. Let $F : M \longrightarrow M'$ be a diffeomorphism between oriented manifolds and v a volume form on M'. The change of variable formula yields

(1.17) $$\int_M F^\star v = \pm \int_{M'} v$$

according whether F preserves orientation or not.

We now state Stokes' formula, which is basic in many contexts. Let K be a compact subset of M with piecewise C^1 boundary. By this, we mean that for each point $a \in \partial K$ there are coordinates $(x_1, \ldots, x_m)$ on a neighborhood V of a, centered at a, such that

$$K \cap V = \big\{x \in V\,;\, x_1 \le 0, \ldots, x_l \le 0\big\}$$

for some index $l \ge 1$. Then $\partial K \cap V$ is a union of smooth hypersurfaces with piecewise C^1 boundaries:

$$\partial K \cap V = \bigcup_{1 \le j \le l} \big\{x \in V\,;\, x_1 \le 0, \ldots, x_j = 0, \ldots, x_l \le 0\big\}.$$

At points of ∂K where $x_j = 0$, then $(x_1, \ldots, \widehat{x_j}, , \ldots, x_m)$ define coordinates on ∂K. We take the orientation of ∂K given by these coordinates or the opposite one, according to the sign $(-1)^{j-1}$. For any differential form u of class C^1 and degree $m-1$ on M, we then have

(1.18) Stokes' formula. $$\int_{\partial K} u = \int_K du.$$

The formula is easily checked by an explicit computation when u has compact support in V: indeed if $u = \sum_{1 \le j \le n} u_j\, dx_1 \wedge \ldots \widehat{dx_j} \ldots dx_m$ and $\partial_j K \cap V$ is the part of $\partial K \cap V$ where $x_j = 0$, a partial integration with respect to x_j yields

$$\int_{\partial_j K \cap V} u_j\, dx_1 \wedge \ldots \widehat{dx_j} \ldots dx_m = \int_V \frac{\partial u_j}{\partial x_j}\, dx_1 \wedge \ldots dx_m,$$

$$\int_{\partial K \cap V} u = \sum_{1 \le j \le m} (-1)^{j-1} \int_{\partial_j K \cap V} u_j\, dx_1 \wedge \ldots \widehat{dx_j} \ldots \wedge dx_m = \int_V du.$$

The general case follows by a partition of unity. In particular, if u has compact support in M, we find $\int_M du = 0$ by choosing $K \supset \operatorname{Supp} u$.

§1.D. Homotopy Formula and Poincaré Lemma

Let u be a differential form on $[0,1]\times M$. For $(t,x)\in[0,1]\times M$, we write

$$u(t,x)=\sum_{|I|=p}u_I(t,x)\,dx_I+\sum_{|J|=p-1}\widetilde u_J(t,x)\,dt\wedge dx_J.$$

We define an operator

$$K:C^s([0,1]\times M,\Lambda^pT^\star_{[0,1]\times M})\longrightarrow C^s(M,\Lambda^{p-1}T^\star_M)$$

$$\text{(1.19)}\quad Ku(x)=\sum_{|J|=p-1}\Big(\int_0^1\widetilde u_J(t,x)\,dt\Big)dx_J$$

and say that Ku is the form obtained by integrating u along $[0,1]$. A computation of the operator $dK+Kd$ shows that all terms involving partial derivatives $\partial\widetilde u_J/\partial x_k$ cancel, hence

$$Kdu+dKu=\sum_{|I|=p}\Big(\int_0^1\frac{\partial u_I}{\partial t}(t,x)\,dt\Big)dx_I=\sum_{|I|=p}\big(u_I(1,x)-u_I(0,x)\big)dx_I,$$

$$\text{(1.20)}\qquad Kdu+dKu=i_1^\star u-i_0^\star u,$$

where $i_t:M\to[0,1]\times M$ is the injection $x\mapsto(t,x)$.

(1.20) Corollary. *Let $F,G:M\longrightarrow M'$ be C^∞ maps. Suppose that F,G are smoothly homotopic, i.e. that there exists a C^∞ map $H:[0,1]\times M\longrightarrow M'$ such that $H(0,x)=F(x)$ and $H(1,x)=G(x)$. Then*

$$F^\star=G^\star:H^p_{\mathrm{DR}}(M',\mathbb R)\longrightarrow H^p_{\mathrm{DR}}(M,\mathbb R).$$

Proof. If v is a p-form on M', then

$$\begin{aligned}G^\star v-F^\star v&=(H\circ i_1)^\star v-(H\circ i_0)^\star v=i_1^\star(H^\star v)-i_0^\star(H^\star v)\\&=d(KH^\star v)+KH^\star(dv)\end{aligned}$$

by (1.20) applied to $u=H^\star v$. If v is closed, then $F^\star v$ and $G^\star v$ differ by an exact form, so they define the same class in $H^p_{\mathrm{DR}}(M,\mathbb R)$. □

(1.21) Corollary. *If the manifold M is contractible, i.e. if there is a smooth homotopy $H:[0,1]\times M\to M$ from a constant map $F:M\to\{x_0\}$ to $G=\mathrm{Id}_X$, then $H^0_{\mathrm{DR}}(M,\mathbb R)=\mathbb R$ and $H^p_{\mathrm{DR}}(M,\mathbb R)=0$ for $p\geq 1$.*

Proof. $F^\star$ is clearly zero in degree $p\geq 1$, while $F^\star:H^0_{\mathrm{DR}}(M,\mathbb R)\xrightarrow{\simeq}\mathbb R$ is induced by the evaluation map $u\mapsto u(x_0)$. The conclusion then follows from the equality $F^\star=G^\star=\mathrm{Id}$ on cohomology groups. □

(1.22) Poincaré lemma. *Let $\Omega \subset \mathbb{R}^m$ be a starshaped open set. If a form $v = \sum v_I dx_I \in C^s(\Omega, \Lambda^p T^\star_\Omega)$, $p \geq 1$, satisfies $dv = 0$, there exists a form $u \in C^s(\Omega, \Lambda^{p-1} T^\star_\Omega)$ such that $du = v$.*

Proof. Let $H(t, x) = tx$ be the homotopy between the identity map $\Omega \to \Omega$ and the constant map $\Omega \to \{0\}$. By the above formula

$$d(KH^\star v) = G^\star v - F^\star v = \begin{cases} v - v(0) & \text{if } p = 0, \\ v & \text{if } p \geq 1. \end{cases}$$

Hence $u = KH^\star v$ is the $(p-1)$-form we are looking for. An explicit computation based on (1.19) easily gives

$$(1.23) \quad u(x) = \sum_{\substack{|I|=p \\ 1 \leq k \leq p}} \Big(\int_0^1 t^{p-1} v_I(tx)\, dt \Big) (-1)^{k-1} x_{i_k} dx_{i_1} \wedge \ldots \widehat{dx_{i_k}} \ldots \wedge dx_{i_p}.$$

§2. Currents on Differentiable Manifolds

§2.A. Definition and Examples

Let M be a C^∞ differentiable manifold, $m = \dim_{\mathbb{R}} M$. All the manifolds considered in Sect. 2 will be assumed to be oriented. We first introduce a topology on the space of differential forms $C^s(M, \Lambda^p T^\star_M)$. Let $\Omega \subset M$ be a coordinate open set and u a p-form on M, written $u(x) = \sum u_I(x)\, dx_I$ on Ω. To every compact subset $L \subset \Omega$ and every integer $s \in \mathbb{N}$, we associate a seminorm

$$(2.1) \quad p^s_L(u) = \sup_{x \in L} \max_{|I|=p, |\alpha| \leq s} |D^\alpha u_I(x)|,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ runs over $\mathbb{N}^m$ and $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}$ is a derivation of order $|\alpha| = \alpha_1 + \cdots + \alpha_m$. This type of multi-index, which will always be denoted by Greek letters, should not be confused with multi-indices of the type $I = (i_1, \ldots, i_p)$ introduced in Sect. 1.

(2.2) Definition.

a) *We denote by $\mathcal{E}^p(M)$ $\big($resp. ${}^s\mathcal{E}^p(M)\big)$ the space $C^\infty(M, \Lambda^p T^\star_M)$ $\big($resp. the space $C^s(M, \Lambda^p T^\star_M)\big)$, equipped with the topology defined by all seminorms p^s_L when s, L, Ω vary (resp. when L, Ω vary).*

b) *If $K \subset M$ is a compact subset, $\mathcal{D}^p(K)$ will denote the subspace of elements $u \in \mathcal{E}^p(M)$ with support contained in K, together with the induced topology; $\mathcal{D}^p(M)$ will stand for the set of all elements with compact support, i.e. $\mathcal{D}^p(M) := \bigcup_K \mathcal{D}^p(K)$.*

c) *The spaces of C^s-forms ${}^s\mathcal{D}^p(K)$ and ${}^s\mathcal{D}^p(M)$ are defined similarly.*

Since our manifolds are assumed to be separable, the topology of $\mathcal{E}^p(M)$ can be defined by means of a countable set of seminorms p^s_L, hence $\mathcal{E}^p(M)$ (and likewise ${}^s\mathcal{E}^p(M)$) is a Fréchet space. The topology of ${}^s\mathcal{D}^p(K)$ is induced by any finite set of seminorms $p^s_{K_j}$ such that the compact sets K_j cover K ; hence ${}^s\mathcal{D}^p(K)$ is a Banach space. It should be observed however that $\mathcal{D}^p(M)$ is not a Fréchet space; in fact $\mathcal{D}^p(M)$ is dense in $\mathcal{E}^p(M)$ and thus non complete for the induced topology. According to (De Rham 1955) spaces of *currents* are defined as the topological duals of the above spaces, in analogy with the usual definition of distributions.

(2.3) Definition. *The space of currents of dimension p (or degree $m-p$) on M is the space $\mathcal{D}'_p(M)$ of linear forms T on $\mathcal{D}^p(M)$ such that the restriction of T to all subspaces $\mathcal{D}^p(K)$, $K \subset\subset M$, is continuous. The degree is indicated by raising the index, hence we set*

$$\mathcal{D}'^{\,m-p}(M) = \mathcal{D}'_p(M) := \textit{topological dual } \big(\mathcal{D}^p(M)\big)'.$$

The space ${}^s\mathcal{D}'_p(M) = {}^s\mathcal{D}'^{\,m-p}(M) := \big({}^s\mathcal{D}^p(M)\big)'$ is defined similarly and is called the space of currents of order s on M.

In the sequel, we let $\langle T, u\rangle$ be the pairing between a current T and a *test form* $u \in \mathcal{D}^p(M)$. It is clear that ${}^s\mathcal{D}'_p(M)$ can be identified with the subspace of currents $T \in \mathcal{D}'_p(M)$ which are continuous for the seminorm p^s_K on $\mathcal{D}^p(K)$ for every compact set K contained in a coordinate patch Ω. The *support* of T, denoted $\operatorname{Supp} T$, is the smallest closed subset $A \subset M$ such that the restriction of T to $\mathcal{D}^p(M \smallsetminus A)$ is zero. The topological dual $\mathcal{E}'_p(M)$ can be identified with the set of currents of $\mathcal{D}'_p(M)$ with compact support: indeed, let T be a linear form on $\mathcal{E}^p(M)$ such that

$$|\langle T, u\rangle| \le C \max\{p^s_{K_j}(u)\}$$

for some $s \in \mathbb{N}$, $C \ge 0$ and a finite number of compact sets K_j ; it follows that $\operatorname{Supp} T \subset \bigcup K_j$. Conversely let $T \in \mathcal{D}'_p(M)$ with support in a compact set K. Let K_j be compact patches such that K is contained in the interior of $\bigcup K_j$ and $\psi \in \mathcal{D}(M)$ equal to 1 on K with $\operatorname{Supp}\psi \subset \bigcup K_j$. For $u \in \mathcal{E}^p(M)$, we define $\langle T, u\rangle = \langle T, \psi u\rangle$; this is independent of ψ and the resulting T is clearly continuous on $\mathcal{E}^p(M)$. The terminology used for the dimension and degree of a current is justified by the following two examples.

(2.4) Example. Let $Z \subset M$ be a closed oriented submanifold of M of dimension p and class C^1 ; Z may have a boundary ∂Z. The *current of integration* over Z, denoted $[Z]$, is defined by

$$\langle [Z], u\rangle = \int_Z u, \qquad u \in {}^0\mathcal{D}^p(M).$$

It is clear that $[Z]$ is a current of order 0 on M and that $\operatorname{Supp}[Z] = Z$. Its dimension is $p = \dim Z$.

(2.5) Example. If f is a differential form of degree q on M with $L^1_{\rm loc}$ coefficients, we can associate to f the current of dimension $m - q$:

$$\langle T_f, u\rangle = \int_M f \wedge u, \qquad u \in {}^0\mathcal{D}^{m-q}(M).$$

T_f is of degree q and of order 0. The correspondence $f \longmapsto T_f$ is injective. In the same way $L^1_{\rm loc}$ functions on $\mathbb{R}^m$ are identified to distributions, we will identify f with its image $T_f \in {}^0\mathcal{D}'^{\,q}(M) = {}^0\mathcal{D}'_{m-q}(M)$.

§2.B. Exterior Derivative and Wedge Product

§2.B.1. Exterior Derivative. Many of the operations available for differential forms can be extended to currents by simple duality arguments. Let $T \in {}^s\mathcal{D}'^{\,q}(M) = {}^s\mathcal{D}'_{m-p}(M)$. The *exterior derivative*

$$dT \in {}^{s+1}\mathcal{D}'^{\,q+1}(M) = {}^{s+1}\mathcal{D}'_{m-q-1}$$

is defined by

$$(2.6)\quad \langle dT, u\rangle = (-1)^{q+1}\,\langle T, du\rangle, \qquad u \in {}^{s+1}\mathcal{D}^{m-q-1}(M).$$

The continuity of the linear form dT on ${}^{s+1}\mathcal{D}^{m-q-1}(M)$ follows from the continuity of the map $d : {}^{s+1}\mathcal{D}^{m-q-1}(K) \longrightarrow {}^s\mathcal{D}^{m-q}(K)$. For all forms $f \in {}^1\mathcal{E}^q(M)$ and $u \in \mathcal{D}^{m-q-1}(M)$, Stokes' formula implies

$$0 = \int_M d(f \wedge u) = \int_M df \wedge u + (-1)^q\, f \wedge du,$$

thus in example (2.5) one actually has $dT_f = T_{df}$ as it should be. In example (2.4), another application of Stokes' formula yields $\int_Z du = \int_{\partial Z} u$, therefore $\langle [Z], du\rangle = \langle [\partial Z], u\rangle$ and

$$(2.7)\quad d[Z] = (-1)^{m-p+1}[\partial Z].$$

§2.B.2. Wedge Product. For $T \in {}^s\mathcal{D}'^{\,q}(M)$ and $g \in {}^s\mathcal{E}^r(M)$, the wedge product $T \wedge g \in {}^s\mathcal{D}'^{\,q+r}(M)$ is defined by

$$(2.8)\quad \langle T \wedge g, u\rangle = \langle T, g \wedge u\rangle, \qquad u \in {}^s\mathcal{D}^{m-q-r}(M).$$

This definition is licit because $u \mapsto g \wedge u$ is continuous in the C^s-topology. The relation

$$d(T \wedge g) = dT \wedge g + (-1)^{\deg T}\, T \wedge dg$$

is easily verified from the definitions.

(2.9) Proposition. *Let $(x_1, \ldots, x_m)$ be a coordinate system on an open subset $\Omega \subset M$. Every current $T \in {}^s\mathcal{D}'^q(M)$ of degree q can be written in a unique way*

$$T = \sum_{|I|=q} T_I \, dx_I \quad \text{on} \quad \Omega,$$

where T_I are distributions of order s on Ω, considered as currents of degree 0.

Proof. If the result is true, for all $f \in {}^s\mathcal{D}^0(\Omega)$ we must have

$$\langle T, f\, dx_{\complement I}\rangle = \langle T_I, dx_I \wedge f\, dx_{\complement I}\rangle = \varepsilon(I, \complement I)\, \langle T_I, f\, dx_1 \wedge \ldots \wedge dx_m\rangle,$$

where $\varepsilon(I, \complement I)$ is the signature of the permutation $(1, \ldots, m) \longmapsto (I, \complement I)$. Conversely, this can be taken as a definition of the coefficient T_I :

(2.10) $\quad T_I(f) = \langle T_I, f\, dx_1 \wedge \ldots \wedge dx_m\rangle := \varepsilon(I, \complement I)\, \langle T, f\, dx_{\complement I}\rangle, \quad f \in {}^s\mathcal{D}^0(\Omega).$

Then T_I is a distribution of order s and it is easy to check that $T = \sum T_I \, dx_I$. □

In particular, currents of order 0 on M can be considered as differential forms with measure coefficients. In order to unify the notations concerning forms and currents, we set

$$\langle T, u\rangle = \int_M T \wedge u$$

whenever $T \in {}^s\mathcal{D}'_p(M) = {}^s\mathcal{D}'^{m-p}(M)$ and $u \in {}^s\mathcal{E}^p(M)$ are such that $\operatorname{Supp} T \cap \operatorname{Supp} u$ is compact. This convention is made so that the notation becomes compatible with the identification of a form f to the current T_f.

§2.C. Direct and Inverse Images

§2.C.1. Direct Images. Assume now that M_1, M_2 are oriented differentiable manifolds of respective dimensions m_1, m_2, and that

(2.11) $\quad F : M_1 \longrightarrow M_2$

is a C^∞ map. The pull-back morphism

(2.12) $\quad {}^s\mathcal{D}^p(M_2) \longrightarrow {}^s\mathcal{E}^p(M_1), \qquad u \longmapsto F^\star u$

is continuous in the C^s topology and we have $\operatorname{Supp} F^\star u \subset F^{-1}(\operatorname{Supp} u)$, but in general $\operatorname{Supp} F^\star u$ is not compact. If $T \in {}^s\mathcal{D}'_p(M_1)$ is such that the restriction of F to $\operatorname{Supp} T$ is *proper*, i.e. if $\operatorname{Supp} T \cap F^{-1}(K)$ is compact for every compact subset $K \subset M_2$, then the linear form $u \longmapsto \langle T, F^\star u\rangle$ is well defined and continuous on ${}^s\mathcal{D}^p(M_2)$. There exists therefore a unique current denoted $F_\star T \in {}^s\mathcal{D}'_p(M_2)$, called *the direct image* of T by F, such that

$$(2.13)\qquad \langle F_\star T, u\rangle = \langle T, F^\star u\rangle, \qquad \forall u \in {}^s\mathcal{D}^p(M_2).$$

We leave the straightforward proof of the following properties to the reader.

(2.14) Theorem. *For every $T \in {}^s\mathcal{D}'_p(M_1)$ such that $F_{\restriction \operatorname{Supp} T}$ is proper, the direct image $F_\star T \in {}^s\mathcal{D}'_p(M_2)$ is such that*

a) $\operatorname{Supp} F_\star T \subset F(\operatorname{Supp} T)$;

b) $d(F_\star T) = F_\star(dT)$;

c) $F_\star(T \wedge F^\star g) = (F_\star T) \wedge g, \quad \forall g \in {}^s\mathcal{E}^q(M_2, \mathbb{R})$;

d) *If $G : M_2 \longrightarrow M_3$ is a C^∞ map such that $(G \circ F)_{\restriction \operatorname{Supp} T}$ is proper, then*

$$G_\star(F_\star T) = (G \circ F)_\star T.$$

Fig. I-2 Local description of a submersion as a projection.

(2.15) Special case. Assume that F is a submersion, i.e. that F is surjective and that for every $x \in M_1$ the differential map $d_x F : T_{M_1,x} \longrightarrow T_{M_2,F(x)}$ is surjective. Let g be a differential form of degree q on M_1, with L^1_{loc} coefficients, such that $F_{\restriction \operatorname{Supp} g}$ is proper. We claim that $F_\star g \in {}^0\mathcal{D}'_{m_1-q}(M_2)$ is the form of degree $q - (m_1 - m_2)$ obtained from g by integration along the fibers of F, also denoted

$$F_\star g(y) = \int_{z \in F^{-1}(y)} g(z).$$

In fact, this assertion is equivalent to the following generalized form of Fubini's theorem:

$$\int_{M_1} g \wedge F^\star u = \int_{y\in M_2} \Big(\int_{z\in F^{-1}(y)} g(z) \Big) \wedge u(y), \qquad \forall u \in {}^0\mathcal{D}^{m_1-q}(M_2).$$

By using a partition of unity on M_1 and the constant rank theorem, the verification of this formula is easily reduced to the case where $M_1 = A \times M_2$ and $F = \mathrm{pr}_2$, cf. Fig. 2. The fibers $F^{-1}(y) \simeq A$ have to be oriented in such a way that the orientation of M_1 is the product of the orientation of A and M_2. Let us write $r = \dim A = m_1 - m_2$ and let $z = (x, y) \in A \times M_2$ be any point of M_1. The above formula becomes

$$\int_{A\times M_2} g(x,y) \wedge u(y) = \int_{y\in M_2} \Big(\int_{x\in A} g(x,y) \Big) \wedge u(y),$$

where the direct image of g is computed from $g = \sum g_{I,J}(x,y)\, dx_I \wedge dy_J$, $|I| + |J| = q$, by the formula

$$\text{(2.16)} \qquad \begin{aligned} F_\star g(y) &= \int_{x\in A} g(x,y) \\ &= \sum_{|J|=q-r} \Big(\int_{x\in A} g_{(1,\ldots,r),J}(x,y)\, dx_1 \wedge \ldots \wedge dx_r \Big) dy_J. \end{aligned}$$

In this situation, we see that $F_\star g$ has L^1_{loc} coefficients on M_2 if g is L^1_{loc} on M_1, and that the map $g \longmapsto F_\star g$ is continuous in the C^s topology.

(2.17) Remark. If $F : M_1 \longrightarrow M_2$ is a diffeomorphism, then we have $F_\star g = \pm (F^{-1})^\star g$ according whether F preserves the orientation or not. In fact formula (1.17) gives

$$\langle F_\star g, u\rangle = \int_{M_1} g \wedge F^\star u = \pm \int_{M_2} (F^{-1})^\star (g \wedge F^\star u) = \pm \int_{M_2} (F^{-1})^\star g \wedge u.$$

§**2.C.2. Inverse Images.** Assume that $F : M_1 \longrightarrow M_2$ is a submersion. As a consequence of the continuity statement after (2.16), one can always define the inverse image $F^\star T \in {}^s\mathcal{D}'^{\,q}(M_1)$ of a current $T \in {}^s\mathcal{D}'^{\,q}(M_2)$ by

$$\langle F^\star T, u\rangle = \langle T, F_\star u\rangle, \qquad u \in {}^s\mathcal{D}^{q+m_1-m_2}(M_1).$$

Then $\dim F^\star T = \dim T + m_1 - m_2$ and Th. 2.14 yields the formulas:

$$\text{(2.18)} \quad d(F^\star T) = F^\star(dT), \qquad F^\star(T \wedge g) = F^\star T \wedge F^\star g, \qquad \forall g \in {}^s\mathcal{D}^\bullet(M_2).$$

Take in particular $T = [Z]$, where Z is an oriented C^1-submanifold of M_2. Then $F^{-1}(Z)$ is a submanifold of M_1 and has a natural orientation given by the isomorphism

$$T_{M_1,x}/T_{F^{-1}(Z),x} \longrightarrow T_{M_2,F(x)}/T_{Z,F(x)},$$

induced by d_xF at every point $x \in Z$. We claim that

(2.19) $F^\star[Z]=[F^{-1}(Z)]$.

Indeed, we have to check that $\int_Z F_\star u=\int_{F^{-1}(Z)} u$ for every $u\in {}^s\mathcal{D}^\bullet(M_1)$. By using a partition of unity on M_1, we may again assume $M_1=A\times M_2$ and $F=\mathrm{pr}_2$. The above equality can be written

$$\int_{y\in Z} F_\star u(y)=\int_{(x,y)\in A\times Z} u(x,y).$$

This follows precisely from (2.16) and Fubini's theorem.

§**2.C.3. Weak Topology.** The weak topology on $\mathcal{D}'_p(M)$ is the topology defined by the collection of seminorms $T\longmapsto|\langle T,f\rangle|$ for all $f\in\mathcal{D}^p(M)$. With respect to the weak topology, all the operations

(2.20) $T\longmapsto dT,\qquad T\longmapsto T\wedge g,\qquad T\longmapsto F_\star T,\qquad T\longmapsto F^\star T$

defined above are continuous. A set $B\subset\mathcal{D}'_p(M)$ is bounded for the weak topology (weakly bounded for short) if and only if $\langle T,f\rangle$ is bounded when T runs over B, for every fixed $f\in\mathcal{D}^p(M)$. The standard Banach-Alaoglu theorem implies that every weakly bounded closed subset $B\subset\mathcal{D}'_p(M)$ is weakly compact.

§2.D. Tensor Products, Homotopies and Poincaré Lemma

§**2.D.1. Tensor Products.** If S, T are currents on manifolds M, M' there exists a unique current on $M\times M'$, denoted $S\otimes T$ and defined in a way analogous to the tensor product of distributions, such that for all $u\in\mathcal{D}^\bullet(M)$ and $v\in\mathcal{D}^\bullet(M')$

(2.21) $\langle S\otimes T,\mathrm{pr}_1^\star u\wedge\mathrm{pr}_2^\star v\rangle=(-1)^{\deg T\deg u}\langle S,u\rangle\,\langle T,v\rangle.$

One verifies easily that $d(S\otimes T)=dS\otimes T+(-1)^{\deg S}S\otimes dT$.

§**2.D.2. Homotopy Formula.** Assume that $H:[0,1]\times M_1\longrightarrow M_2$ is a C^∞ homotopy from $F(x)=H(0,x)$ to $G(x)=H(1,x)$ and that $T\in\mathcal{D}'_\bullet(M_1)$ is a current such that $H_{\restriction[0,1]\times\mathrm{Supp}\,T}$ is proper. If $[0,1]$ is considered as the current of degree 0 on $\mathbb{R}$ associated to its characteristic function, we find $d[0,1]=\delta_0-\delta_1$, thus

$$\begin{aligned}d\big(H_\star([0,1]\otimes T)\big)&=H_\star(\delta_0\otimes T-\delta_1\otimes T+[0,1]\otimes dT)\\&=F_\star T-G_\star T+H_\star([0,1]\otimes dT).\end{aligned}$$

Therefore we obtain the *homotopy formula*

(2.22) $F_\star T-G_\star T=d\big(H_\star([0,1]\otimes T)\big)-H_\star([0,1]\otimes dT).$

When T is closed, i.e. $dT=0$, we see that $F_\star T$ and $G_\star T$ are cohomologous on M_2, i.e. they differ by an exact current dS.

§2.D.3. Regularization of Currents. Let $\rho \in C^\infty(\mathbb{R}^m)$ be a function with support in $B(0,1)$, such that $\rho(x)$ depends only on $|x| = (\sum |x_i|^2)^{1/2}$, $\rho \geq 0$ and $\int_{\mathbb{R}^m} \rho(x)\,dx = 1$. We associate to ρ the family of functions (ρ_ε) such that

$$(2.23) \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^m}\,\rho\Big(\frac{x}{\varepsilon}\Big), \qquad \operatorname{Supp}\rho_\varepsilon \subset B(0,\varepsilon), \qquad \int_{\mathbb{R}^m} \rho_\varepsilon(x)\,dx = 1.$$

We shall refer to this construction by saying that (ρ_ε) is a *family of smoothing kernels*. For every current $T = \sum T_I\,dx_I$ on an open subset $\Omega \subset \mathbb{R}^m$, the family of smooth forms

$$T \star \rho_\varepsilon = \sum_I (T_I \star \rho_\varepsilon)\,dx_I,$$

defined on $\Omega_\varepsilon = \{x \in \mathbb{R}^m \,;\, d(x, \complement\Omega) > \varepsilon\}$, converges weakly to T as ε tends to 0. Indeed, $\langle T \star \rho_\varepsilon, f\rangle = \langle T, \rho_\varepsilon \star f\rangle$ and $\rho_\varepsilon \star f$ converges to f in $\mathcal{D}^p(\Omega)$ with respect to all seminorms p^s_K.

§2.D.4. Poincaré Lemma for Currents. Let $T \in {}^s\mathcal{D}'^{\,q}(\Omega)$ be a closed current on an open set $\Omega \subset \mathbb{R}^m$. We first show that T is cohomologous to a smooth form. In fact, let $\psi \in C^\infty(\mathbb{R}^m)$ be a cut-off function such that $\operatorname{Supp}\psi \subset \overline{\Omega}$, $0 < \psi \leq 1$ and $|d\psi| \leq 1$ on Ω. For any vector $v \in B(0,1)$ we set

$$F_v(x) = x + \psi(x)v.$$

Since $x \mapsto \psi(x)v$ is a contraction, F_v is a diffeomorphism of $\mathbb{R}^m$ which leaves $\complement\Omega$ invariant pointwise, so $F_v(\Omega) = \Omega$. This diffeomorphism is homotopic to the identity through the homotopy $H_v(t,x) = F_{tv}(x) : [0,1] \times \Omega \longrightarrow \Omega$ which is proper for every v. Formula (2.22) implies

$$(F_v)_\star T - T = d\big((H_v)_\star([0,1] \otimes T)\big).$$

After averaging with a smoothing kernel $\rho_\varepsilon(v)$ we get $\Theta - T = dS$ where

$$\Theta = \int_{B(0,\varepsilon)} (F_v)_\star T\,\rho_\varepsilon(v)\,dv, \qquad S = \int_{B(0,\varepsilon)} (H_v)_\star([0,1] \otimes T)\,\rho_\varepsilon(v)\,dv.$$

Then S is a current of the same order s as T and Θ is smooth. Indeed, for $u \in \mathcal{D}^p(\Omega)$ we have

$$\langle \Theta, u\rangle = \langle T, u_\varepsilon\rangle \quad \text{where} \quad u_\varepsilon(x) = \int_{B(0,\varepsilon)} F_v^\star u(x)\,\rho_\varepsilon(v)\,dv\ ;$$

we can make a change of variable $z = F_v(x) \Leftrightarrow v = \psi(x)^{-1}(z - x)$ in the last integral and perform derivatives on ρ_ε to see that each seminorm $p^t_K(u_\varepsilon)$ is controlled by the sup norm of u. Thus Θ and all its derivatives are currents of order 0, so Θ is smooth. Now we have $d\Theta = 0$ and by the usual Poincaré lemma (1.22) applied to Θ we obtain

(2.24) Theorem. *Let $\Omega \subset \mathbb{R}^m$ be a starshaped open subset and $T \in {}^s\mathcal{D}'^{\,q}(\Omega)$ a current of degree $q \geq 1$ and order s such that $dT = 0$. There exists a current $S \in {}^s\mathcal{D}'^{\,q-1}(\Omega)$ of degree $q-1$ and order $\leq s$ such that $dS = T$ on Ω.* □

§3. Holomorphic Functions and Complex Manifolds

§3.A. Cauchy Formula in One Variable

We start by recalling a few elementary facts in one complex variable theory. Let $\Omega \subset \mathbb{C}$ be an open set and let $z = x + \mathrm{i}y$ be the complex variable, where $x, y \in \mathbb{R}$. If f is a function of class C^1 on Ω, we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \overline{z}}\,d\overline{z}$$

with the usual notations

$$(3.1)\qquad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big).$$

The function f is holomorphic on Ω if df is $\mathbb{C}$-linear, that is, $\partial f/\partial\overline{z} = 0$.

(3.2) Cauchy formula. *Let $K \subset \mathbb{C}$ be a compact set with piecewise C^1 boundary ∂K. Then for every $f \in C^1(K, \mathbb{C})$*

$$f(w) = \frac{1}{2\pi\mathrm{i}}\int_{\partial K}\frac{f(z)}{z-w}\,dz - \int_K \frac{1}{\pi(z-w)}\,\frac{\partial f}{\partial\overline{z}}\,d\lambda(z), \qquad w \in K^\circ$$

where $d\lambda(z) = \frac{\mathrm{i}}{2}dz\wedge d\overline{z} = dx\wedge dy$ is the Lebesgue measure on $\mathbb{C}$.

Proof. Assume for simplicity $w = 0$. As the function $z \mapsto 1/z$ is locally integrable at $z = 0$, we get

$$\begin{aligned}\int_K \frac{1}{\pi z}\,\frac{\partial f}{\partial\overline{z}}\,d\lambda(z) &= \lim_{\varepsilon\to 0}\int_{K\smallsetminus D(0,\varepsilon)}\frac{1}{\pi z}\,\frac{\partial f}{\partial\overline{z}}\,\frac{\mathrm{i}}{2}dz\wedge d\overline{z}\\ &= \lim_{\varepsilon\to 0}\int_{K\smallsetminus D(0,\varepsilon)} d\Big[\frac{1}{2\pi\mathrm{i}}\,f(z)\,\frac{dz}{z}\Big]\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\partial K} f(z)\,\frac{dz}{z} - \lim_{\varepsilon\to 0}\frac{1}{2\pi\mathrm{i}}\int_{\partial D(0,\varepsilon)} f(z)\,\frac{dz}{z}\end{aligned}$$

by Stokes' formula. The last integral is equal to $\frac{1}{2\pi}\int_0^{2\pi} f(\varepsilon e^{\mathrm{i}\theta})\,d\theta$ and converges to $f(0)$ as ε tends to 0. □

When f is holomorphic on Ω, we get the usual Cauchy formula

(3.3) $$f(w) = \frac{1}{2\pi\mathrm{i}}\int_{\partial K} \frac{f(z)}{z-w}\,dz, \qquad w \in K^\circ,$$

from which many basic properties of holomorphic functions can be derived: power and Laurent series expansions, Cauchy residue formula, ... Another interesting consequence is:

(3.4) Corollary. *The L^1_{loc} function $E(z) = 1/\pi z$ is a fundamental solution of the operator $\partial/\partial\overline{z}$ on $\mathbb{C}$, i.e. $\partial E/\partial\overline{z} = \delta_0$ (Dirac measure at 0). As a consequence, if v is a distribution with compact support in $\mathbb{C}$, then the convolution $u = (1/\pi z) \star v$ is a solution of the equation $\partial u/\partial\overline{z} = v$.*

Proof. Apply (3.2) with $w = 0$, $f \in \mathcal{D}(\mathbb{C})$ and $K \supset \operatorname{Supp} f$, so that $f = 0$ on the boundary ∂K and $f(0) = \langle 1/\pi z, -\partial f/\partial\overline{z}\rangle$. □

(3.5) Remark. It should be observed that this formula cannot be used to solve the equation $\partial u/\partial\overline{z} = v$ when $\operatorname{Supp} v$ is not compact; moreover, if $\operatorname{Supp} v$ is compact, a solution u with compact support need not always exist. Indeed, we have a necessary condition

$$\langle v, z^n\rangle = -\langle u, \partial z^n/\partial\overline{z}\rangle = 0$$

for all integers $n \ge 0$. Conversely, when the necessary condition $\langle v, z^n\rangle = 0$ is satisfied, the canonical solution $u = (1/\pi z) \star v$ has compact support: this is easily seen by means of the power series expansion $(w-z)^{-1} = \sum z^n w^{-n-1}$, if we suppose that $\operatorname{Supp} v$ is contained in the disk $|z| < R$ and that $|w| > R$.

§3.B. Holomorphic Functions of Several Variables

Let $\Omega \subset \mathbb{C}^n$ be an open set. A function $f : \Omega \to \mathbb{C}$ is said to be holomorphic if f is continuous and separately holomorphic with respect to each variable, i.e. $z_j \mapsto f(\ldots, z_j, \ldots)$ is holomorphic when $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ are fixed. The set of holomorphic functions on Ω is a ring and will be denoted $\mathcal{O}(\Omega)$. We first extend the Cauchy formula to the case of polydisks. The open polydisk $D(z_0, R)$ of center $(z_{0,1}, \ldots, z_{0,n})$ and (multi)radius $R = (R_1, \ldots, R_n)$ is defined as the product of the disks of center $z_{0,j}$ and radius $R_j > 0$ in each factor $\mathbb{C}$:

(3.6) $$D(z_0, R) = D(z_{0,1}, R_1) \times \ldots \times D(z_{0,n}, R_n) \subset \mathbb{C}^n.$$

The *distinguished boundary* of $D(z_0, R)$ is by definition the product of the boundary circles

(3.7) $$\Gamma(z_0, R) = \Gamma(z_{0,1}, R_1) \times \ldots \times \Gamma(z_{0,n}, R_n).$$

It is important to observe that the distinguished boundary is smaller than the topological boundary $\partial D(z_0, R) = \bigcup_j \{z \in \overline{D}(z_0, R)\,;\, |z_j - z_{0,j}| = R_j\}$ when $n \ge 2$. By induction on n, we easily get the

(3.8) Cauchy formula on polydisks. *If $\overline{D}(z_0, R)$ is a closed polydisk contained in Ω and $f \in \mathcal{O}(\Omega)$, then for all $w \in D(z_0, R)$ we have*

$$f(w) = \frac{1}{(2\pi\mathrm{i})^n} \int_{\Gamma(z_0,R)} \frac{f(z_1, \ldots, z_n)}{(z_1 - w_1) \ldots (z_n - w_n)}\, dz_1 \ldots dz_n. \qquad \square$$

The expansion $(z_j - w_j)^{-1} = \sum (w_j - z_{0,j})^{\alpha_j} (z_j - z_{0,j})^{-\alpha_j - 1}$, $\alpha_j \in \mathbb{N}$, $1 \leq j \leq n$, shows that f can be expanded as a convergent power series $f(w) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (w - z_0)^\alpha$ over the polydisk $D(z_0, R)$, with the standard notations $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and with

$$(3.9) \quad a_\alpha = \frac{1}{(2\pi\mathrm{i})^n} \int_{\Gamma(z_0,R)} \frac{f(z_1, \ldots, z_n)\, dz_1 \ldots dz_n}{(z_1 - z_{0,1})^{\alpha_1 + 1} \ldots (z_n - z_{0,n})^{\alpha_n + 1}} = \frac{f^{(\alpha)}(z_0)}{\alpha!}.$$

As a consequence, f is holomorphic over Ω if and only if f is $\mathbb{C}$-analytic. Arguments similar to the one variable case easily yield the

(3.10) Analytic continuation theorem. *If Ω is connected and if there exists a point $z_0 \in \Omega$ such that $f^{(\alpha)}(z_0) = 0$ for all $\alpha \in \mathbb{N}^n$, then $f = 0$ on Ω.* $\square$

Another consequence of (3.9) is the *Cauchy inequality*

$$(3.11) \quad |f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{R^\alpha} \sup_{\Gamma(z_0,R)} |f|, \qquad \overline{D}(z_0, R) \subset \Omega,$$

From this, it follows that every bounded holomorphic function on $\mathbb{C}^n$ is constant (Liouville's theorem), and more generally, every holomorphic function F on $\mathbb{C}^n$ such that $|F(z)| \leq A(1 + |z|)^B$ with suitable constants $A, B \geq 0$ is in fact a polynomial of total degree $\leq B$.

We endow $\mathcal{O}(\Omega)$ with the topology of uniform convergence on compact sets $K \subset\subset \Omega$, that is, the topology induced by $C^0(\Omega, \mathbb{C})$. Then $\mathcal{O}(\Omega)$ is closed in $C^0(\Omega, \mathbb{C})$. The Cauchy inequalities (3.11) show that all derivations D^α are continuous operators on $\mathcal{O}(\Omega)$ and that any sequence $f_j \in \mathcal{O}(\Omega)$ that is uniformly bounded on all compact sets $K \subset\subset \Omega$ is locally equicontinuous. By Ascoli's theorem, we obtain

(3.12) Montel's theorem. *Every locally uniformly bounded sequence (f_j) in $\mathcal{O}(\Omega)$ has a convergent subsequence $(f_{j(\nu)})$.*

In other words, bounded subsets of the Fréchet space $\mathcal{O}(\Omega)$ are relatively compact (a Fréchet space possessing this property is called a Montel space).

§3.C. Differential Calculus on Complex Analytic Manifolds

A *complex analytic manifold* X of dimension $\dim_{\mathbb{C}} X = n$ is a differentiable manifold equipped with a holomorphic atlas (τ_α) with values in $\mathbb{C}^n$; this means by definition that the transition maps $\tau_{\alpha\beta}$ are holomorphic. The tangent spaces $T_{X,x}$ then have a natural complex vector space structure, given by the coordinate isomorphisms

$$d\tau_\alpha(x) : T_{X,x} \longrightarrow \mathbb{C}^n, \qquad U_\alpha \ni x\,;$$

the induced complex structure on $T_{X,x}$ is indeed independent of α since the differentials $d\tau_{\alpha\beta}$ are $\mathbb{C}$-linear isomorphisms. We denote by $T_X^{\mathbb{R}}$ the underlying real tangent space and by $J \in \mathrm{End}(T_X^{\mathbb{R}})$ the *almost complex structure*, i.e. the operator of multiplication by $\mathrm{i} = \sqrt{-1}$. If $(z_1, \ldots, z_n)$ are complex analytic coordinates on an open subset $\Omega \subset X$ and $z_k = x_k + \mathrm{i}y_k$, then $(x_1, y_1, \ldots, x_n, y_n)$ define real coordinates on Ω, and $T^{\mathbb{R}}_{X\restriction\Omega}$ admits $(\partial/\partial x_1$, $\partial/\partial y_1$, …, $\partial/\partial x_n$, $\partial/\partial y_n)$ as a basis; the almost complex structure is given by $J(\partial/\partial x_k) = \partial/\partial y_k$, $J(\partial/\partial y_k) = -\partial/\partial x_k$. The complexified tangent space $\mathbb{C}\otimes T_X = \mathbb{C}\otimes_{\mathbb{R}} T_X^{\mathbb{R}} = T_X^{\mathbb{R}} \oplus \mathrm{i}T_X^{\mathbb{R}}$ splits into conjugate complex subspaces which are the eigenspaces of the complexified endomorphism $\mathrm{Id}\otimes J$ associated to the eigenvalues i and $-$i. These subspaces have respective bases

$$(3.13) \qquad \frac{\partial}{\partial z_k} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} - \mathrm{i}\frac{\partial}{\partial y_k}\Big), \qquad \frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} + \mathrm{i}\frac{\partial}{\partial y_k}\Big), \qquad 1 \le k \le n$$

and are denoted $T^{1,0}X$ (*holomorphic vectors* or *vectors of type* $(1,0)$) and $T^{0,1}X$ (*antiholomorphic vectors* or *vectors of type* $(0,1)$). The subspaces $T^{1,0}X$ and $T^{0,1}X$ are canonically isomorphic to the complex tangent space T_X (with complex structure J) and its conjugate $\overline{T_X}$ (with conjugate complex structure $-J$), via the $\mathbb{C}$-linear embeddings

$$\begin{array}{rlrl} T_X \longrightarrow & T_X^{1,0} \subset \mathbb{C}\otimes T_X, & \overline{T_X} \longrightarrow & T_X^{0,1} \subset \mathbb{C}\otimes T_X \\ \xi \longmapsto & \frac{1}{2}(\xi - \mathrm{i}J\xi), & \xi \longmapsto & \frac{1}{2}(\xi + \mathrm{i}J\xi). \end{array}$$

We thus have a canonical decomposition $\mathbb{C}\otimes T_X = T_X^{1,0} \oplus T_X^{0,1} \simeq T_X \oplus \overline{T_X}$, and by duality a decomposition

$$\mathrm{Hom}_{\mathbb{R}}(T_X^{\mathbb{R}};\mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\otimes T_X;\mathbb{C}) \simeq T_X^\star \oplus \overline{T_X^\star}$$

where $T_X^\star$ is the space of $\mathbb{C}$-linear forms and $\overline{T_X^\star}$ the space of conjugate $\mathbb{C}$-linear forms. With these notations, (dx_k, dy_k) is a basis of $\mathrm{Hom}_{\mathbb{R}}(T_{\mathbb{R}}X, \mathbb{C})$, (dz_j) a basis of $T_X^\star$, $(d\overline{z}_j)$ a basis of $\overline{T_X^\star}$, and the differential of a function $f \in C^1(\Omega, \mathbb{C})$ can be written

$$(3.14) \quad df = \sum_{k=1}^n \frac{\partial f}{\partial x_k}\,dx_k + \frac{\partial f}{\partial y_k}\,dy_k = \sum_{k=1}^n \frac{\partial f}{\partial z_k}\,dz_k + \frac{\partial f}{\partial \overline{z}_k}\,d\overline{z}_k.$$

The function f is holomorphic on Ω if and only if df is $\mathbb{C}$-linear, i.e. if and only if f satisfies the *Cauchy-Riemann equations* $\partial f/\partial\overline{z}_k = 0$ on Ω, $1 \le k \le n$. We still denote here by $\mathcal{O}(X)$ the algebra of holomorphic functions on X.

Now, we study the basic rules of complex differential calculus. The complexified exterior algebra $\mathbb{C} \otimes_{\mathbb{R}} \Lambda^\bullet_{\mathbb{R}}(T^{\mathbb{R}}_X)^\star = \Lambda^\bullet_{\mathbb{C}}(\mathbb{C} \otimes T_X)^\star$ is given by

$$\Lambda^k(\mathbb{C} \otimes T_X)^\star = \Lambda^k\big(T_X \oplus \overline{T_X}\big)^\star = \bigoplus_{p+q=k} \Lambda^{p,q} T^\star_X, \qquad 0 \le k \le 2n$$

where the exterior products are taken over $\mathbb{C}$, and where the components $\Lambda^{p,q}T^\star_X$ are defined by

$$(3.15) \quad \Lambda^{p,q}T^\star_X = \Lambda^p T^\star_X \otimes \Lambda^q \overline{T^\star_X}.$$

A complex differential form u on X is said to be of *bidegree* or *type* (p,q) if its value at every point lies in the component $\Lambda^{p,q}T^\star_X$; we shall denote by $C^s(\Omega, \Lambda^{p,q}T^\star_X)$ the space of differential forms of bidegree (p,q) and class C^s on any open subset Ω of X. If Ω is a coordinate open set, such a form can be written

$$u(z) = \sum_{|I|=p,|J|=q} u_{I,J}(z)\, dz_I \wedge d\overline{z}_J, \qquad u_{I,J} \in C^s(\Omega, \mathbb{C}).$$

This writing is usually much more convenient than the expression in terms of the real basis $(dx_I \wedge dy_J)_{|I|+|J|=k}$ which is not compatible with the splitting of $\Lambda^k T^\star_{\mathbb{C}} X$ in its (p,q) components. Formula (3.14) shows that the exterior derivative d splits into $d = d' + d''$, where

$$d' : C^\infty(X, \Lambda^{p,q}T^\star_X) \longrightarrow C^\infty(X, \Lambda^{p+1,q}T^\star_X),$$
$$d'' : C^\infty(X, \Lambda^{p,q}T^\star_X) \longrightarrow C^\infty(X, \Lambda^{p,q+1}T^\star_X),$$

$$(3.16') \quad d'u = \sum_{I,J} \sum_{1\le k\le n} \frac{\partial u_{I,J}}{\partial z_k}\, dz_k \wedge dz_I \wedge d\overline{z}_J,$$

$$(3.16'') \quad d''u = \sum_{I,J} \sum_{1\le k\le n.} \frac{\partial u_{I,J}}{\partial \overline{z}_k}\, d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.$$

The identity $d^2 = (d' + d'')^2 = 0$ is equivalent to

$$(3.17) \quad d'^2 = 0, \qquad d'd'' + d''d' = 0, \qquad d''^2 = 0,$$

since these three operators send (p,q)-forms in $(p+2,q)$, $(p+1,q+1)$ and $(p,q+2)$-forms, respectively. In particular, the operator d'' defines for each $p = 0, 1, \ldots, n$ a complex, called the *Dolbeault complex*

$$C^\infty(X, \Lambda^{p,0}T^\star_X) \xrightarrow{d''} \cdots \longrightarrow C^\infty(X, \Lambda^{p,q}T^\star_X) \xrightarrow{d''} C^\infty(X, \Lambda^{p,q+1}T^\star_X)$$

and corresponding *Dolbeault cohomology groups*

(3.18) $$H^{p,q}(X,\mathbb{C}) = \frac{\operatorname{Ker} d''^{\,p,q}}{\operatorname{Im} d''^{\,p,q-1}},$$

with the convention that the image of d'' is zero for $q=0$. The cohomology group $H^{p,0}(X,\mathbb{C})$ consists of $(p,0)$-forms $u=\sum_{|I|=p} u_I(z)\,dz_I$ such that $\partial u_I/\partial\overline{z}_k = 0$ for all I,k, i.e. such that all coefficients u_I are holomorphic. Such a form is called a *holomorphic p-form* on X.

Let $F : X_1 \longrightarrow X_2$ be a holomorphic map between complex manifolds. The pull-back $F^\star u$ of a (p,q)-form u of bidegree (p,q) on X_2 is again homogeneous of bidegree (p,q), because the components F_k of F in any coordinate chart are holomorphic, hence $F^\star dz_k = dF_k$ is $\mathbb{C}$-linear. In particular, the equality $dF^\star u = F^\star du$ implies

(3.19) $$d'F^\star u = F^\star d'u, \qquad d''F^\star u = F^\star d''u.$$

Note that these commutation relations are no longer true for a non holomorphic change of variable. As in the case of the De Rham cohomology groups, we get a pull-back morphism

$$F^\star : H^{p,q}(X_2,\mathbb{C}) \longrightarrow H^{p,q}(X_1,\mathbb{C}).$$

The rules of complex differential calculus can be easily extended to currents. We use the following notation.

(3.20) Definition. *There are decompositions*

$$\mathcal{D}^k(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{D}^{p,q}(X,\mathbb{C}), \qquad \mathcal{D}'_k(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{D}'_{p,q}(X,\mathbb{C}).$$

The space $\mathcal{D}'_{p,q}(X,\mathbb{C})$ is called the space of currents of bidimension (p,q) and bidegree $(n-p,n-q)$ on X, and is also denoted $\mathcal{D}'^{\,n-p,n-q}(X,\mathbb{C})$.

§3.D. Newton and Bochner-Martinelli Kernels

The *Newton kernel* is the elementary solution of the usual Laplace operator $\Delta = \sum \partial^2/\partial x_j^2$ in $\mathbb{R}^m$. We first recall a construction of the Newton kernel.

Let $d\lambda = dx_1 \ldots dx_m$ be the Lebesgue measure on $\mathbb{R}^m$. We denote by $B(a,r)$ the euclidean open ball of center a and radius r in $\mathbb{R}^m$ and by $S(a,r) = \partial B(a,r)$ the corresponding sphere. Finally, we set $\alpha_m = \operatorname{Vol}\big(B(0,1)\big)$ and $\sigma_{m-1} = m\alpha_m$ so that

(3.21) $$\operatorname{Vol}\big(B(a,r)\big) = \alpha_m r^m, \qquad \operatorname{Area}\big(S(a,r)\big) = \sigma_{m-1} r^{m-1}.$$

The second equality follows from the first by derivation. An explicit computation of the integral $\int_{\mathbb{R}^m} e^{-|x|^2} d\lambda(x)$ in polar coordinates shows that $\alpha_m = \pi^{m/2}/(m/2)!$ where $x! = \Gamma(x+1)$ is the Euler Gamma function. The *Newton kernel* is then given by:

$$(3.22)\quad \begin{cases} N(x) = \dfrac{1}{2\pi}\log|x| & \text{if } m=2,\\ N(x) = -\dfrac{1}{(m-2)\sigma_{m-1}}\,|x|^{2-m} & \text{if } m\neq 2.\end{cases}$$

The function $N(x)$ is locally integrable on $\mathbb{R}^m$ and satisfies $\Delta N = \delta_0$. When $m=2$, this follows from Cor. 3.4 and the fact that $\Delta = 4\partial^2/\partial z\partial\overline{z}$. When $m\neq 2$, this can be checked by computing the weak limit

$$\begin{aligned}\lim_{\varepsilon\to 0}\ \Delta(|x|^2+\varepsilon^2)^{1-m/2} &= \lim_{\varepsilon\to 0}\ m(2-m)\varepsilon^2(|x|^2+\varepsilon^2)^{-1-m/2}\\ &= m(2-m)\,I_m\,\delta_0\end{aligned}$$

with $I_m = \int_{\mathbb{R}^m}(|x|^2+1)^{-1-m/2}\,d\lambda(x)$. The last equality is easily seen by performing the change of variable $y=\varepsilon x$ in the integral

$$\int_{\mathbb{R}^m}\varepsilon^2(|x|^2+\varepsilon^2)^{-1-m/2}\,f(x)\,d\lambda(x) = \int_{\mathbb{R}^m}(|y|^2+1)^{-1-m/2}\,f(\varepsilon y)\,d\lambda(y),$$

where f is an arbitrary test function. Using polar coordinates, we find that $I_m = \sigma_{m-1}/m$ and our formula follows.

The *Bochner-Martinelli kernel* is the $(n,n-1)$-differential form on $\mathbb{C}^n$ with L^1_{loc} coefficients defined by

$$(3.23)\quad \begin{aligned} k_{\mathrm{BM}}(z) &= c_n\sum_{1\le j\le n}(-1)^j\,\frac{\overline{z}_j\,dz_1\wedge\ldots dz_n\wedge d\overline{z}_1\wedge\ldots\widehat{d\overline{z}_j}\ldots\wedge d\overline{z}_n}{|z|^{2n}},\\ c_n &= (-1)^{n(n-1)/2}\,\frac{(n-1)!}{(2\pi\mathrm{i})^n}.\end{aligned}$$

(3.24) Lemma. *$d''k_{\mathrm{BM}} = \delta_0$ on $\mathbb{C}^n$.*

Proof. Since the Lebesgue measure on $\mathbb{C}^n$ is

$$d\lambda(z) = \bigwedge_{1\le j\le n}\frac{\mathrm{i}}{2}dz_j\wedge d\overline{z}_j = \Big(\frac{\mathrm{i}}{2}\Big)^n(-1)^{\frac{n(n-1)}{2}}\,dz_1\wedge\ldots dz_n\wedge d\overline{z}_1\wedge\ldots d\overline{z}_n,$$

we find

$$\begin{aligned} d''k_{\mathrm{BM}} &= -\frac{(n-1)!}{\pi^n}\sum_{1\le j\le n}\frac{\partial}{\partial\overline{z}_j}\Big(\frac{\overline{z}_j}{|z|^{2n}}\Big)d\lambda(z)\\ &= -\frac{1}{n(n-1)\alpha_{2n}}\sum_{1\le j\le n}\frac{\partial^2}{\partial z_j\partial\overline{z}_j}\Big(\frac{1}{|z|^{2n-2}}\Big)d\lambda(z)\\ &= \Delta N(z)d\lambda(z) = \delta_0. \end{aligned}$$

□

We let $K_{\mathrm{BM}}(z,\zeta)$ be the pull-back of k_{BM} by the map $\pi:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$, $(z,\zeta)\longmapsto z-\zeta$. Then Formula (2.19) implies

(3.25) $\quad d''K_{\mathrm{BM}} = \pi^\star\delta_0 = [\Delta],$

where $[\Delta]$ denotes the current of integration on the diagonal $\Delta \subset \mathbb{C}^n \times \mathbb{C}^n$.

(3.26) Koppelman formula. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with piecewise C^1 boundary. Then for every (p,q)-form v of class C^1 on $\overline{\Omega}$ we have*

$$v(z) = \int_{\partial\Omega} K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta) \\ + d''_z \int_{\Omega} K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta) + \int_{\Omega} K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge d''v(\zeta)$$

on Ω, where $K^{p,q}_{\mathrm{BM}}(z,\zeta)$ denotes the component of $K_{\mathrm{BM}}(z,\zeta)$ of type (p,q) in z and $(n-p, n-q-1)$ in ζ.

Proof. Given $w \in \mathcal{D}^{n-p,n-q}(\Omega)$, we consider the integral

$$\int_{\partial\Omega\times\Omega} K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z).$$

It is well defined since K_{BM} has no singularities on $\partial\Omega \times \operatorname{Supp} v \subset\subset \partial\Omega\times\Omega$. Since $w(z)$ vanishes on $\partial\Omega$ the integral can be extended as well to $\partial(\Omega\times\Omega)$. As $K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z)$ is of total bidegree $(2n, 2n-1)$, its differential d' vanishes. Hence Stokes' formula yields

$$\int_{\partial\Omega\times\Omega} K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z) = \int_{\Omega\times\Omega} d''\big(K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z)\big) \\ = \int_{\Omega\times\Omega} d''K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z) - K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge d''v(\zeta)\wedge w(z) \\ - (-1)^{p+q}\int_{\Omega\times\Omega} K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge d''w(z).$$

By (3.25) we have

$$\int_{\Omega\times\Omega} d''K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta)\wedge w(z) = \int_{\Omega\times\Omega} [\Delta]\wedge v(\zeta)\wedge w(z) = \int_{\Omega} v(z)\wedge w(z)$$

Denoting $\langle\ ,\ \rangle$ the pairing between currents and test forms on Ω, the above equality is thus equivalent to

$$\langle \int_{\partial\Omega} K_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta), w(z)\rangle = \langle v(z) - \int_{\Omega} K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge d''v(\zeta), w(z)\rangle \\ - (-1)^{p+q}\langle \int_{\Omega} K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta), d''w(z)\rangle,$$

which is itself equivalent to the Koppelman formula by integrating $d''v$ by parts. $\square$

(3.27) Corollary. *Let $v \in {}^s\mathcal{D}^{p,q}(\mathbb{C}^n)$ be a form of class C^s with compact support such that $d''v = 0$, $q \geq 1$. Then the $(p, q-1)$-form*

$$u(z) = \int_{\mathbb{C}^n} K_{\mathrm{BM}}^{p,q-1}(z,\zeta) \wedge v(\zeta)$$

is a C^s solution of the equation $d''u = v$. Moreover, if $(p,q) = (0,1)$ and $n \geq 2$ then u has compact support, thus the Dolbeault cohomology group with compact support $H_c^{0,1}(\mathbb{C}^n, \mathbb{C})$ vanishes for $n \geq 2$.

Proof. Apply the Koppelman formula on a sufficiently large ball $\overline{\Omega} = \overline{B}(0,R)$ containing $\operatorname{Supp} v$. Then the formula immediately gives $d''u = v$. Observe that the coefficients of $K_{\mathrm{BM}}(z,\zeta)$ are $O(|z-\zeta|^{-(2n-1)})$, hence $|u(z)| = O(|z|^{-(2n-1)})$ at infinity. If $q = 1$, then u is holomorphic on $\mathbb{C}^n \smallsetminus \overline{B}(0,R)$. Now, this complement is a union of complex lines when $n \geq 2$, hence $u = 0$ on $\mathbb{C}^n \smallsetminus \overline{B}(0,R)$ by Liouville's theorem. □

(3.28) Hartogs extension theorem. *Let Ω be an open set in $\mathbb{C}^n$, $n \geq 2$, and let $K \subset \Omega$ be a compact subset such that $\Omega \smallsetminus K$ is connected. Then every holomorphic function $f \in \mathcal{O}(\Omega \smallsetminus K)$ extends into a function $\widetilde{f} \in \mathcal{O}(\Omega)$.*

Proof. Let $\psi \in \mathcal{D}(\Omega)$ be a cut-off function equal to 1 on a neighborhood of K. Set $f_0 = (1-\psi)f \in C^\infty(\Omega)$, defined as 0 on K. Then $v = d''f_0 = -f\,d''\psi$ can be extended by 0 outside Ω, and can thus be seen as a smooth $(0,1)$-form with compact support in $\mathbb{C}^n$, such that $d''v = 0$. By Cor. 3.27, there is a smooth function u with compact support in $\mathbb{C}^n$ such that $d''u = v$. Then $\widetilde{f} = f_0 - u \in \mathcal{O}(\Omega)$. Now u is holomorphic outside $\operatorname{Supp}\psi$, so u vanishes on the unbounded component G of $\mathbb{C}^n \smallsetminus \operatorname{Supp}\psi$. The boundary ∂G is contained in $\partial \operatorname{Supp}\psi \subset \Omega \smallsetminus K$, so $\widetilde{f} = (1-\psi)f - u$ coincides with f on the non empty open set $\Omega \cap G \subset \Omega \smallsetminus K$. Therefore $\widetilde{f} = f$ on the connected open set $\Omega \smallsetminus K$. □

A refined version of the Hartogs extension theorem due to Bochner will be given in Exercise 8.13. It shows that f need only be given as a C^1 function on $\partial\Omega$, satisfying the tangential Cauchy-Riemann equations (a so-called *CR-function*). Then f extends as a holomorphic function $\widetilde{f} \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$, provided that $\partial\Omega$ is connected.

§3.E. The Dolbeault-Grothendieck Lemma

We are now in a position to prove the Dolbeault-Grothendieck lemma (Dolbeault 1953), which is the analogue for d'' of the Poincaré lemma. The proof given below makes use of the Bochner-Martinelli kernel. Many other proofs can be given, e.g. by using a reduction to the one dimensional case in combination with the Cauchy formula (3.2), see Exercise 8.5 or (Hörmander 1966).

(3.29) Dolbeault-Grothendieck lemma. *Let Ω be a neighborhood of* 0 *in* $\mathbb{C}^n$ *and* $v \in {}^s\mathcal{E}^{p,q}(\Omega,\mathbb{C})$, *[resp.* $v \in {}^s\mathcal{D}'^{p,q}(\Omega,\mathbb{C})$*], such that* $d''v = 0$, *where* $1 \le s \le \infty$.

a) *If* $q = 0$, *then* $v(z) = \sum_{|I|=p} v_I(z)\,dz_I$ *is a holomorphic* p*-form, i.e. a form whose coefficients are holomorphic functions.*

b) *If* $q \ge 1$, *there exists a neighborhood* $\omega \subset \Omega$ *of* 0 *and a form* u *in* ${}^s\mathcal{E}^{p,q-1}(\omega,\mathbb{C})$ *[resp. a current* $u \in {}^s\mathcal{D}'^{p,q-1}(\omega,\mathbb{C})$*] such that* $d''u = v$ *on* ω.

Proof. We assume that Ω is a ball $B(0,r) \subset \mathbb{C}^n$ and take for simplicity $r > 1$ (possibly after a dilation of coordinates). We then set $\omega = B(0,1)$. Let $\psi \in \mathcal{D}(\Omega)$ be a cut-off function equal to 1 on ω. The Koppelman formula (3.26) applied to the form ψv on Ω gives

$$\psi(z)v(z) = d''_z \int_\Omega K_{\mathrm{BM}}^{p,q-1}(z,\zeta)\wedge\psi(\zeta)v(\zeta) + \int_\Omega K_{\mathrm{BM}}^{p,q}(z,\zeta)\wedge d''\psi(\zeta)\wedge v(\zeta).$$

This formula is valid even when v is a current, because we may regularize v as $v \star \rho_\varepsilon$ and take the limit. We introduce on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$ the kernel

$$K(z,w,\zeta) = c_n \sum_{j=1}^n \frac{(-1)^j(w_j - \overline{\zeta}_j)}{((z-\zeta)\cdot(w-\overline{\zeta}))^n} \bigwedge_k (dz_k - d\zeta_k) \wedge \bigwedge_{k\ne j} (dw_k - d\overline{\zeta}_k).$$

By construction, $K_{\mathrm{BM}}(z,\zeta)$ is the result of the substitution $w = \overline{z}$ in $K(z,w,\zeta)$, i.e. $K_{\mathrm{BM}} = h^\star K$ where $h(z,\zeta) = (z,\overline{z},\zeta)$. We denote by $K^{p,q}$ the component of K of bidegree $(p,0)$ in z, $(q,0)$ in w and $(n-p, n-q-1)$ in ζ. Then $K_{\mathrm{BM}}^{p,q} = h^\star K^{p,q}$ and we find

$$v = d''u_0 + g^\star v_1 \quad \text{on } \omega,$$

where $g(z) = (z,\overline{z})$ and

$$u_0(z) = \int_\Omega K_{\mathrm{BM}}^{p,q-1}(z,\zeta)\wedge\psi(\zeta)v(\zeta),$$

$$v_1(z,w) = \int_\Omega K^{p,q}(z,w,\zeta)\wedge d''\psi(\zeta)\wedge v(\zeta).$$

By definition of $K^{p,q}(z,w,\zeta)$, v_1 is holomorphic on the open set

$$U = \big\{(z,w)\in\omega\times\omega\,;\,\forall\zeta\notin\omega,\ \mathrm{Re}(z-\zeta)\cdot(w-\overline{\zeta}) > 0\big\},$$

which contains the "conjugate-diagonal" points $(z,\overline{z})$ as well as the points $(z,0)$ and $(0,w)$ in $\omega\times\omega$. Moreover U clearly has convex slices $(\{z\}\times\mathbb{C}^n)\cap U$ and $(\mathbb{C}^n\times\{w\})\cap U$. In particular U is starshaped with respect to w, i.e.

$$(z,w)\in U \Longrightarrow (z,tw)\in U, \qquad \forall t\in[0,1].$$

As u_1 is of type $(p,0)$ in z and $(q,0)$ in w, we get $d''_z(g^\star v_1) = g^\star d_w v_1 = 0$, hence $d_w v_1 = 0$. For $q=0$ we have $K_{\rm BM}^{p,q-1} = 0$, thus $u_0 = 0$, and v_1 does not depend on w, thus v is holomorphic on ω. For $q \geq 1$, we can use the homotopy formula (1.23) with respect to w (considering z as a parameter) to get a holomorphic form $u_1(z,w)$ of type $(p,0)$ in z and $(q-1,0)$ in w, such that $d_w u_1(z,w) = v_1(z,w)$. Then we get $d'' g^\star u_1 = g^\star d_w u_1 = g^\star v_1$, hence

$$v = d''(u_0 + g^\star u_1) \quad \text{on } \omega.$$

Finally, the coefficients of u_0 are obtained as linear combinations of convolutions of the coefficients of ψv with $L^1_{\rm loc}$ functions of the form $\overline{\zeta}_j|\zeta|^{-2n}$. Hence u_0 is of class C^s (resp. is a current of order s), if v is. □

(3.30) Corollary. *The operator d'' is hypoelliptic in bidegree $(p,0)$, i.e. if a current $f \in \mathcal{D}'^{p,0}(X,\mathbb{C})$ satisfies $d''f \in \mathcal{E}^{p,1}(X,\mathbb{C})$, then $f \in \mathcal{E}^{p,0}(X,\mathbb{C})$.*

Proof. The result is local, so we may assume that $X = \Omega$ is a neighborhood of 0 in $\mathbb{C}^n$. The $(p,1)$-form $v = d''f \in \mathcal{E}^{p,1}(X,\mathbb{C})$ satisfies $d''v = 0$, hence there exists $u \in \mathcal{E}^{p,0}(\widetilde{\Omega},\mathbb{C})$ such that $d''u = d''f$. Then $f-u$ is holomorphic and $f = (f-u)+u \in \mathcal{E}^{p,0}(\widetilde{\Omega},\mathbb{C})$. □

§4. Subharmonic Functions

A *harmonic* (resp. *subharmonic*) function on an open subset of $\mathbb{R}^m$ is essentially a function (or distribution) u such that $\Delta u = 0$ (resp. $\Delta u \geq 0$). A fundamental example of subharmonic function is given by the Newton kernel N, which is actually harmonic on $\mathbb{R}^m \smallsetminus \{0\}$. Subharmonic functions are an essential tool of harmonic analysis and potential theory. Before giving their precise definition and properties, we derive a basic integral formula involving the Green kernel of the Laplace operator on the ball.

§4.A. Construction of the Green Kernel

The *Green kernel* $G_\Omega(x,y)$ of a smoothly bounded domain $\Omega \subset\subset \mathbb{R}^m$ is the solution of the following *Dirichlet boundary problem* for the Laplace operator Δ on Ω:

(4.1) Definition. *The Green kernel of a smoothly bounded domain $\Omega \subset\subset \mathbb{R}^m$ is a function $G_\Omega(x,y) : \overline{\Omega} \times \overline{\Omega} \to [-\infty, 0]$ with the following properties:*

a) $G_\Omega(x,y)$ *is* C^∞ *on* $\overline{\Omega} \times \overline{\Omega} \smallsetminus \mathrm{Diag}_\Omega$ $\quad$ ($\mathrm{Diag}_\Omega =$ *diagonal*) ;

b) $G_\Omega(x,y) = G_\Omega(y,x)$;

c) $G_\Omega(x,y) < 0$ *on* $\Omega \times \Omega$ *and* $G_\Omega(x,y) = 0$ *on* $\partial\Omega \times \Omega$;

d) $\Delta_x G_\Omega(x,y) = \delta_y$ *on* Ω *for every fixed* $y \in \Omega$.

It can be shown that G_Ω always exists and is unique. The uniqueness is an easy consequence of the maximum principle (see Th. 4.14 below). In the case where $\Omega = B(0,r)$ is a ball (the only case we are going to deal with), the existence can be shown through explicit calculations. In fact the Green kernel $G_r(x,y)$ of $B(0,r)$ is

$$(4.2)\quad G_r(x,y) = N(x-y) - N\Big(\frac{|y|}{r}\Big(x - \frac{r^2}{|y|^2}\,y\Big)\Big), \qquad x,y \in \overline{B}(0,r).$$

A substitution of the explicit value of $N(x)$ yields:

$$G_r(x,y) = \frac{1}{4\pi}\log\frac{|x-y|^2}{r^2 - 2\langle x,y\rangle + \frac{1}{r^2}|x|^2\,|y|^2} \quad \text{if } m=2, \quad \text{otherwise}$$

$$G_r(x,y) = \frac{-1}{(m-2)\sigma_{m-1}}\Big(|x-y|^{2-m} - \big(r^2 - 2\langle x,y\rangle + \frac{1}{r^2}|x|^2\,|y|^2\big)^{1-m/2}\Big).$$

(4.3) Theorem. *The above defined function* G_r *satisfies all four properties* (4.1 a–d) *on* $\Omega = B(0,r)$*, thus* G_r *is the Green kernel of* $B(0,r)$*.*

Proof. The first three properties are immediately verified on the formulas, because

$$r^2 - 2\langle x,y\rangle + \frac{1}{r^2}|x|^2\,|y|^2 = |x-y|^2 + \frac{1}{r^2}\big(r^2-|x|^2\big)\big(r^2-|y|^2\big).$$

For property d), observe that $r^2y/|y|^2 \notin \overline{B}(0,r)$ whenever $y \in B(0,r) \smallsetminus \{0\}$. The second Newton kernel in the right hand side of (4.1) is thus harmonic in x on $B(0,r)$, and

$$\Delta_x G_r(x,y) = \Delta_x N(x-y) = \delta_y \quad \text{on} \quad B(0,r). \qquad \square$$

§4.B. Green-Riesz Representation Formula and Dirichlet Problem

§4.B.1. Green-Riesz Formula. For all smooth functions u, v on a smoothly bounded domain $\Omega \subset\subset \mathbb{R}^m$, we have

$$(4.4)\qquad \int_\Omega (u\,\Delta v - v\,\Delta u)\,d\lambda = \int_{\partial\Omega}\Big(u\,\frac{\partial v}{\partial\nu} - v\,\frac{\partial u}{\partial\nu}\Big)\,d\sigma$$

where $\partial/\partial\nu$ is the derivative along the outward normal unit vector ν of $\partial\Omega$ and $d\sigma$ the euclidean area measure. Indeed

$$(-1)^{j-1}\,dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge dx_m{}_{\restriction\partial\Omega} = \nu_j\,d\sigma,$$

for the wedge product of $\langle \nu, dx\rangle$ with the left hand side is $\nu_j\,d\lambda$. Therefore

$$\frac{\partial v}{\partial \nu}\, d\sigma = \sum_{j=1}^{m} \frac{\partial v}{\partial x_j}\,\nu_j\, d\sigma = \sum_{j=1}^{m} (-1)^{j-1}\frac{\partial v}{\partial x_j}\, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_m.$$

Formula (4.4) is then an easy consequence of Stokes' theorem. Observe that (4.4) is still valid if v is a distribution with singular support relatively compact in Ω. For $\Omega = B(0,r)$, $u \in C^2\big(\overline{B}(0,r),\mathbb{R}\big)$ and $v(y) = G_r(x,y)$, we get the *Green-Riesz representation formula*:

$$(4.5)\quad u(x) = \int_{B(0,r)} \Delta u(y)\, G_r(x,y)\, d\lambda(y) + \int_{S(0,r)} u(y)\, P_r(x,y)\, d\sigma(y)$$

where $P_r(x,y) = \partial G_r(x,y)/\partial\nu(y)$, $(x,y) \in B(0,r) \times S(0,r)$. The function $P_r(x,y)$ is called the *Poisson kernel*. It is smooth and satisfies $\Delta_x P_r(x,y) = 0$ on $B(0,r)$ by (4.1 d). A simple computation left to the reader yields:

$$(4.6)\quad P_r(x,y) = \frac{1}{\sigma_{m-1} r}\,\frac{r^2 - |x|^2}{|x-y|^m}.$$

Formula (4.5) for $u \equiv 1$ shows that $\int_{S(0,r)} P_r(x,y)\, d\sigma(y) = 1$. When x in $B(0,r)$ tends to $x_0 \in S(0,r)$, we see that $P_r(x,y)$ converges uniformly to 0 on every compact subset of $S(0,r) \smallsetminus \{x_0\}$; it follows that the measure $P_r(x,y)\, d\sigma(y)$ converges weakly to δ_{x_0} on $S(0,r)$.

§4.B.2. Solution of the Dirichlet Problem. For any bounded measurable function v on $S(a,r)$ we define

$$(4.7)\quad P_{a,r}[v](x) = \int_{S(a,r)} v(y)\, P_r(x-a, y-a)\, d\sigma(y), \qquad x \in B(a,r).$$

If $u \in C^0\big(\overline{B}(a,r),\mathbb{R}\big) \cap C^2\big(B(a,r),\mathbb{R}\big)$ is harmonic, i.e. $\Delta u = 0$ on $B(a,r)$, then (4.5) gives $u = P_{a,r}[u]$ on $B(a,r)$, i.e. the Poisson kernel reproduces harmonic functions. Suppose now that $v \in C^0\big(S(a,r),\mathbb{R}\big)$ is given. Then $P_r(x-a, y-a)\, d\sigma(y)$ converges weakly to δ_{x_0} when x tends to $x_0 \in S(a,r)$, so $P_{a,r}[v](x)$ converges to $v(x_0)$. It follows that the function u defined by

$$\begin{cases} u = P_{a,r}[v] & \text{on } \; B(a,r), \\ u = v & \text{on } \; S(a,r) \end{cases}$$

is continuous on $\overline{B}(a,r)$ and harmonic on $B(a,r)$; thus u is the solution of the Dirichlet problem with boundary values v.

§4.C. Definition and Basic Properties of Subharmonic Functions

§4.C.1. Definition. Mean Value Inequalities. If u is a Borel function on $\overline{B}(a,r)$ which is bounded above or below, we consider the mean values of u over the ball or sphere:

$$\mu_B(u\,;a,r) = \frac{1}{\alpha_m r^m} \int_{B(a,r)} u(x)\,d\lambda(x), \tag{4.8}$$

$$\mu_S(u\,;a,r) = \frac{1}{\sigma_{m-1} r^{m-1}} \int_{S(a,r)} u(x)\,d\sigma(x). \tag{4.8$'$}$$

As $d\lambda = dr\,d\sigma$ these mean values are related by

$$\begin{aligned} \mu_B(u\,;a,r) &= \frac{1}{\alpha_m r^m} \int_0^r \sigma_{m-1} t^{m-1}\,\mu_S(u\,;a,t)\,dt \\ &= m \int_0^1 t^{m-1}\,\mu_S(u\,;a,rt)\,dt. \end{aligned} \tag{4.9}$$

Now, apply formula (4.5) with $x = 0$. We get $P_r(0,y) = 1/\sigma_{m-1}r^{m-1}$ and $G_r(0,y) = (|y|^{2-m} - r^{2-m})/(2-m)\sigma_{m-1} = -(1/\sigma_{m-1}) \int_{|y|}^r t^{1-m} dt$, thus

$$\begin{aligned} \int_{B(0,r)} \Delta u(y)\,G_r(0,y)\,d\lambda(y) &= -\frac{1}{\sigma_{m-1}} \int_0^r \frac{dt}{t^{m-1}} \int_{|y|<t} \Delta u(y)\,d\lambda(y) \\ &= -\frac{1}{m} \int_0^r \mu_B(\Delta u\,;0,t)\,t\,dt \end{aligned}$$

thanks to the Fubini formula. By translating $S(0,r)$ to $S(a,r)$, (4.5) implies the *Gauss formula*

$$\mu_S(u\,;a,r) = u(a) + \frac{1}{m} \int_0^r \mu_B(\Delta u\,;a,t)\,t\,dt. \tag{4.10}$$

Let Ω be an open subset of $\mathbb{R}^m$ and $u \in C^2(\Omega,\mathbb{R})$. If $a \in \Omega$ and $\Delta u(a) > 0$ (resp. $\Delta u(a) < 0$), Formula (4.10) shows that $\mu_S(u\,;a,r) > u(a)$ (resp. $\mu_S(u\,;a,r) < u(a)$) for r small enough. In particular, u is harmonic (i.e. $\Delta u = 0$) if and only if u satisfies the *mean value equality*

$$\mu_S(u\,;a,r) = u(a), \qquad \forall \overline{B}(a,r) \subset \Omega.$$

Now, observe that if (ρ_ε) is a family of radially symmetric smoothing kernels associated with $\rho(x) = \widetilde{\rho}(|x|)$ and if u is a Borel locally bounded function, an easy computation yields

$$\begin{aligned} u \star \rho_\varepsilon(a) &= \int_{B(0,1)} u(a + \varepsilon x)\,\rho(x)\,d\lambda \\ &= \sigma_{m-1} \int_0^1 \mu_S(u\,;a,\varepsilon t)\,\widetilde{\rho}(t)\,t^{m-1}\,dt. \end{aligned} \tag{4.11}$$

Thus, if u is a Borel locally bounded function satisfying the mean value equality on Ω, (4.11) shows that $u \star \rho_\varepsilon = u$ on Ω_ε, in particular u must be smooth. Similarly, if we replace the mean value equality by an inequality, the relevant regularity property to be required for u is just semicontinuity.

(4.12) Theorem and definition. *Let $u : \Omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function. The following various forms of mean value inequalities are equivalent:*

a) $u(x) \le P_{a,r}[u](x), \quad \forall \overline{B}(a,r) \subset \Omega, \quad \forall x \in B(a,r)$;

b) $u(a) \le \mu_S\,(u\,; a, r), \quad \forall \overline{B}(a,r) \subset \Omega$;

c) $u(a) \le \mu_B(u\,; a, r), \quad \forall \overline{B}(a,r) \subset \Omega$;

d) *for every $a \in \Omega$, there exists a sequence (r_ν) decreasing to 0 such that*

$$u(a) \le \mu_B(u\,; a, r_\nu) \qquad \forall \nu \;;$$

e) *for every $a \in \Omega$, there exists a sequence (r_ν) decreasing to 0 such that*

$$u(a) \le \mu_S(u\,; a, r_\nu) \qquad \forall \nu.$$

A function u satisfying one of the above properties is said to be subharmonic on Ω. The set of subharmonic functions will be denoted by $\mathrm{Sh}(\Omega)$.

By (4.10) we see that a function $u \in C^2(\Omega, \mathbb{R})$ is subharmonic if and only if $\Delta u \ge 0$: in fact $\mu_S(u\,; a, r) < u(a)$ for r small if $\Delta u(a) < 0$. It is also clear on the definitions that every (locally) convex function on Ω is subharmonic.

Proof. We have obvious implications

$$\text{a)} \Longrightarrow \text{b)} \Longrightarrow \text{c)} \Longrightarrow \text{d)} \Longrightarrow \text{e)},$$

the second and last ones by (4.10) and the fact that $\mu_B(u\,; a, r_\nu) \le \mu_S(u\,; a, t)$ for at least one $t \in \,]0, r_\nu[$. In order to prove e) $\Longrightarrow$ a), we first need a suitable version of the maximum principle.

(4.13) Lemma. *Let $u : \Omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function satisfying property* 4.12 e). *If u attains its supremum at a point $x_0 \in \Omega$, then u is constant on the connected component of x_0 in Ω.*

Proof. We may assume that Ω is connected. Let

$$W = \{x \in \Omega \;;\; u(x) < u(x_0)\}.$$

W is open by the upper semicontinuity, and distinct from Ω since $x_0 \notin W$. We want to show that $W = \emptyset$. Otherwise W has a non empty connected component W_0, and W_0 has a boundary point $a \in \Omega$. We have $a \in \Omega \smallsetminus W$, thus $u(a) = u(x_0)$. By assumption 4.12 e), we get $u(a) \le \mu_S(u\,; a, r_\nu)$ for some sequence $r_\nu \to 0$. For r_ν small enough, W_0 intersects $\Omega \smallsetminus \overline{B}(a, r_\nu)$ and $B(a, r_\nu)$; as W_0 is connected, we also have $S(a, r_\nu) \cap W_0 \ne \emptyset$. Since $u \le u(x_0)$ on the sphere $S(a, r_\nu)$ and $u < u(x_0)$ on its open subset $S(a, r_\nu) \cap W_0$, we get $u(a) \le \mu_S(u\,; a, r) < u(x_0)$, a contradiction. □

(4.14) Maximum principle. *If u is subharmonic in Ω (in the sense that u satisfies the weakest property* 4.12 e)*), then*

$$\sup_\Omega u = \limsup_{\Omega \ni z \to \partial\Omega \cup \{\infty\}} u(z),$$

and $\sup_K u = \sup_{\partial K} u(z)$ for every compact subset $K \subset \Omega$.

Proof. We have of course $\limsup_{z \to \partial\Omega \cup \{\infty\}} u(z) \le \sup_\Omega u$. If the inequality is strict, this means that the supremum is achieved on some compact subset $L \subset \Omega$. Thus, by the upper semicontinuity, there is $x_0 \in L$ such that $\sup_\Omega u = \sup_L u = u(x_0)$. Lemma 4.13 shows that u is constant on the connected component Ω_0 of x_0 in Ω, hence

$$\sup_\Omega u = u(x_0) = \limsup_{\Omega_0 \ni z \to \partial\Omega_0 \cup \{\infty\}} u(z) \le \limsup_{\Omega \ni z \to \partial\Omega \cup \{\infty\}} u(z),$$

contradiction. The statement involving a compact subset K is obtained by applying the first statement to $\Omega' = K^\circ$. □

Proof of (4.12) e) $\Longrightarrow$ a) Let u be an upper semicontinuous function satisfying 4.12 e) and $\overline{B}(a,r) \subset \Omega$ an arbitrary closed ball. One can find a decreasing sequence of continuous functions $v_k \in C^0\big(S(a,r), \mathbb{R}\big)$ such that $\lim v_k = u$. Set $h_k = P_{a,r}[v_k] \in C^0\big(\overline{B}(a,r), \mathbb{R}\big)$. As h_k is harmonic on $B(a,r)$, the function $u - h_k$ satisfies 4.12 e) on $B(a,r)$. Furthermore $\limsup_{x \to \xi \in S(a,r)} u(x) - h_k(x) \le u(\xi) - v_k(\xi) \le 0$, so $u - h_k \le 0$ on $B(a,r)$ by Th. 4.14. By monotone convergence, we find $u \le P_{a,r}[u]$ on $B(a,r)$ when k tends to $+\infty$. □

§**4.C.2. Basic Properties.** Here is a short list of the most basic properties.

(4.15) Theorem. *For any decreasing sequence (u_k) of subharmonic functions, the limit $u = \lim u_k$ is subharmonic.*

Proof. A decreasing limit of upper semicontinuous functions is again upper semicontinuous, and the mean value inequalities 4.12 remain valid for u by Lebesgue's monotone convergence theorem. □

(4.16) Theorem. *Let $u_1, \ldots, u_p \in \mathrm{Sh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is non decreasing in each t_j. If χ is extended by continuity into a function $[-\infty, +\infty[^p \longrightarrow [-\infty, +\infty[$, then*

$$\chi(u_1, \ldots, u_p) \in \mathrm{Sh}(\Omega).$$

In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p}) \in \mathrm{Sh}(\Omega)$.

Proof. Every convex function is continuous, hence $\chi(u_1, \ldots, u_p)$ is upper semicontinuous. One can write

$$\chi(t) = \sup_{i \in I} A_i(t)$$

where $A_i(t) = a_1 t_1 + \cdots + a_p t_p + b$ is the family of affine functions that define supporting hyperplanes of the graph of χ. As $\chi(t_1, \ldots, t_p)$ is non-decreasing in each t_j, we have $a_j \geq 0$, thus

$$\sum_{1 \leq j \leq p} a_j u_j(x) + b \leq \mu_B\big(\sum a_j u_j + b\,; x, r\big) \leq \mu_B\big(\chi(u_1, \ldots, u_p)\,; x, r\big)$$

for every ball $\overline{B}(x, r) \subset \Omega$. If one takes the supremum of this inequality over all the A_i 's, it follows that $\chi(u_1, \ldots, u_p)$ satisfies the mean value inequality 4.12 c). In the last example, the function $\chi(t_1, \ldots, t_p) = \log(e^{t_1} + \cdots + e^{t_p})$ is convex because

$$\sum_{1 \leq j,k \leq p} \frac{\partial^2 \chi}{\partial t_j \partial t_k}\, \xi_j \xi_k = e^{-\chi} \sum \xi_j^2\, e^{t_j} - e^{-2\chi}\big(\sum \xi_j\, e^{t_j}\big)^2$$

and $\big(\sum \xi_j\, e^{t_j}\big)^2 \leq \big(\sum \xi_j^2\, e^{t_j}\big)\, e^{\chi}$ by the Cauchy-Schwarz inequality. □

(4.17) Theorem. *If Ω is connected and $u \in \mathrm{Sh}(\Omega)$, then either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}(\Omega)$.*

Proof. Note that a subharmonic function is always locally bounded above. Let W be the set of points $x \in \Omega$ such that u is integrable in a neighborhood of x. Then W is open by definition and $u > -\infty$ almost everywhere on W. If $x \in \overline{W}$, one can choose $a \in W$ such that $|a - x| < r = \frac{1}{2} d(x, \complement\Omega)$ and $u(a) > -\infty$. Then $B(a, r)$ is a neighborhood of x, $\overline{B}(a, r) \subset \Omega$ and $\mu_B(u\,; a, r) \geq u(a) > -\infty$. Therefore $x \in W$, W is also closed. We must have $W = \Omega$ or $W = \emptyset$; in the last case $u \equiv -\infty$ by the mean value inequality. □

(4.18) Theorem. *Let $u \in \mathrm{Sh}(\Omega)$ be such that $u \not\equiv -\infty$ on each connected component of Ω. Then*

a) $r \longmapsto \mu_S(u\,; a, r)$, $r \longmapsto \mu_B(u\,; a, r)$ *are non decreasing functions in the interval* $]0, d(a, \complement\Omega)[$, *and* $\mu_B(u\,; a, r) \leq \mu_S(u\,; a, r)$.

b) *For any family (ρ_ε) of smoothing kernels, $u \star \rho_\varepsilon \in \mathrm{Sh}(\Omega_\varepsilon) \cap C^\infty(\Omega_\varepsilon, \mathbb{R})$, the family $(u \star \rho_\varepsilon)$ is non decreasing in ε and $\lim_{\varepsilon \to 0} u \star \rho_\varepsilon = u$.*

Proof. We first verify statements a) and b) when $u \in C^2(\Omega, \mathbb{R})$. Then $\Delta u \geq 0$ and $\mu_S(u\,; a, r)$ is non decreasing in virtue of (4.10). By (4.9), we find that $\mu_B(u\,; a, r)$ is also non decreasing and that $\mu_B(u\,; a, r) \leq \mu_S(u\,; a, r)$. Furthermore, Formula (4.11) shows that $\varepsilon \longmapsto u \star \rho_\varepsilon(a)$ is non decreasing (provided that ρ_ε is radially symmetric).

In the general case, we first observe that property 4.12 c) is equivalent to the inequality

$$u \leq u \star \mu_r \quad \text{on} \quad \Omega_r, \qquad \forall r > 0,$$

where μ_r is the probability measure of uniform density on $B(0,r)$. This inequality implies $u\star\rho_\varepsilon \leq u\star\rho_\varepsilon\star\mu_r$ on $(\Omega_r)_\varepsilon = \Omega_{r+\varepsilon}$, thus $u\star\rho_\varepsilon \in C^\infty(\Omega_\varepsilon, \mathbb{R})$ is subharmonic on Ω_ε. It follows that $u \star \rho_\varepsilon \star \rho_\eta$ is non decreasing in η ; by symmetry, it is also non decreasing in ε, and so is $u \star \rho_\varepsilon = \lim_{\eta\to 0} u \star \rho_\varepsilon \star \rho_\eta$. We have $u \star \rho_\varepsilon \geq u$ by (4.19) and $\limsup_{\varepsilon\to 0} u \star \rho_\varepsilon \leq u$ by the upper semicontinuity. Hence $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u$. Property a) for u follows now from its validity for $u \star \rho_\varepsilon$ and from the monotone convergence theorem. □

(4.19) Corollary. *If $u \in \mathrm{Sh}(\Omega)$ is such that $u \not\equiv -\infty$ on each connected component of Ω, then Δu computed in the sense of distribution theory is a positive measure.*

Indeed $\Delta(u\star\rho_\varepsilon) \geq 0$ as a function, and $\Delta(u\star\rho_\varepsilon)$ converges weakly to Δu in $\mathcal{D}'(\Omega)$. Corollary 4.19 has a converse, but the correct statement is slightly more involved than for the direct property:

(4.20) Theorem. *If $v \in \mathcal{D}'(\Omega)$ is such that Δv is a positive measure, there exists a unique function $u \in \mathrm{Sh}(\Omega)$ locally integrable such that v is the distribution associated to u.*

We must point out that u need not coincide everywhere with v, even when v is a locally integrable upper semicontinuous function: for example, if v is the characteristic function of a compact subset $K \subset \Omega$ of measure 0, the subharmonic representant of v is $u = 0$.

Proof. Set $v_\varepsilon = v \star \rho_\varepsilon \in C^\infty(\Omega_\varepsilon, \mathbb{R})$. Then $\Delta v_\varepsilon = (\Delta v) \star \rho_\varepsilon \geq 0$, thus $v_\varepsilon \in \mathrm{Sh}(\Omega_\varepsilon)$. Arguments similar to those in the proof of Th. 4.18 show that (v_ε) is non decreasing in ε. Then $u := \lim_{\varepsilon\to 0}\ v_\varepsilon \in \mathrm{Sh}(\Omega)$ by Th. 4.15. Since v_ε converges weakly to v, the monotone convergence theorem shows that

$$\langle v, f\rangle = \lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon\, f\, d\lambda = \int_\Omega u\, f\, d\lambda, \qquad \forall f \in \mathcal{D}(\Omega), \quad f \geq 0,$$

which concludes the existence part. The uniqueness of u is clear from the fact that u must satisfy $u = \lim u \star \rho_\varepsilon = \lim v \star \rho_\varepsilon$. □

The most natural topology on the space $\mathrm{Sh}(\Omega)$ of subharmonic functions is the topology induced by the vector space topology of $L^1_{\mathrm{loc}}(\Omega)$ (Fréchet topology of convergence in L^1 norm on every compact subset of Ω).

(4.21) Proposition. *The convex cone $\mathrm{Sh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ is closed in $L^1_{\mathrm{loc}}(\Omega)$, and it has the property that every bounded subset is relatively compact.*

Proof. Let (u_j) be a sequence in $\mathrm{Sh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$. If $u_j \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ then $\Delta u_j \to \Delta u$ in the weak topology of distributions, hence $\Delta u \geq 0$ and u can be represented by a subharmonic function thanks to Th. 4.20. Now, suppose that $\|u_j\|_{L^1(K)}$ is uniformly bounded for every compact subset K of Ω. Let $\mu_j = \Delta u_j \geq 0$. If $\psi \in \mathcal{D}(\Omega)$ is a test function equal to 1 on a neighborhood ω of K and such that $0 \leq \psi \leq 1$ on Ω, we find

$$\mu_j(K) \leq \int_\Omega \psi\, \Delta u_j\, d\lambda = \int_\Omega \Delta\psi\, u_j\, d\lambda \leq C\|u_j\|_{L^1(K')},$$

where $K' = \operatorname{Supp}\psi$, hence the sequence of measures (μ_j) is uniformly bounded in mass on every compact subset of Ω. By weak compactness, there is a subsequence (μ_{j_ν}) which converges weakly to a positive measure μ on Ω. We claim that $f \star (\psi\mu_{j_\nu})$ converges to $f \star (\psi\mu)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^m)$ for every function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. In fact, this is clear if $f \in C^\infty(\mathbb{R}^m)$, and in general we use an approximation of f by a smooth function g together with the estimate

$$\|(f-g) \star (\psi\mu_{j_\nu})\|_{L^1(A)} \leq \|(f-g)\|_{L^1(A+K')}\mu_{j_\nu}(K'), \qquad \forall A \subset\subset \mathbb{R}^m$$

to get the conclusion. We apply this when $f = N$ is the Newton kernel. Then $h_j = u_j - N \star (\psi\mu_j)$ is harmonic on ω and bounded in $L^1(\omega)$. As $h_j = h_j \star \rho_\varepsilon$ for any smoothing kernel ρ_ε, we see that all derivatives $D^\alpha h_j = h_j \star (D^\alpha \rho_\varepsilon)$ are in fact uniformly locally bounded in ω. Hence, after extracting a new subsequence, we may suppose that h_{j_ν} converges uniformly to a limit h on ω. Then $u_{j_\nu} = h_{j_\nu} + N \star (\psi\mu_{j_\nu})$ converges to $u = h + N \star (\psi\mu)$ in $L^1_{\mathrm{loc}}(\omega)$, as desired. □

We conclude this subsection by stating a generalized version of the Green-Riesz formula.

(4.22) Proposition. *Let $u \in \mathrm{Sh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ and $\overline{B}(0,r) \subset \Omega$.*

a) *The Green-Riesz formula still holds true for such an u, namely, for every $x \in B(0,r)$*

$$u(x) = \int_{B(0,r)} \Delta u(y)\, G_r(x,y)\, d\lambda(y) + \int_{S(0,r)} u(y)\, P_r(x,y)\, d\sigma(y).$$

b) (Harnack inequality)
If $u \geq 0$ on $\overline{B}(0,r)$, then for all $x \in B(0,r)$

$$0 \leq u(x) \leq \int_{S(0,r)} u(y)\, P_r(x,y)\, d\sigma(y) \leq \frac{r^{m-2}(r+|x|)}{(r-|x|)^{m-1}}\, \mu_S(u\,;0,r).$$

If $u \leq 0$ on $\overline{B}(0,r)$, then for all $x \in B(0,r)$

$$u(x) \leq \int_{S(0,r)} u(y)\, P_r(x,y)\, d\sigma(y) \leq \frac{r^{m-2}(r-|x|)}{(r+|x|)^{m-1}}\, \mu_S(u\,;0,r) \leq 0.$$

Proof. We know that a) holds true if u is of class C^2. In general, we replace u by $u \star \rho_\varepsilon$ and take the limit. We only have to check that

$$\int_{B(0,r)} \mu \star \rho_\varepsilon(y)\, G_r(x,y)\, d\lambda(y) = \lim_{\varepsilon \to 0} \int_{B(0,r)} \mu(y)\, G_r(x,y)\, d\lambda(y)$$

for the positive measure $\mu = \Delta u$. Let us denote by $\widetilde{G}_x(y)$ the function such that

$$\widetilde{G}_x(y) = \begin{cases} G_r(x,y) & \text{if } x \in B(0,r) \\ 0 & \text{if } x \notin B(0,r). \end{cases}$$

Then

$$\begin{aligned} \int_{B(0,r)} \mu \star \rho_\varepsilon(y)\, G_r(x,y)\, d\lambda(y) &= \int_{\mathbb{R}^m} \mu \star \rho_\varepsilon(y)\, \widetilde{G}_x(y)\, d\lambda(y) \\ &= \int_{\mathbb{R}^m} \mu(y)\, \widetilde{G}_x \star \rho_\varepsilon(y)\, d\lambda(y). \end{aligned}$$

However $\widetilde{G}_x$ is continuous on $\mathbb{R}^m \smallsetminus \{x\}$ and subharmonic in a neighborhood of x, hence $\widetilde{G}_x \star \rho_\varepsilon$ converges uniformly to $\widetilde{G}_x$ on every compact subset of $\mathbb{R}^m \smallsetminus \{x\}$, and converges pointwise monotonically in a neighborhood of x. The desired equality follows by the monotone convergence theorem. Finally, b) is a consequence of a), for the integral involving Δu is nonpositive and

$$\frac{1}{\sigma_{m-1} r^{m-1}} \frac{r^{m-2}(r - |x|)}{(r + |x|)^{m-1}} \le P_r(x,y) \le \frac{1}{\sigma_{m-1} r^{m-1}} \frac{r^{m-2}(r + |x|)}{(r - |x|)^{m-1}}$$

by (4.6) combined with the obvious inequality $(r - |x|)^m \le |x - y|^m \le (r + |x|)^m$. □

§4.C.3. Upper Envelopes and Choquet's Lemma. Let $\Omega \subset \mathbb{R}^n$ and let $(u_\alpha)_{\alpha \in I}$ be a family of upper semicontinuous functions $\Omega \longrightarrow [-\infty, +\infty[$. We assume that (u_α) is locally uniformly bounded above. Then the upper envelope

$$u = \sup u_\alpha$$

need not be upper semicontinuous, so we consider its *upper semicontinuous regularization*:

$$u^\star(z) = \lim_{\varepsilon \to 0} \sup_{B(z,\varepsilon)} u \ge u(z).$$

It is easy to check that $u^\star$ is the smallest upper semicontinuous function which is $\ge u$. Our goal is to show that $u^\star$ can be computed with a countable subfamily of (u_α). Let $B(z_j, \varepsilon_j)$ be a countable basis of the topology of Ω. For each j, let (z_{jk}) be a sequence of points in $B(z_j, \varepsilon_j)$ such that

$$\sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u,$$

and for each pair (j,k), let $\alpha(j,k,l)$ be a sequence of indices $\alpha \in I$ such that $u(z_{jk}) = \sup_l u_{\alpha(j,k,l)}(z_{jk})$. Set

$$v = \sup_{j,k,l} u_{\alpha(j,k,l)}.$$

Then $v \le u$ and $v^\star \le u^\star$. On the other hand

$$\sup_{B(z_j,\varepsilon_j)} v \ge \sup_k v(z_{jk}) \ge \sup_{k,l} u_{\alpha(j,k,l)}(z_{jk}) = \sup_k u(z_{jk}) = \sup_{B(z_j,\varepsilon_j)} u.$$

As every ball $B(z,\varepsilon)$ is a union of balls $B(z_j,\varepsilon_j)$, we easily conclude that $v^\star \ge u^\star$, hence $v^\star = u^\star$. Therefore:

(4.23) Choquet's lemma. *Every family (u_α) has a countable subfamily $(v_j) = (u_{\alpha(j)})$ such that its upper envelope v satisfies $v \le u \le u^\star = v^\star$.* □

(4.24) Proposition. *If all u_α are subharmonic, the upper regularization $u^\star$ is subharmonic and equal almost everywhere to u.*

Proof. By Choquet's lemma we may assume that (u_α) is countable. Then $u = \sup u_\alpha$ is a Borel function. As each u_α satisfies the mean value inequality on every ball $\overline{B}(z,r) \subset \Omega$, we get

$$u(z) = \sup u_\alpha(z) \le \sup \mu_B(u_\alpha\,;\, z,r) \le \mu_B(u\,;\, z,r).$$

The right-hand side is a continuous function of z, so we infer

$$u^\star(z) \le \mu_B(u\,;\, z,r) \le \mu_B(u^\star\,;\, z,r)$$

and $u^\star$ is subharmonic. By the upper semicontinuity of $u^\star$ and the above inequality we find $u^\star(z) = \lim_{r\to 0} \mu_B(u\,;\, z,r)$, thus $u^\star = u$ almost everywhere by Lebesgue's lemma. □

§5. Plurisubharmonic Functions

§5.A. Definition and Basic Properties

Plurisubharmonic functions have been introduced independently by (Lelong 1942) and (Oka 1942) for the study of holomorphic convexity. They are the complex counterparts of subharmonic functions.

(5.1) Definition. *A function $u : \Omega \longrightarrow [-\infty, +\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic if*

a) *u is upper semicontinuous* ;

b) *for every complex line $L \subset \mathbb{C}^n$, $u_{\restriction \Omega \cap L}$ is subharmonic on $\Omega \cap L$.*

The set of plurisubharmonic functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

An equivalent way of stating property b) is: for all $a \in \Omega$, $\xi \in \mathbb{C}^n$, $|\xi| < d(a, \complement\Omega)$, then

$$(5.2) \quad u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{\mathrm{i}\theta}\,\xi)\,d\theta.$$

An integration of (5.2) over $\xi \in S(0,r)$ yields $u(a) \le \mu_S(u\,; a, r)$, therefore

$$(5.3) \quad \mathrm{Psh}(\Omega) \subset \mathrm{Sh}(\Omega).$$

The following results have already been proved for subharmonic functions and are easy to extend to the case of plurisubharmonic functions:

(5.4) Theorem. *For any decreasing sequence of plurisubharmonic functions $u_k \in \mathrm{Psh}(\Omega)$, the limit $u = \lim u_k$ is plurisubharmonic on Ω.*

(5.5) Theorem. *Let $u \in \mathrm{Psh}(\Omega)$ be such that $u \not\equiv -\infty$ on every connected component of Ω. If (ρ_ε) is a family of smoothing kernels, then $u \star \rho_\varepsilon$ is C^∞ and plurisubharmonic on Ω_ε, the family $(u \star \rho_\varepsilon)$ is non decreasing in ε and* $\lim_{\varepsilon \to 0} u \star \rho_\varepsilon = u$.

(5.6) Theorem. *Let $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is non decreasing in each t_j. Then $\chi(u_1, \ldots, u_p)$ is plurisubharmonic on Ω. In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p})$ are plurisubharmonic on Ω.*

(5.7) Theorem. *Let $\{u_\alpha\} \subset \mathrm{Psh}(\Omega)$ be locally uniformly bounded from above and $u = \sup u_\alpha$. Then the regularized upper envelope $u^\star$ is plurisubharmonic and is equal to u almost everywhere.*

Proof. By Choquet's lemma, we may assume that (u_α) is countable. Then u is a Borel function which clearly satisfies (5.2), and thus $u \star \rho_\varepsilon$ also satisfies (5.2). Hence $u \star \rho_\varepsilon$ is plurisubharmonic. By Proposition 4.24, $u^\star = u$ almost everywhere and $u^\star$ is subharmonic, so

$$u^\star = \lim u^\star \star \rho_\varepsilon = \lim u \star \rho_\varepsilon$$

is plurisubharmonic. □

If $u \in C^2(\Omega, \mathbb{R})$, the subharmonicity of restrictions of u to complex lines, $\mathbb{C} \ni w \longmapsto u(a + w\xi)$, $a \in \Omega$, $\xi \in \mathbb{C}^n$, is equivalent to

$$\frac{\partial^2}{\partial w \partial \overline{w}} u(a + w\xi) = \sum_{1 \le j,k \le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(a + w\xi)\, \xi_j \overline{\xi}_k \ge 0.$$

Therefore, u is plurisubharmonic on Ω if and only if the hermitian form $\sum \partial^2 u / \partial z_j \partial \overline{z}_k(a)\, \xi_j \overline{\xi}_k$ is semipositive at every point $a \in \Omega$. This equivalence is still true for arbitrary plurisubharmonic functions, under the following form:

(5.8) Theorem. *If $u \in \mathrm{Psh}(\Omega)$, $u \not\equiv -\infty$ on every connected component of Ω, then for all $\xi \in \mathbb{C}^n$*

$$Hu(\xi) := \sum_{1 \le j,k \le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\, \xi_j \overline{\xi}_k \in \mathcal{D}'(\Omega)$$

is a positive measure. Conversely, if $v \in \mathcal{D}'(\Omega)$ is such that $Hv(\xi)$ is a positive measure for every $\xi \in \mathbb{C}^n$, there exists a unique function $u \in \mathrm{Psh}(\Omega)$ locally integrable on Ω such that v is the distribution associated to u.

Proof. If $u \in \mathrm{Psh}(\Omega)$, then $Hu(\xi) = \text{weak } \lim H(u \star \rho_\varepsilon)(\xi) \ge 0$. Conversely, $Hv \ge 0$ implies $H(v \star \rho_\varepsilon) = (Hv) \star \rho_\varepsilon \ge 0$, thus $v \star \rho_\varepsilon \in \mathrm{Psh}(\Omega)$, and also $\Delta v \ge 0$, hence $(v \star \rho_\varepsilon)$ is non decreasing in ε and $u = \lim_{\varepsilon \to 0} v \star \rho_\varepsilon \in \mathrm{Psh}(\Omega)$ by Th. 5.4. □

(5.9) Proposition. *The convex cone $\mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ is closed in $L^1_{\mathrm{loc}}(\Omega)$, and it has the property that every bounded subset is relatively compact.*

§5.B. Relations with Holomorphic Functions

In order to get a better geometric insight, we assume more generally that u is a C^2 function on a complex n-dimensional manifold X. The *complex Hessian* of u at a point $a \in X$ is the hermitian form on T_X defined by

$$(5.10) \qquad Hu_a = \sum_{1 \le j,k \le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(a)\, dz_j \otimes d\overline{z}_k.$$

If $F : X \longrightarrow Y$ is a holomorphic mapping and if $v \in C^2(Y, \mathbb{R})$, we have $d'd''(v \circ F) = F^\star d'd''v$. In equivalent notations, a direct calculation gives for all $\xi \in T_{X,a}$

$$H(v \circ F)_a(\xi) = \sum_{j,k,l,m} \frac{\partial^2 v\big(F(a)\big)}{\partial z_l \partial \overline{z}_m} \frac{\partial F_l(a)}{\partial z_j} \xi_j \overline{\frac{\partial F_m(a)}{\partial z_k} \xi_k} = Hv_{F(a)}\big(F'(a).\xi\big).$$

In particular Hu_a does not depend on the choice of coordinates $(z_1, \ldots, z_n)$ on X, and $Hv_a \ge 0$ on Y implies $H(v \circ F)_a \ge 0$ on X. Therefore, the notion of plurisubharmonic function makes sense on any complex manifold.

(5.11) Theorem. *If $F : X \longrightarrow Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v \circ F \in \mathrm{Psh}(X)$.*

Proof. It is enough to prove the result when $X = \Omega_1 \subset \mathbb{C}^n$ and $X = \Omega_2 \subset \mathbb{C}^p$ are open subsets . The conclusion is already known when v is of class C^2, and it can be extended to an arbitrary upper semicontinuous function v by using Th. 5.4 and the fact that $v = \lim v \star \rho_\varepsilon$. □

(5.12) Example. By (3.22) we see that $\log|z|$ is subharmonic on $\mathbb{C}$, thus $\log|f| \in \mathrm{Psh}(X)$ for every holomorphic function $f \in \mathcal{O}(X)$. More generally

$$\log\big(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\big) \in \mathrm{Psh}(X)$$

for every $f_j \in \mathcal{O}(X)$ and $\alpha_j \geq 0$ (apply Th. 5.6 with $u_j = \alpha_j \log|f_j|$).

§5.C. Convexity Properties

The close analogy of plurisubharmonicity with the concept of convexity strongly suggests that there are deeper connections between these notions. We describe here a few elementary facts illustrating this philosophy. Another interesting connection between plurisubharmonicity and convexity will be seen in § 7.B (Kiselman's minimum principle).

(5.13) Theorem. *If $\Omega = \omega + \mathrm{i}\omega'$ where ω, ω' are open subsets of $\mathbb{R}^n$, and if $u(z)$ is a plurisubharmonic function on Ω that depends only on $x = \mathrm{Re}\, z$, then $\omega \ni x \longmapsto u(x)$ is convex.*

Proof. This is clear when $u \in C^2(\Omega, \mathbb{R})$, for $\partial^2 u/\partial z_j \partial\overline{z}_k = \frac{1}{4}\,\partial^2 u/\partial x_j \partial x_k$. In the general case, write $u = \lim u \star \rho_\varepsilon$ and observe that $u \star \rho_\varepsilon(z)$ depends only on x. □

(5.14) Corollary. *If u is a plurisubharmonic function in the open polydisk $D(a,R) = \prod D(a_j, R_j) \subset \mathbb{C}^n$, then*

$$\begin{aligned}
\mu(u\,; r_1, \ldots, r_n) &= \frac{1}{(2\pi)^n} \int_0^{2\pi} u(a_1 + r_1 e^{\mathrm{i}\theta_1}, \ldots, a_n + r_n e^{\mathrm{i}\theta_n})\, d\theta_1 \ldots d\theta_n, \\
m(u\,; r_1, \ldots, r_n) &= \sup_{z \in D(a,r)} u(z_1, \ldots, z_n), \qquad r_j < R_j
\end{aligned}$$

are convex functions of $(\log r_1, \ldots, \log r_n)$ that are non decreasing in each variable.

Proof. That μ is non decreasing follows from the subharmonicity of u along every coordinate axis. Now, it is easy to verify that the functions

$$\widetilde{\mu}(z_1, \ldots, z_n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} u(a_1 + e^{z_1} e^{\mathrm{i}\theta_1}, \ldots, a_n + e^{z_n} e^{\mathrm{i}\theta_n})\, d\theta_1 \ldots d\theta_n,$$
$$\widetilde{m}(z_1, \ldots, z_n) = \sup_{|w_j| \leq 1} u(a_1 + e^{z_1} w_1, \ldots, a_n + e^{z_n} w_n)$$

are upper semicontinuous, satisfy the mean value inequality, and depend only on $\operatorname{Re} z_j \in \,]0, \log R_j[$. Therefore $\widetilde{\mu}$ and $\widetilde{M}$ are convex. Cor. 5.14 follows from the equalities

$$\mu(u\,;\, r_1, \ldots, r_n) = \widetilde{\mu}(\log r_1, \ldots, \log r_n),$$
$$m(u\,;\, r_1, \ldots, r_n) = \widetilde{m}(\log r_1, \ldots, \log r_n). \qquad \square$$

§5.D. Pluriharmonic Functions

Pluriharmonic functions are the counterpart of harmonic functions in the case of functions of complex variables:

(5.15) Definition. *A function u is said to be pluriharmonic if u and $-u$ are plurisubharmonic.*

A pluriharmonic function is harmonic (in particular smooth) in any $\mathbb{C}$-analytic coordinate system, and is characterized by the condition $Hu = 0$, i.e. $d'd''u = 0$ or

$$\partial^2 u / \partial z_j \partial \overline{z}_k = 0 \quad \text{for all} \quad j, k.$$

If $f \in \mathcal{O}(X)$, it follows that the functions $\operatorname{Re} f$, $\operatorname{Im} f$ are pluriharmonic. Conversely:

(5.16) Theorem. *If the De Rham cohomology group $H^1_{\mathrm{DR}}(X, \mathbb{R})$ is zero, every pluriharmonic function u on X can be written $u = \operatorname{Re} f$ where f is a holomorphic function on X.*

Proof. By hypothesis $H^1_{\mathrm{DR}}(X, \mathbb{R}) = 0$, $u \in C^\infty(X)$ and $d(d'u) = d''d'u = 0$, hence there exists $g \in C^\infty(X)$ such that $dg = d'u$. Then dg is of type $(1, 0)$, i.e. $g \in \mathcal{O}(X)$ and

$$d(u - 2 \operatorname{Re} g) = d(u - g - \overline{g}) = (d'u - dg) + (d''u - d\overline{g}) = 0.$$

Therefore $u = \operatorname{Re}(2g + C)$, where C is a locally constant function. $\square$

§5.E. Global Regularization of Plurisubharmonic Functions

We now study a very efficient regularization and patching procedure for continuous plurisubharmonic functions, essentially due to (Richberg 1968). The main idea is contained in the following lemma:

(5.17) Lemma. *Let $u_\alpha \in \mathrm{Psh}(\Omega_\alpha)$ where $\Omega_\alpha \subset\subset X$ is a locally finite open covering of X. Assume that for every index β*

$$\limsup_{\zeta \to z} u_\beta(\zeta) < \max_{\Omega_\alpha \ni z} \{u_\alpha(z)\}$$

at all points $z \in \partial\Omega_\beta$. Then the function

$$u(z) = \max_{\Omega_\alpha \ni z} u_\alpha(z)$$

is plurisubharmonic on X.

Proof. Fix $z_0 \in X$. Then the indices β such that $z_0 \in \partial\Omega_\beta$ or $z_0 \notin \overline{\Omega}_\beta$ do not contribute to the maximum in a neighborhood of z_0. Hence there is a a finite set I of indices α such that $\Omega_\alpha \ni z_0$ and a neighborhood $V \subset \bigcap_{\alpha \in I} \Omega_\alpha$ on which $u(z) = \max_{\alpha \in I} u_\alpha(z)$. Therefore u is plurisubharmonic on V. □

The above patching procedure produces functions which are in general only continuous. When smooth functions are needed, one has to use a regularized max function. Let $\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ be a nonnegative function with support in $[-1, 1]$ such that $\int_\mathbb{R} \theta(h)\,dh = 1$ and $\int_\mathbb{R} h\theta(h)\,dh = 0$.

(5.18) Lemma. *For arbitrary $\eta = (\eta_1, \ldots, \eta_p) \in {]0, +\infty[^p}$, the function*

$$M_\eta(t_1, \ldots, t_p) = \int_{\mathbb{R}^n} \max\{t_1 + h_1, \ldots, t_p + h_p\} \prod_{1 \le j \le n} \theta(h_j/\eta_j)\,dh_1 \ldots dh_p$$

possesses the following properties:

a) *$M_\eta(t_1, \ldots, t_p)$ is non decreasing in all variables, smooth and convex on $\mathbb{R}^n$;*

b) $\max\{t_1, \ldots, t_p\} \le M_\eta(t_1, \ldots, t_p) \le \max\{t_1 + \eta_1, \ldots, t_p + \eta_p\}$;

c) $M_\eta(t_1, \ldots, t_p) = M_{(\eta_1, \ldots, \widehat{\eta_j}, \ldots, \eta_p)}(t_1, \ldots, \widehat{t_j}, , \ldots, t_p)$
if $t_j + \eta_j \le \max_{k \ne j}\{t_k - \eta_k\}$;

d) $M_\eta(t_1 + a, \ldots, t_p + a) = M_\eta(t_1, \ldots, t_p) + a, \qquad \forall a \in \mathbb{R}$;

e) *if $u_1, \ldots, u_p$ are plurisubharmonic and satisfy $H(u_j)_z(\xi) \ge \gamma_z(\xi)$ where $z \mapsto \gamma_z$ is a continuous hermitian form on T_X, then $u = M_\eta(u_1, \ldots, u_p)$ is plurisubharmonic and satisfies $Hu_z(\xi) \ge \gamma_z(\xi)$.*

Proof. The change of variables $h_j \mapsto h_j - t_j$ shows that M_η is smooth. All properties are immediate consequences of the definition, except perhaps e).

That $M_\eta(u_1,\ldots,u_p)$ is plurisubharmonic follows from a) and Th. 5.6. Fix a point z_0 and $\varepsilon>0$. All functions $u'_j(z)=u_j(z)-\gamma_{z_0}(z-z_0)+\varepsilon|z-z_0|^2$ are plurisubharmonic near z_0. It follows that

$$M_\eta(u'_1,\ldots,u'_p)=u-\gamma_{z_0}(z-z_0)+\varepsilon|z-z_0|^2$$

is also plurisubharmonic near z_0. Since $\varepsilon>0$ was arbitrary, e) follows. □

(5.19) Corollary. *Let $u_\alpha\in C^\infty(\overline{\Omega}_\alpha)\cap\mathrm{Psh}(\Omega_\alpha)$ where $\Omega_\alpha\subset\subset X$ is a locally finite open covering of X. Assume that $u_\beta(z)<\max\{u_\alpha(z)\}$ at every point $z\in\partial\Omega_\beta$, when α runs over the indices such that $\Omega_\alpha\ni z$. Choose a family (η_α) of positive numbers so small that $u_\beta(z)+\eta_\beta\le\max_{\Omega_\alpha\ni z}\{u_\alpha(z)-\eta_\alpha\}$ for all β and $z\in\partial\Omega_\beta$. Then the function defined by*

$$\widetilde{u}(z)=M_{(\eta_\alpha)}\big(u_\alpha(z)\big)\qquad\text{for } \alpha \text{ such that }\ \Omega_\alpha\ni z$$

is smooth and plurisubharmonic on X. □

(5.20) Definition. *A function $u\in\mathrm{Psh}(X)$ is said to be strictly plurisubharmonic if $u\in L^1_{\mathrm{loc}}(X)$ and if for every point $x_0\in X$ there exists a neighborhood Ω of x_0 and $c>0$ such that $u(z)-c|z|^2$ is plurisubharmonic on Ω, i.e. $\sum(\partial^2u/\partial z_j\partial\overline{z}_k)\xi_j\overline{\xi}_k\ge c|\xi|^2$ (as distributions on Ω) for all $\xi\in\mathbb{C}^n$.*

(5.21) Theorem (Richberg 1968)**.** *Let $u\in\mathrm{Psh}(X)$ be a continuous function which is strictly plurisubharmonic on an open subset $\Omega\subset X$, with $Hu\ge\gamma$ for some continuous positive hermitian form γ on Ω. For any continuous function $\lambda\in C^0(\Omega)$, $\lambda>0$, there exists a plurisubharmonic function $\widetilde{u}$ in $C^0(X)\cap C^\infty(\Omega)$ such that $u\le\widetilde{u}\le u+\lambda$ on Ω and $\widetilde{u}=u$ on $X\smallsetminus\Omega$, which is strictly plurisubharmonic on Ω and satisfies $H\widetilde{u}\ge(1-\lambda)\gamma$. In particular, $\widetilde{u}$ can be chosen strictly plurisubharmonic on X if u has the same property.*

Proof. Let (Ω_α) be a locally finite open covering of Ω by relatively compact open balls contained in coordinate patches of X. Choose concentric balls $\Omega''_\alpha\subset\Omega'_\alpha\subset\Omega_\alpha$ of respective radii $r''_\alpha<r'_\alpha<r_\alpha$ and center $z=0$ in the given coordinates $z=(z_1,\ldots,z_n)$ near $\overline{\Omega}_\alpha$, such that Ω''_α still cover Ω. We set

$$u_\alpha(z)=u\star\rho_{\varepsilon_\alpha}(z)+\delta_\alpha(r'^2_\alpha-|z|^2)\qquad\text{on }\ \overline{\Omega}_\alpha.$$

For $\varepsilon_\alpha<\varepsilon_{\alpha,0}$ and $\delta_\alpha<\delta_{\alpha,0}$ small enough, we have $u_\alpha\le u+\lambda/2$ and $Hu_\alpha\ge(1-\lambda)\gamma$ on $\overline{\Omega}_\alpha$. Set

$$\eta_\alpha=\delta_\alpha\,\min\{r'^2_\alpha-r''^2_\alpha,(r^2_\alpha-r'^2_\alpha)/2\}.$$

Choose first $\delta_\alpha<\delta_{\alpha,0}$ such that $\eta_\alpha<\min_{\overline{\Omega}_\alpha}\lambda/2$, and then $\varepsilon_\alpha<\varepsilon_{\alpha,0}$ so small that $u\le u\star\rho_{\varepsilon_\alpha}<u+\eta_\alpha$ on $\overline{\Omega}_\alpha$. As $\delta_\alpha(r'^2-|z|^2)$ is $\le-2\eta_\alpha$ on $\partial\Omega_\alpha$

and $> \eta_\alpha$ on $\overline{\Omega}''_\alpha$, we have $u_\alpha < u - \eta_\alpha$ on $\partial\Omega_\alpha$ and $u_\alpha > u + \eta_\alpha$ on $\overline{\Omega}''_\alpha$, so that the condition required in Corollary 5.19 is satisfied. We define

$$\widetilde{u} = \begin{cases} u & \text{on } X \setminus \Omega, \\ M_{(\eta_\alpha)}(u_\alpha) & \text{on } \Omega. \end{cases}$$

By construction, $\widetilde{u}$ is smooth on Ω and satisfies $u \le \widetilde{u} \le u + \lambda$, $Hu \ge (1-\lambda)\gamma$ thanks to 5.18 (b,e). In order to see that $\widetilde{u}$ is plurisubharmonic on X, observe that $\widetilde{u}$ is the uniform limit of $\widetilde{u}_\alpha$ with

$$\widetilde{u}_\alpha = \max\{u\,,\, M_{(\eta_1\ldots\eta_\alpha)}(u_1\ldots u_\alpha)\} \quad \text{on} \quad \bigcup_{1\le\beta\le\alpha} \Omega_\beta$$

and $\widetilde{u}_\alpha = u$ on the complement. $\square$

§5.F. Polar and Pluripolar Sets.

Polar and pluripolar sets are sets of $-\infty$ poles of subharmonic and plurisubharmonic functions. Although these functions possess a large amount of flexibility, pluripolar sets have some properties which remind their loose relationship with holomorphic functions.

(5.22) Definition. *A set $A \subset \Omega \subset \mathbb{R}^m$ (resp. $A \subset X$, $\dim_{\mathbb{C}} X = n$) is said to be polar (resp. pluripolar) if for every point $x \in \Omega$ there exist a connected neighborhood W of x and $u \in \mathrm{Sh}(W)$ (resp. $u \in \mathrm{Psh}(W)$), $u \not\equiv -\infty$, such that $A \cap W \subset \{x \in W\ ;\ u(x) = -\infty\}$.*

Theorem 4.17 implies that a polar or pluripolar set is of zero Lebesgue measure. Now, we prove a simple extension theorem.

(5.23) Theorem. *Let $A \subset \Omega$ be a closed polar set and $v \in \mathrm{Sh}(\Omega \setminus A)$ such that v is bounded above in a neighborhood of every point of A. Then v has a unique extension $\widetilde{v} \in \mathrm{Sh}(\Omega)$.*

Proof. The uniqueness is clear because A has zero Lebesgue measure. On the other hand, every point of A has a neighborhood W such that

$$A \cap W \subset \{x \in W\ ;\ u(x) = -\infty\}, \qquad u \in \mathrm{Sh}(W), \qquad u \not\equiv -\infty.$$

After shrinking W and subtracting a constant to u, we may assume $u \le 0$. Then for every $\varepsilon > 0$ the function $v_\varepsilon = v + \varepsilon u \in \mathrm{Sh}(W \setminus A)$ can be extended as an upper semicontinuous on W by setting $v_\varepsilon = -\infty$ on $A \cap W$. Moreover, v_ε satisfies the mean value inequality $v_\varepsilon(a) \le \mu_S(v_\varepsilon\,; a, r)$ if $a \in W \setminus A$, $r < d(a, A \cup \complement W)$, and also clearly if $a \in A$, $r < d(a, \complement W)$. Therefore $v_\varepsilon \in \mathrm{Sh}(W)$ and $\widetilde{v} = (\sup v_\varepsilon)^\star \in \mathrm{Sh}(W)$. Clearly $\widetilde{v}$ coincides with v on $W \setminus A$. A similar proof gives:

(5.24) Theorem. *Let A be a closed pluripolar set in a complex analytic manifold X. Then every function $v \in \mathrm{Psh}(X \smallsetminus A)$ that is locally bounded above near A extends uniquely into a function $\widetilde{v} \in \mathrm{Psh}(X)$.* □

(5.25) Corollary. *Let $A \subset X$ be a closed pluripolar set. Every holomorphic function $f \in \mathcal{O}(X \smallsetminus A)$ that is locally bounded near A extends to a holomorphic function $\widetilde{f} \in \mathcal{O}(X)$.*

Proof. Apply Th. 5.24 to $\pm \mathrm{Re}\, f$ and $\pm \mathrm{Im}\, f$. It follows that $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ have pluriharmonic extensions to X, in particular f extends to $\widetilde{f} \in C^\infty(X)$. By density of $X \smallsetminus A$, $d''\widetilde{f} = 0$ on X. □

(5.26) Corollary. *Let $A \subset \Omega$ (resp. $A \subset X$) be a closed (pluri)polar set. If Ω (resp. X) is connected, then $\Omega \smallsetminus A$ (resp. $X \smallsetminus A$) is connected.*

Proof. If $\Omega \smallsetminus A$ (resp. $X \smallsetminus A$) is a disjoint union $\Omega_1 \cup \Omega_2$ of non empty open subsets, the function defined by $f \equiv 0$ on Ω_1, $f \equiv 1$ on Ω_2 would have a harmonic (resp. holomorphic) extension through A, a contradiction. □

§6. Domains of Holomorphy and Stein Manifolds

§6.A. Domains of Holomorphy in $\mathbb{C}^n$. Examples

Loosely speaking, a domain of holomorphy is an open subset Ω in $\mathbb{C}^n$ such that there is no part of $\partial\Omega$ across which all functions $f \in \mathcal{O}(\Omega)$ can be extended. More precisely:

(6.1) Definition. *Let $\Omega \subset \mathbb{C}^n$ be an open subset. Ω is said to be a domain of holomorphy if for every connected open set $U \subset \mathbb{C}^n$ which meets $\partial\Omega$ and every connected component V of $U \cap \Omega$ there exists $f \in \mathcal{O}(\Omega)$ such that $f_{\restriction V}$ has no holomorphic extension to U.*

Under the hypotheses made on U, we have $\emptyset \neq \partial V \cap U \subset \partial\Omega$. In order to show that Ω is a domain of holomorphy, it is thus sufficient to find for every $z_0 \in \partial\Omega$ a function $f \in \mathcal{O}(\Omega)$ which is unbounded near z_0.

(6.2) Examples. Every open subset $\Omega \subset \mathbb{C}$ is a domain of holomorphy (for any $z_0 \in \partial\Omega$, $f(z) = (z - z_0)^{-1}$ cannot be extended at z_0). In $\mathbb{C}^n$, every *convex* open subset is a domain of holomorphy: if $\mathrm{Re}\langle z - z_0, \xi_0\rangle = 0$ is a supporting hyperplane of $\partial\Omega$ at z_0, the function $f(z) = (\langle z - z_0, \xi_0\rangle)^{-1}$ is holomorphic on Ω but cannot be extended at z_0.

(6.3) Hartogs figure. Assume that $n \geq 2$. Let $\omega \subset \mathbb{C}^{n-1}$ be a connected open set and $\omega' \subsetneq \omega$ an open subset. Consider the open sets in $\mathbb{C}^n$:

$$\begin{aligned}&\Omega = \big((D(R) \smallsetminus \overline{D}(r)) \times \omega\big) \cup \big(D(R) \times \omega'\big) &&\text{(Hartogs figure)},\\ &\widetilde{\Omega} = D(R) \times \omega &&\text{(filled Hartogs figure)}.\end{aligned}$$

where $0 \le r < R$ and $D(r) \subset \mathbb{C}$ denotes the open disk of center 0 and radius r in $\mathbb{C}$.

Then every function $f \in \mathcal{O}(\Omega)$ can be extended to $\widetilde{\Omega} = \omega \times D(R)$ by means of the Cauchy formula:

$$\widetilde{f}(z_1, z') = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta_1|=\rho} \frac{f(\zeta_1, z')}{\zeta_1 - z_1} d\zeta_1, \qquad z \in \widetilde{\Omega}, \quad \max\{|z_1|, r\} < \rho < R.$$

In fact $\widetilde{f} \in \mathcal{O}(D(R) \times \omega)$ and $\widetilde{f} = f$ on $D(R) \times \omega'$, so we must have $\widetilde{f} = f$ on Ω since Ω is connected. It follows that Ω is not a domain of holomorphy. Let us quote two interesting consequences of this example.

(6.4) Corollary (Riemann's extension theorem). *Let X be a complex analytic manifold, and S a closed submanifold of codimension ≥ 2. Then every $f \in \mathcal{O}(X \smallsetminus S)$ extends holomorphically to X.*

Proof. This is a local result. We may choose coordinates $(z_1, \ldots, z_n)$ and a polydisk $D(R)^n$ in the corresponding chart such that $S \cap D(R)^n$ is given by equations $z_1 = \ldots = z_p = 0$, $p = \operatorname{codim} S \ge 2$. Then, denoting $\omega = D(R)^{n-1}$ and $\omega' = \omega \smallsetminus \{z_2 = \ldots = z_p = 0\}$, the complement $D(R)^n \smallsetminus S$ can be written as the Hartogs figure

$$D(R)^n \smallsetminus S = \big((D(R) \smallsetminus \{0\}) \times \omega\big) \cup \big(D(R) \times \omega'\big).$$

It follows that f can be extended to $\widetilde{\Omega} = D(R)^n$. □

§6.B. Holomorphic Convexity and Pseudoconvexity

Let X be a complex manifold. We first introduce the notion of holomorphic hull of a compact set $K \subset X$. This can be seen somehow as the complex analogue of the notion of (affine) convex hull for a compact set in a real vector space. It is shown that domains of holomorphy in $\mathbb{C}^n$ are characterized a property of holomorphic convexity. Finally, we prove that holomorphic convexity implies pseudoconvexity – a complex analogue of the geometric notion of convexity.

(6.5) Definition. *Let X be a complex manifold and let K be a compact subset of X. Then the holomorphic hull of K in X is defined to be*

$$\widehat{K} = \widehat{K}_{\mathcal{O}(X)} = \big\{z \in X\,;\, |f(z)| \le \sup_K |f|,\ \forall f \in \mathcal{O}(X)\big\}.$$

(6.6) Elementary properties.

Fig. I-3 Hartogs figure

a) $\widehat{K}$ is a closed subset of X containing K. Moreover we have

$$\sup_{\widehat{K}} |f| = \sup_{K} |f|, \qquad \forall f \in \mathcal{O}(X),$$

hence $\widehat{\widehat{K}} = \widehat{K}$.

b) If $h : X \to Y$ is a holomorphic map and $K \subset X$ is a compact set, then $h(\widehat{K}_{\mathcal{O}(X)}) \subset \widehat{h(K)}_{\mathcal{O}(Y)}$. In particular, if $X \subset Y$, then $\widehat{K}_{\mathcal{O}(X)} \subset \widehat{K}_{\mathcal{O}(Y)} \cap X$. This is immediate from the definition.

c) $\widehat{K}$ contains the union of K with all relatively compact connected components of $X \smallsetminus K$ (thus $\widehat{K}$ "fills the holes" of K). In fact, for every connected component U of $X \smallsetminus K$ we have $\partial U \subset \partial K$, hence if $\overline{U}$ is compact the maximum principle yields

$$\sup_{\overline{U}} |f| = \sup_{\partial U} |f| \leq \sup_{K} |f|, \qquad \text{for all } f \in \mathcal{O}(X).$$

d) More generally, suppose that there is a holomorphic map $h : U \to X$ defined on a relatively compact open set U in a complex manifold S, such that h extends as a continuous map $h : \overline{U} \to X$ and $h(\partial U) \subset K$. Then $h(\overline{U}) \subset \widehat{K}$. Indeed, for $f \in \mathcal{O}(X)$, the maximum principle again yields

$$\sup_{\overline{U}} |f \circ h| = \sup_{\partial U} |f \circ h| \leq \sup_{K} |f|.$$

This is especially useful when U is the unit disk in $\mathbb{C}$.

e) Suppose that $X = \Omega \subset \mathbb{C}^n$ is an open set. By taking $f(z) = \exp(A(z))$ where A is an arbitrary affine function, we see that $\widehat{K}_{\mathcal{O}(\Omega)}$ is contained in the intersection of all affine half-spaces containing K. Hence $\widehat{K}_{\mathcal{O}(\Omega)}$ is contained in the affine convex hull $\widehat{K}_{\mathrm{aff}}$. As a consequence $\widehat{K}_{\mathcal{O}(\Omega)}$ is always bounded and $\widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ is a compact set. However, when Ω is arbitrary, $\widehat{K}_{\mathcal{O}(\Omega)}$ is not always compact; for example, in case $\Omega = \mathbb{C}^n \setminus \{0\}$, $n \geq 2$, then $\mathcal{O}(\Omega) = \mathcal{O}(\mathbb{C}^n)$ and the holomorphic hull of $K = S(0,1)$ is the non compact set $\widehat{K} = \overline{B}(0,1) \setminus \{0\}$.

(6.7) Definition. *A complex manifold X is said to be holomorphically convex if the holomorphic hull $\widehat{K}_{\mathcal{O}(X)}$ of every compact set $K \subset X$ is compact.*

(6.8) Remark. *A complex manifold X is holomorphically convex if and only if there is an exhausting sequence of holomorphically compact subsets $K_\nu \subset X$, i.e. compact sets such that*

$$X = \bigcup K_\nu, \qquad \widehat{K}_\nu = K_\nu, \qquad K_\nu^\circ \supset K_{\nu-1}.$$

Indeed, if X is holomorphically convex, we may define K_ν inductively by $K_0 = \emptyset$ and $K_{\nu+1} = (K'_\nu \cup L_\nu)^\wedge_{\mathcal{O}(X)}$, where K'_ν is a neighborhood of K_ν and L_ν a sequence of compact sets of X such that $X = \bigcup L_\nu$. The converse is obvious: if such a sequence (K_ν) exists, then every compact subset $K \subset X$ is contained in some K_ν, hence $\widehat{K} \subset \widehat{K}_\nu = K_\nu$ is compact. □

We now concentrate on domains of holomorphy in $\mathbb{C}^n$. We denote by d and $B(z,r)$ the distance and the open balls associated to an arbitrary norm on $\mathbb{C}^n$, and we set for simplicity $B = B(0,1)$.

(6.9) Proposition. *If Ω is a domain of holomorphy and $K \subset \Omega$ is a compact subset, then $d(\widehat{K}, \complement\Omega) = d(K, \complement\Omega)$ and $\widehat{K}$ is compact.*

Proof. Let $f \in \mathcal{O}(\Omega)$. Given $r < d(K, \complement\Omega)$, we denote by M the supremum of $|f|$ on the compact subset $K + r\overline{B} \subset \Omega$. Then for every $z \in K$ and $\xi \in \overline{B}$, the function

$$(6.10) \quad \mathbb{C} \ni t \longmapsto f(z + t\xi) = \sum_{k=0}^{+\infty} \frac{1}{k!} D^k f(z)(\xi)^k\, t^k$$

is analytic in the disk $|t| < r$ and bounded by M. The Cauchy inequalities imply

$$|D^k f(z)(\xi)^k| \leq M k!\, r^{-k}, \qquad \forall z \in K, \quad \forall \xi \in \overline{B}.$$

As the left hand side is an analytic fuction of z in Ω, the inequality must also hold for $z \in \widehat{K}$, $\xi \in \overline{B}$. Every $f \in \mathcal{O}(\Omega)$ can thus be extended to any ball $B(z,r)$, $z \in \widehat{K}$, by means of the power series (6.10). Hence $B(z,r)$ must be contained in Ω, and this shows that $d(\widehat{K}, \complement\Omega) \geq r$. As $r < d(K, \complement\Omega)$ was arbitrary, we get $d(\widehat{K}, \complement\Omega) \geq d(K, \complement\Omega)$ and the converse inequality is clear, so $d(\widehat{K}, \complement\Omega) = d(K, \complement\Omega)$. As $\widehat{K}$ is bounded and closed in Ω, this shows that $\widehat{K}$ is compact. □

(6.11) Theorem. *Let Ω be an open subset of $\mathbb{C}^n$. The following properties are equivalent:*

a) *Ω is a domain of holomorphy;*

b) *Ω is holomorphically convex;*

c) *For every countable subset $\{z_j\}_{j\in\mathbb{N}} \subset \Omega$ without accumulation points in Ω and every sequence of complex numbers (a_j), there exists an interpolation function $F \in \mathcal{O}(\Omega)$ such that $F(z_j) = a_j$.*

d) *There exists a function $F \in \mathcal{O}(\Omega)$ which is unbounded on any neighborhood of any point of $\partial\Omega$.*

Proof. d) $\Longrightarrow$ a) is obvious and a) $\Longrightarrow$ b) is a consequence of Prop. 6.9.

c) $\Longrightarrow$ d). If $\Omega = \mathbb{C}^n$ there is nothing to prove. Otherwise, select a dense sequence (ζ_j) in $\partial\Omega$ and take $z_j \in \Omega$ such that $d(z_j, \zeta_j) < 2^{-j}$. Then the interpolation function $F \in \mathcal{O}(\Omega)$ such that $F(z_j) = j$ satisfies d).

b) $\Longrightarrow$ c). Let $K_\nu \subset \Omega$ be an exhausting sequence of holomorphically convex compact sets as in Remark 6.8. Let $\nu(j)$ be the unique index ν such that $z_j \in K_{\nu(j)+1} \smallsetminus K_{\nu(j)}$. By the definition of a holomorphic hull, we can find a function $g_j \in \mathcal{O}(\Omega)$ such that

$$\sup_{K_{\nu(j)}} |g_j| < |g_j(z_j)|.$$

After multiplying g_j by a constant, we may assume that $g_j(z_j) = 1$. Let $P_j \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial equal to 1 at z_j and to 0 at $z_0, z_1, \ldots, z_{j-1}$. We set

$$F = \sum_{j=0}^{+\infty} \lambda_j P_j g_j^{m_j},$$

where $\lambda_j \in \mathbb{C}$ and $m_j \in \mathbb{N}$ are chosen inductively such that

$$\lambda_j = a_j - \sum_{0\leq k<j} \lambda_k P_k(z_j) g_k(z_j)^{m_k},$$

$$|\lambda_j P_j g_j^{m_j}| \leq 2^{-j} \quad \text{on} \quad K_{\nu(j)}\ ;$$

once λ_j has been chosen, the second condition holds as soon as m_j is large enough. Since $\{z_j\}$ has no accumulation point in Ω, the sequence $\nu(j)$ tends to $+\infty$, hence the series converges uniformly on compact sets. □

We now show that a holomorphically convex manifold must satisfy some more geometric convexity condition, known as pseudoconvexity, which is most easily described in terms of the existence of plurisubharmonic exhaustion functions.

(6.12) Definition. *A function $\psi : X \longrightarrow [-\infty, +\infty[$ on a topological space X is said to be an exhaustion if all sublevel sets $X_c := \{z \in X\,;\, \psi(z) < c\}$, $c \in \mathbb{R}$, are relatively compact. Equivalently, ψ is an exhaustion if and only if ψ tends to $+\infty$ relatively to the filter of complements $X \smallsetminus K$ of compact subsets of X.*

A function ψ on an open set $\Omega \subset \mathbb{R}^n$ is thus an exhaustion if and only if $\psi(x) \to +\infty$ as $x \to \partial\Omega$ or $x \to \infty$. It is easy to check, cf. Exercise 8.8, that a connected open set $\Omega \subset \mathbb{R}^n$ is convex if and only if Ω has a locally convex exhaustion function. Since plurisubharmonic functions appear as the natural generalization of convex functions in complex analysis, we are led to the following definition.

(6.13) Definition. *Let X be a complex n-dimensional manifold. Then X is said to be*

a) *weakly pseudoconvex if there exists a smooth plurisubharmonic exhaustion function $\psi \in \mathrm{Psh}(X) \cap C^\infty(X)$;*
b) *strongly pseudoconvex if there exists a smooth strictly plurisubharmonic exhaustion function $\psi \in \mathrm{Psh}(X) \cap C^\infty(X)$, i.e. $H\psi$ is positive definite at every point.*

(6.14) Theorem. *Every holomorphically convex manifold X is weakly pseudoconvex.*

Proof. Let (K_ν) be an exhausting sequence of holomorphically convex compact sets as in Remark 6.8. For every point $a \in L_\nu := K_{\nu+2} \smallsetminus K^\circ_{\nu+1}$, one can select $g_{\nu,a} \in \mathcal{O}(\Omega)$ such that $\sup_{K_\nu} |g_{\nu,a}| < 1$ and $|g_{\nu,a}(a)| > 1$. Then $|g_{\nu,a}(z)| > 1$ in a neighborhood of a ; by the Borel-Lebesgue lemma, one can find finitely many functions $(g_{\nu,a})_{a\in I_\nu}$ such that

$$\max_{a\in I_\nu}\big\{|g_{\nu,a}(z)|\big\} > 1 \quad \text{for} \quad z \in L_\nu, \qquad \max_{a\in I_\nu}\big\{|g_{\nu,a}(z)|\big\} < 1 \quad \text{for} \quad z \in K_\nu.$$

For a sufficiently large exponent $p(\nu)$ we get

$$\sum_{a\in I_\nu} |g_{\nu,a}|^{2p(\nu)} \geq \nu \quad \text{on} \quad L_\nu, \qquad \sum_{a\in I_\nu} |g_{\nu,a}|^{2p(\nu)} \leq 2^{-\nu} \quad \text{on} \quad K_\nu.$$

It follows that the series

$$\psi(z) = \sum_{\nu\in\mathbb{N}} \sum_{a\in I_\nu} |g_{\nu,a}(z)|^{2p(\nu)}$$

converges uniformly to a real analytic function $\psi \in \mathrm{Psh}(X)$ (see Exercise 8.11). By construction $\psi(z) \geq \nu$ for $z \in L_\nu$, hence ψ is an exhaustion. $\square$

(6.15) Example. The converse to Theorem 6.14 does not hold. In fact let $X = \mathbb{C}^2/\Gamma$ be the quotient of $\mathbb{C}^2$ by the free abelian group of rank 2 generated by the affine automorphisms

$$g_1(z,w) = (z+1, e^{\mathrm{i}\theta_1}w), \qquad g_2(z,w) = (z+\mathrm{i}, e^{\mathrm{i}\theta_2}w), \qquad \theta_1,\, \theta_2 \in \mathbb{R}.$$

Since Γ acts properly discontinuously on $\mathbb{C}^2$, the quotient has a structure of a complex (non compact) 2-dimensional manifold. The function $w \mapsto |w|^2$ is Γ-invariant, hence it induces a function $\psi((z,w)^\sim) = |w|^2$ on X which is in fact a plurisubharmonic exhaustion function. Therefore X is weakly pseudoconvex. On the other hand, any holomorphic function $f \in \mathcal{O}(X)$ corresponds to a Γ-invariant holomorphic function $\widetilde{f}(z,w)$ on $\mathbb{C}^2$. Then $z \mapsto \widetilde{f}(z,w)$ is bounded for w fixed, because $\widetilde{f}(z,w)$ lies in the image of the compact set $K \times \overline{D}(0,|w|)$, $K =$ unit square in $\mathbb{C}$. By Liouville's theorem, $\widetilde{f}(z,w)$ does not depend on z. Hence functions $f \in \mathcal{O}(X)$ are in one-to-one correspondence with holomorphic functions $\widetilde{f}(w)$ on $\mathbb{C}$ such that $\widetilde{f}(e^{\mathrm{i}\theta_j}w) = \widetilde{f}(w)$. By looking at the Taylor expansion at the origin, we conclude that $\widetilde{f}$ must be a constant if $\theta_1 \notin \mathbb{Q}$ or $\theta_1 \notin \mathbb{Q}$ (if $\theta_1, \theta_2 \in \mathbb{Q}$ and m is the least common denominator of θ_1, θ_2, then $\widetilde{f}$ is a power series of the form $\sum \alpha_k w^{mk}$). From this, it follows easily that X is holomorphically convex if and only if $\theta_1, \theta_2 \in \mathbb{Q}$.

§6.C. Stein Manifolds

The class of holomorphically convex manifolds contains two types of manifolds of a rather different nature:

- domains of holomorphy $X = \Omega \subset \mathbb{C}^n$;
- compact complex manifolds.

In the first case we have a lot of holomorphic functions, in fact the functions in $\mathcal{O}(\Omega)$ separate any pair of points of Ω. On the other hand, if X is compact and connected, the sets $\mathrm{Psh}(X)$ and $\mathcal{O}(X)$ consist of constant functions merely (by the maximum principle). It is therefore desirable to introduce a clear distinction between these two subclasses. For this purpose, (Stein 1951) introduced the class of manifolds which are now called Stein manifolds.

(6.16) Definition. *A complex manifold X is said to be a Stein manifold if*

a) *X is holomorphically convex;*

b) *$\mathcal{O}(X)$ locally separates points in X, i.e. every point $x \in X$ has a neighborhood V such that for any $y \in V \smallsetminus \{x\}$ there exists $f \in \mathcal{O}(X)$ with $f(y) \neq f(x)$.*

The second condition is automatic if $X = \Omega$ is an open subset of $\mathbb{C}^n$. Hence an open set $\Omega \subset \mathbb{C}^n$ is Stein if and only if Ω is a domain of holomorphy.

(6.17) Lemma. *If a complex manifold X satisfies the axiom* (6.16 b) *of local separation, there exists a smooth nonnegative strictly plurisubharmonic function $u \in \mathrm{Psh}(X)$.*

Proof. Fix $x_0 \in X$. We first show that there exists a smooth nonnegative function $u_0 \in \mathrm{Psh}(X)$ which is strictly plurisubharmonic on a neighborhood of x_0. Let $(z_1, \ldots, z_n)$ be local analytic coordinates centered at x_0, and if necessary, replace z_j by λz_j so that the closed unit ball $\overline{B} = \{\sum |z_j|^2 \le 1\}$ is contained in the neighborhood $V \ni x_0$ on which (6.16 b) holds. Then, for every point $y \in \partial B$, there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $f(y) \ne f(x_0)$. Replacing f with $\lambda(f - f(x_0))$, we can achieve $f(x_0) = 0$ and $|f(y)| > 1$. By compactness of ∂B, we find finitely many functions $f_1, \ldots, f_N \in \mathcal{O}(X)$ such that $v_0 = \sum |f_j|^2$ satisfies $v_0(x_0) = 0$, while $v_0 \ge 1$ on ∂B. Now, we set

$$u_0(z) = \begin{cases} v_0(z) & \text{on } X \smallsetminus B, \\ M_\varepsilon\{v_0(z), (|z|^2+1)/3\} & \text{on } B. \end{cases}$$

where M_ε are the regularized max functions defined in 5.18. Then u_0 is smooth and plurisubharmonic, coincides with v_0 near ∂B and with $(|z|^2+1)/3$ on a neighborhood of x_0. We can cover X by countably many neighborhoods $(V_j)_{j \ge 1}$, for which we have a smooth plurisubharmonic functions $u_j \in \mathrm{Psh}(X)$ such that u_j is strictly plurisubharmonic on V_j. Then select a sequence $\varepsilon_j > 0$ converging to 0 so fast that $u = \sum \varepsilon_j u_j \in C^\infty(X)$. The function u is nonnegative and strictly plurisubharmonic everywhere on X. □

(6.18) Theorem. *Every Stein manifold is strongly pseudoconvex.*

Proof. By Th. 6.14, there is a smooth exhaustion function $\psi \in \mathrm{Psh}(X)$. If $u \ge 0$ is strictly plurisubharmonic, then $\psi' = \psi + u$ is a strictly plurisubharmonic exhaustion. □

The converse problem to know whether every strongly pseudoconvex manifold is actually a Stein manifold is known as the *Levi problem*, and was raised by (Levi 1910) in the case of domains $\Omega \subset \mathbb{C}^n$. In that case, the problem has been solved in the affirmative independently by (Oka 1953), (Norguet 1954) and (Bremermann 1954). The general solution of the Levi problem has been obtained by (Grauert 1958). Our proof will rely on the theory of L^2 estimates for d'', which will be available only in Chapter VIII.

(6.19) Remark. It will be shown later that Stein manifolds always have enough holomorphic functions to separate finitely many points, and one can

Fig. I-4 Hartogs figure with excrescence

even interpolate given values of a function and its derivatives of some fixed order at any discrete set of points. In particular, we might have replaced condition (6.16 b) by the stronger requirement that $\mathcal{O}(X)$ separates any pair of points. On the other hand, there are examples of manifolds satisfying the local separation condition (6.16 b), but not global separation. A simple example is obtained by attaching an excrescence inside a Hartogs figure, in such a way that the resulting map $\pi : X \to D = D(0,1)^2$ is not one-to-one (see Figure I-4 above); then $\mathcal{O}(X)$ coincides with $\pi^\star\mathcal{O}(D)$.

§6.D. Heredity Properties

Holomorphic convexity and pseudoconvexity are preserved under quite a number of natural constructions. The main heredity properties can be summarized in the following Proposition.

(6.20) Proposition. *Let $\mathcal{C}$ denote the class of holomorphically convex* (*resp. of Stein, or weakly pseudoconvex, strongly pseudoconvex manifolds*)*.*

a) *If $X, Y \in \mathcal{C}$, then $X \times Y \in \mathcal{C}$.*

b) *If $X \in \mathcal{C}$ and S is a closed complex submanifold of X, then $S \in \mathcal{C}$.*

c) *If $(S_j)_{1\le j\le N}$ is a collection of* (*not necessarily closed*) *submanifolds of a complex manifold X such that $S = \bigcap S_j$ is a submanifold of X, and if $S_j \in \mathcal{C}$ for all j, then $S \in \mathcal{C}$.*

d) *If $F : X \to Y$ is a holomorphic map and $S \subset X$, $S' \subset Y$ are* (*not necessarily closed*) *submanifolds in the class $\mathcal{C}$, then $S \cap F^{-1}(S')$ is in $\mathcal{C}$, as long as it is a submanifold of X.*

e) *If X is a weakly* (*resp. strongly*) *pseudoconvex manifold and u is a smooth plurisubharmonic function on X, then the open set $\Omega = u^{-1}(]-\infty, c[$ is weakly* (*resp. strongly*) *pseudoconvex. In particular the sublevel sets*

$$X_c = \psi^{-1}(]-\infty, c[)$$

of a (strictly) plurisubharmonic exhaustion function are weakly (resp. strongly) pseudoconvex.

Proof. All properties are more or less immediate to check, so we only give the main facts.

a) For $K \subset X$, $L \subset Y$ compact, we have $(K \times L)^{\wedge}_{\mathcal{O}(X\times Y)} = \widehat{K}_{\mathcal{O}(X)} \times \widehat{K}_{\mathcal{O}(Y)}$, and if φ, ψ are plurisubharmonic exhaustions of X, Y, then $\varphi(x) + \psi(y)$ is a plurisubharmonic exhaustion of $X \times Y$.

b) For a compact set $K \subset S$, we have $\widehat{K}_{\mathcal{O}(S)} \subset \widehat{K}_{\mathcal{O}(X)} \cap S$, and if $\psi \in \mathrm{Psh}(X)$ is an exhaustion, then $\psi\restriction S \in \mathrm{Psh}(S)$ is an exhaustion (since S is closed).

c) $\bigcap S_j$ is a closed submanifold in $\prod S_j$ (equal to its intersection with the diagonal of X^N).

d) For a compact set $K \subset S \cap F^{-1}(S')$, we have

$$\widehat{K}_{\mathcal{O}(S\cap F^{-1}(S'))} \subset \widehat{K}_{\mathcal{O}(S)} \cap F^{-1}(\widehat{F(K)}_{\mathcal{O}(S')}),$$

and if φ, ψ are plurisubharmonic exhaustions of S, S', then $\varphi + \psi \circ F$ is a plurisubharmonic exhaustion of $S \cap F^{-1}(S')$.

e) $\varphi(z) := \psi(z) + 1/(c - u(z))$ is a (strictly) plurisubharmonic exhaustion function on Ω. $\square$

§7. Pseudoconvex Open Sets in $\mathbb{C}^n$

§7.A. Geometric Characterizations of Pseudoconvex Open Sets

We first discuss some characterizations of pseudoconvex open sets in $\mathbb{C}^n$. We will need the following elementary criterion for plurisubharmonicity.

(7.1) Criterion. *Let $v : \Omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function. Then v is plurisubharmonic if and only if for every closed disk $\overline{\Delta} = z_0 + \overline{D}(1)\eta \subset \Omega$ and every polynomial $P \in \mathbb{C}[t]$ such that $v(z_0 + t\eta) \le \mathrm{Re}\, P(t)$ for $|t| = 1$, then $v(z_0) \le \mathrm{Re}\, P(0)$.*

Proof. The condition is necessary because $t \longmapsto v(z_0 + t\eta) - \mathrm{Re}\, P(t)$ is subharmonic in a neighborhood of $\overline{D}(1)$, so it satisfies the maximum principle on $D(1)$ by Th. 4.14. Let us prove now the sufficiency. The upper semicontinuity of v implies $v = \lim_{\nu\to+\infty} v_\nu$ on $\partial\Delta$ where (v_ν) is a strictly decreasing sequence of continuous functions on $\partial\Delta$. As trigonometric polynomials are dense in $C^0(S^1, \mathbb{R})$, we may assume $v_\nu(z_0 + e^{\mathrm{i}\theta}\eta) = \mathrm{Re}\, P_\nu(e^{\mathrm{i}\theta})$, $P_\nu \in \mathbb{C}[t]$. Then $v(z_0 + t\eta) \le \mathrm{Re}\, P_\nu(t)$ for $|t| = 1$, and the hypothesis implies

$$v(z_0) \le \operatorname{Re} P_\nu(0) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} P_\nu(e^{\mathrm{i}\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} v_\nu(z_0 + e^{\mathrm{i}\theta}\eta)\,d\theta.$$

Taking the limit when ν tends to $+\infty$ shows that v satisfies the mean value inequality (5.2). □

For any $z \in \Omega$ and $\xi \in \mathbb{C}^n$, we denote by

$$\delta_\Omega(z,\xi) = \sup\big\{r > 0\ ;\ z + D(r)\,\xi \subset \Omega\big\}$$

the distance from z to $\partial\Omega$ in the complex direction ξ.

(7.2) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be an open subset. The following properties are equivalent:*

a) *Ω is strongly pseudoconvex* (*according to Def. 6.13* b);

b) *Ω is weakly pseudoconvex*;

c) *Ω has a plurisubharmonic exhaustion function ψ.*

d) $-\log\delta_\Omega(z,\xi)$ *is plurisubharmonic on* $\Omega \times \mathbb{C}^n$;

e) $-\log d(z, \complement\Omega)$ *is plurisubharmonic on Ω.*

If one of these properties hold, Ω is said to be a pseudoconvex open set.

Proof. The implications a) $\Longrightarrow$ b) $\Longrightarrow$ c) are obvious. For the implication c) $\Longrightarrow$ d), we use Criterion 7.1. Consider a disk $\overline{\Delta} = (z_0,\xi_0) + \overline{D}(1)\,(\eta,\alpha)$ in $\Omega \times \mathbb{C}^n$ and a polynomial $P \in \mathbb{C}[t]$ such that

$$-\log\delta_\Omega(z_0 + t\eta, \xi_0 + t\alpha) \le \operatorname{Re} P(t) \quad \text{for} \quad |t| = 1.$$

We have to verify that the inequality also holds when $|t| < 1$. Consider the holomorphic mapping $h : \mathbb{C}^2 \longrightarrow \mathbb{C}^n$ defined by

$$h(t,w) = z_0 + t\eta + we^{-P(t)}(\xi_0 + t\alpha).$$

By hypothesis

$$h\big(\overline{D}(1) \times \{0\}\big) = \mathrm{pr}_1(\overline{\Delta}) \subset \Omega,$$
$$h\big(\partial D(1) \times D(1)\big) \subset \Omega \quad (\text{since } |e^{-P}| \le \delta_\Omega \ \text{ on } \partial\Delta),$$

and the desired conclusion is that $h\big(\overline{D}(1) \times D(1)\big) \subset \Omega$. Let J be the set of radii $r \ge 0$ such that $h\big(\overline{D}(1) \times \overline{D}(r)\big) \subset \Omega$. Then J is an open interval $[0, R[$, $R > 0$. If $R < 1$, we get a contradiction as follows. Let $\psi \in \mathrm{Psh}(\Omega)$ be an exhaustion function and

$$K = h\big(\partial D(1) \times \overline{D}(R)\big) \subset\subset \Omega, \qquad c = \sup_K \psi.$$

As $\psi \circ h$ is plurisubharmonic on a neighborhood of $\overline{D}(1) \times D(R)$, the maximum principle applied with respect to t implies

$$\psi \circ h(t,w) \le c \quad \text{on} \quad \overline{D}(1) \times D(R),$$

hence $h\big(\overline{D}(1) \times D(R)\big) \subset \Omega_c \subset\subset \Omega$ and $h\big(\overline{D}(1) \times \overline{D}(R+\varepsilon)\big) \subset \Omega$ for some $\varepsilon > 0$, a contradiction.

d) $\Longrightarrow$ e). The function $-\log d(z, \complement\Omega)$ is continuous on Ω and satisfies the mean value inequality because

$$-\log d(z, \complement\Omega) = \sup_{\xi \in \overline{B}} \big(-\log \delta_\Omega(z,\xi)\big).$$

e) $\Longrightarrow$ a). It is clear that

$$u(z) = |z|^2 + \max\{\log d(z, \complement\Omega)^{-1}, 0\}$$

is a continuous strictly plurisubharmonic exhaustion function. Richberg's theorem 5.21 implies that there exists $\psi \in C^\infty(\Omega)$ strictly plurisubharmonic such that $u \le \psi \le u+1$. Then ψ is the required exhaustion function. □

(7.3) Proposition.

a) *Let $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^p$ be pseudoconvex. Then $\Omega \times \Omega'$ is pseudoconvex. For every holomorphic map $F : \Omega \to \mathbb{C}^p$ the inverse image $F^{-1}(\Omega')$ is pseudoconvex.*

b) *If $(\Omega_\alpha)_{\alpha \in I}$ is a family of pseudoconvex open subsets of $\mathbb{C}^n$, the interior of the intersection $\Omega = \big(\bigcap_{\alpha \in I} \Omega_\alpha\big)^\circ$ is pseudoconvex.*

c) *If $(\Omega_j)_{j \in \mathbb{N}}$ is a non decreasing sequence of pseudoconvex open subsets of $\mathbb{C}^n$, then $\Omega = \bigcup_{j \in \mathbb{N}} \Omega_j$ is pseudoconvex.*

Proof. a) Let φ, ψ be smooth plurisubharmonic exhaustions of Ω, Ω'. Then $(z,w) \longmapsto \varphi(z) + \psi(w)$ is an exhaustion of $\Omega \times \Omega'$ and $z \longmapsto \varphi(z) + \psi(F(z))$ is an exhaustion of $F^{-1}(\Omega')$.

b) We have $-\log d(z, \complement\Omega) = \sup_{\alpha \in I} -\log d(z, \complement\Omega_\alpha)$, so this function is plurisubharmonic.

c) The limit $-\log d(z, \complement\Omega) = \lim\!\downarrow_{j \to +\infty} -\log d(z, \complement\Omega_j)$ is plurisubharmonic, hence Ω is pseudoconvex. This result cannot be generalized to strongly pseudoconvex manifolds: J.E. Fornaess in (Fornaess 1977) has constructed an increasing sequence of 2-dimensional Stein (even affine algebraic) manifolds X_ν whose union is not Stein; see Exercise 8.16. □

(7.4) Examples.

a) An *analytic polyhedron* in $\mathbb{C}^n$ is an open subset of the form

$$P = \{z \in \mathbb{C}^n \,;\, |f_j(z)| < 1,\ 1 \le j \le N\}$$

where $(f_j)_{1 \le j \le N}$ is a family of analytic functions on $\mathbb{C}^n$. By 7.3 a), every analytic polyhedron is pseudoconvex.

b) Let $\omega \subset \mathbb{C}^{n-1}$ be pseudoconvex and let $u : \omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function. Then the *Hartogs domain*

$$\Omega = \big\{(z_1, z') \in \mathbb{C} \times \omega\,;\ \log|z_1| + u(z') < 0\big\}$$

is pseudoconvex if and and only if u is plurisubharmonic. To see that the plurisubharmonicity of u is necessary, observe that

$$u(z') = -\log \delta_\Omega\big((0, z'), (1, 0)\big).$$

Conversely, assume that u is plurisubharmonic and continuous. If ψ is a plurisubharmonic exhaustion of ω, then

$$\psi(z') + \big|\log|z_1| + u(z')\big|^{-1}$$

is an exhaustion of Ω. This is no longer true if u is not continuous, but in this case we may apply Property 7.3 c) to conclude that

$$\Omega_\varepsilon = \big\{(z_1, z')\,;\ d(z', \complement\omega) > \varepsilon,\ \log|z_1| + u \star \rho_\varepsilon(z') < 0\big\}, \qquad \Omega = \bigcup \Omega_\varepsilon$$

are pseudoconvex.

c) An open set $\Omega \subset \mathbb{C}^n$ is called a *tube* of base ω if $\Omega = \omega + \mathrm{i}\mathbb{R}^n$ for some open subset $\omega \subset \mathbb{R}^n$. Then of course $-\log d(z, \complement\Omega) = -\log(x, \complement\omega)$ depends only on the real part $x = \mathrm{Re}\, z$. By Th. 5.13, this function is plurisubharmonic if and only if it is locally convex in x. Therefore Ω if pseudoconvex if and only if every connected component of ω is convex.

d) An open set $\Omega \subset \mathbb{C}^n$ is called a *Reinhardt domain* if $(e^{\mathrm{i}\theta_1} z_1, \ldots, e^{\mathrm{i}\theta_n} z_n)$ is in Ω for every $z = (z_1, \ldots, z_n) \in \Omega$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}^n$. For such a domain, we consider the *logarithmic indicatrix*

$$\omega^\star = \Omega^\star \cap \mathbb{R}^n \quad \text{with} \quad \Omega^\star = \{\zeta \in \mathbb{C}^n\,;\ (e^{\zeta_1}, \ldots, e^{\zeta_n}) \in \Omega\}.$$

It is clear that $\Omega^\star$ is a tube of base $\omega^\star$. Therefore every connected component of $\omega^\star$ must be convex if Ω is pseudoconvex. The converse is not true: $\Omega = \mathbb{C}^n \smallsetminus \{0\}$ is not pseudoconvex for $n \geq 2$ although $\omega^\star = \mathbb{R}^n$ is convex. However, the Reinhardt open set

$$\Omega^\bullet = \big\{(z_1, \ldots, z_n) \in (\mathbb{C} \smallsetminus \{0\})^n\,;\ (\log|z_1|, \ldots, \log|z_n|) \in \omega^\star\big\} \subset \Omega$$

is easily seen to be pseudoconvex if $\omega^\star$ is convex: if χ is a convex exhaustion of $\omega^\star$, then $\psi(z) = \chi(\log|z_1|, \ldots, \log|z_n|)$ is a plurisubharmonic exhaustion of $\Omega^\bullet$. Similarly, if $\omega^\star$ is convex and such that $x \in \omega^\star \Longrightarrow y \in \omega^\star$ for $y_j \leq x_j$, we can take χ increasing in all variables and tending to $+\infty$ on $\partial\omega^\star$, hence the set

$$\widetilde{\Omega} = \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n\,;\ |z_j| \leq e^{x_j} \ \text{ for some } \ x \in \omega^\star\big\}$$

is a pseudoconvex Reinhardt open set containing 0. $\square$

§7.B. Kiselman's Minimum Principle

We already know that a maximum of plurisubharmonic functions is plurisubharmonic. However, if v is a plurisubharmonic function on $X \times \mathbb{C}^n$, the partial minimum function on X defined by $u(\zeta) = \inf_{z\in\Omega} v(\zeta, z)$ need not be plurisubharmonic. A simple counterexample in $\mathbb{C} \times \mathbb{C}$ is given by

$$v(\zeta, z) = |z|^2 + 2\,\mathrm{Re}(z\zeta) = |z + \overline{\zeta}|^2 - |\zeta|^2, \qquad u(\zeta) = -|\zeta|^2.$$

It follows that the image $F(\Omega)$ of a pseudoconvex open set Ω by a holomorphic map F need not be pseudoconvex. In fact, if

$$\Omega = \{(t, \zeta, z) \in \mathbb{C}^3 \,;\, \log|t| + v(\zeta, z) < 0\}$$

and if $\Omega' \subset \mathbb{C}^2$ is the image of Ω by the projection map $(t, \zeta, z) \longmapsto (t, \zeta)$, then $\Omega' = \{(t, \zeta) \in \mathbb{C}^2 \,;\, \log|t| + u(\zeta) < 0\}$ is not pseudoconvex. However, the minimum property holds true when $v(\zeta, z)$ depends only on $\mathrm{Re}\, z$:

(7.5) Theorem (Kiselman 1978). *Let $\Omega \subset \mathbb{C}^p \times \mathbb{C}^n$ be a pseudoconvex open set such that each slice*

$$\Omega_\zeta = \{z \in \mathbb{C}^n \,;\, (\zeta, z) \in \Omega\}, \qquad \zeta \in \mathbb{C}^p,$$

is a convex tube $\omega_\zeta + \mathrm{i}\mathbb{R}^n$, $\omega_\zeta \subset \mathbb{R}^n$. For every plurisubharmonic function $v(\zeta, z)$ on Ω that does not depend on $\mathrm{Im}\, z$, *the function*

$$u(\zeta) = \inf_{z\in\Omega_\zeta} v(\zeta, z)$$

is plurisubharmonic or locally $\equiv -\infty$ on $\Omega' = \mathrm{pr}_{\mathbb{C}^n}(\Omega)$.

Proof. The hypothesis implies that $v(\zeta, z)$ is convex in $x = \mathrm{Re}\, z$. In addition, we first assume that v is smooth, plurisubharmonic in (ζ, z), strictly convex in x and $\lim_{x\to\{\infty\}\cup\partial\omega_\zeta} v(\zeta, x) = +\infty$ for every $\zeta \in \Omega'$. Then $x \longmapsto v(\zeta, x)$ has a unique minimum point $x = g(\zeta)$, solution of the equations $\partial v/\partial x_j(x, \zeta) = 0$. As the matrix $(\partial^2 v/\partial x_j \partial x_k)$ is positive definite, the implicit function theorem shows that g is smooth. Now, if $\mathbb{C} \ni w \longmapsto \zeta_0 + wa$, $a \in \mathbb{C}^n$, $|w| \le 1$ is a complex disk Δ contained in Ω, there exists a holomorphic function f on the unit disk, smooth up to the boundary, whose real part solves the Dirichlet problem

$$\mathrm{Re}\, f(e^{\mathrm{i}\theta}) = g(\zeta_0 + e^{\mathrm{i}\theta} a).$$

Since $v(\zeta_0 + wa, f(w))$ is subharmonic in w, we get the mean value inequality

$$v(\zeta_0, f(0)) \le \frac{1}{2\pi} \int_0^{2\pi} v\big(\zeta_0 + e^{\mathrm{i}\theta} a, f(e^{\mathrm{i}\theta})\big) d\theta = \frac{1}{2\pi} \int_{\partial\Delta} v(\zeta, g(\zeta)) d\theta.$$

The last equality holds because $\operatorname{Re} f = g$ on $\partial\Delta$ and $v(\zeta, z) = v(\zeta, \operatorname{Re} z)$ by hypothesis. As $u(\zeta_0) \le v(\zeta_0, f(0))$ and $u(\zeta) = v(\zeta, g(\zeta))$, we see that u satisfies the mean value inequality, thus u is plurisubharmonic.

Now, this result can be extended to arbitrary functions v as follows: let $\psi(\zeta, z) \ge 0$ be a continuous plurisubharmonic function on Ω which is independent of $\operatorname{Im} z$ and is an exhaustion of $\Omega \cap (\mathbb{C}^p \times \mathbb{R}^n)$, e.g.

$$\psi(\zeta, z) = \max\{|\zeta|^2 + |\operatorname{Re} z|^2, -\log \delta_\Omega(\zeta, z)\}.$$

There is slowly increasing sequence $C_j \to +\infty$ such that each function $\psi_j = (C_j - \psi\star\rho_{1/j})^{-1}$ is an "exhaustion" of a pseudoconvex open set $\Omega_j \subset\subset \Omega$ whose slices are convex tubes and such that $d(\Omega_j, \complement\Omega) > 2/j$. Then

$$v_j(\zeta, z) = v \star \rho_{1/j}(\zeta, z) + \frac{1}{j}|\operatorname{Re} z|^2 + \psi_j(\zeta, z)$$

is a decreasing sequence of plurisubharmonic functions on Ω_j satisfying our previous conditions. As $v = \lim v_j$, we see that $u = \lim u_j$ is plurisubharmonic. □

(7.6) Corollary. *Let $\Omega \subset \mathbb{C}^p \times \mathbb{C}^n$ be a pseudoconvex open set such that all slices Ω_ζ, $\zeta \in \mathbb{C}^p$, are convex tubes in $\mathbb{C}^n$. Then the projection Ω' of Ω on $\mathbb{C}^p$ is pseudoconvex.*

Proof. Take $v \in \operatorname{Psh}(\Omega)$ equal to the function ψ defined in the proof of Th. 7.5. Then u is a plurisubharmonic exhaustion of Ω'. □

§7.C. Levi Form of the Boundary

For an arbitrary domain in $\mathbb{C}^n$, we first show that pseudoconvexity is a local property of the boundary.

(7.7) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be an open subset such that every point $z_0 \in \partial\Omega$ has a neighborhood V such that $\Omega \cap V$ is pseudoconvex. Then Ω is pseudoconvex.*

Proof. As $d(z, \complement\Omega)$ coincides with $d\big(z, \complement(\Omega \cap V)\big)$ in a neighborhood of z_0, we see that there exists a neighborhood U of $\partial\Omega$ such that $-\log d(z, \complement\Omega)$ is plurisubharmonic on $\Omega \cap U$. Choose a convex increasing function χ such that

$$\chi(r) > \sup_{(\Omega\smallsetminus U)\cap\overline{B}(0,r)} -\log d(z, \complement\Omega), \qquad \forall r \ge 0.$$

Then the function

$$\psi(z) = \max\big\{\chi(|z|), -\log d(z, \complement\Omega)\big\}$$

coincides with $\chi(|z|)$ in a neighborhood of $\Omega \smallsetminus U$. Therefore $\psi \in \mathrm{Psh}(\Omega)$, and ψ is clearly an exhaustion. □

Now, we give a geometric characterization of the pseudoconvexity property when $\partial\Omega$ is of class C^2. Let $\rho \in C^2(\overline{\Omega})$ be a *defining function* of Ω, i.e. a function such that

$$(7.9)\quad \rho < 0 \ \text{ on } \ \Omega, \qquad \rho = 0 \ \text{ and } \ d\rho \neq 0 \ \text{ on } \ \partial\Omega.$$

The *holomorphic tangent space* to $\partial\Omega$ is by definition the largest complex subspace which is contained in the tangent space $T_{\partial\Omega}$ to the boundary:

$$(7.9)\quad {}^hT_{\partial\Omega} = T_{\partial\Omega} \cap JT_{\partial\Omega}.$$

It is easy to see that ${}^hT_{\partial\Omega,z}$ is the complex hyperplane of vectors $\xi \in \mathbb{C}^n$ such that

$$d'\rho(z)\cdot\xi = \sum_{1\le j\le n} \frac{\partial\rho}{\partial z_j}\,\xi_j = 0.$$

The *Levi form* on ${}^hT_{\partial\Omega}$ is defined at every point $z \in \partial\Omega$ by

$$(7.10)\quad L_{\partial\Omega,z}(\xi) = \frac{1}{|\nabla\rho(z)|}\sum_{j,k} \frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\,\xi_j\overline{\xi}_k, \qquad \xi \in {}^hT_{\partial\Omega,z}.$$

The Levi form does not depend on the particular choice of ρ, as can be seen from the following intrinsic computation of $L_{\partial\Omega}$ (we still denote by $L_{\partial\Omega}$ the associated sesquilinear form).

(7.11) Lemma. *Let ξ, η be C^1 vector fields on $\partial\Omega$ with values in ${}^hT_{\partial\Omega}$. Then*

$$\langle[\xi,\eta], J\nu\rangle = 4\,\mathrm{Im}\, L_{\partial\Omega}(\xi,\eta)$$

where ν is the outward normal unit vector to $\partial\Omega$, $[\ ,\ ]$ the Lie bracket of vector fields and $\langle\ ,\ \rangle$ the hermitian inner product.

Proof. Extend first ξ,η as vector fields in a neighborhood of $\partial\Omega$ and set

$$\xi' = \sum \xi_j \frac{\partial}{\partial z_j} = \frac{1}{2}(\xi - \mathrm{i}J\xi), \qquad \eta'' = \sum \overline{\eta}_k \frac{\partial}{\partial\overline{z}_k} = \frac{1}{2}(\eta + \mathrm{i}J\eta).$$

As $\xi, J\xi, \eta, J\eta$ are tangent to $\partial\Omega$, we get on $\partial\Omega$:

$$0 = \xi'.(\eta''.\rho) + \eta''.(\xi'.\rho) = \sum_{1\le j,k\le n} 2\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\,\xi_j\overline{\eta}_k + \xi_j\frac{\partial\overline{\eta}_k}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k} + \overline{\eta}_k\frac{\partial\xi_j}{\partial\overline{z}_k}\frac{\partial\rho}{\partial z_j}.$$

Since $[\xi,\eta]$ is also tangent to $\partial\Omega$, we have $\mathrm{Re}\langle[\xi,\eta],\nu\rangle = 0$, hence $\langle J[\xi,\eta],\nu\rangle$ is real and

$$\langle[\xi,\eta],J\nu\rangle=-\langle J[\xi,\eta],\nu\rangle=-\frac{1}{|\nabla\rho|}\big(J[\xi,\eta].\rho\big)=-\frac{2}{|\nabla\rho|}\,\mathrm{Re}\,\big(J[\xi',\eta''].\rho\big)$$

because $J[\xi',\eta']=i[\xi',\eta']$ and its conjugate $J[\xi'',\eta'']$ are tangent to $\partial\Omega$. We find now

$$J[\xi',\eta'']=-\mathrm{i}\sum\xi_j\frac{\partial\overline{\eta}_k}{\partial z_j}\frac{\partial}{\partial\overline{z}_k}+\overline{\eta}_k\frac{\partial\xi_j}{\partial\overline{z}_k}\frac{\partial}{\partial\overline{z}_j},$$

$$\mathrm{Re}\,\big(J[\xi',\eta''].\rho\big)=\mathrm{Im}\sum\xi_j\frac{\partial\overline{\eta}_k}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k}+\overline{\eta}_k\frac{\partial\xi_j}{\partial\overline{z}_k}\frac{\partial\rho}{\partial z_j}=-2\,\mathrm{Im}\sum\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\,\xi_j\overline{\eta}_k,$$

$$\langle[\xi,\eta],J\nu\rangle=\frac{4}{|\nabla\rho|}\,\mathrm{Im}\sum\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}\,\xi_j\overline{\eta}_k=4\,\mathrm{Im}\,L_{\partial\Omega}(\xi,\eta).\qquad\square$$

(7.12) Theorem. *An open subset $\Omega\subset\mathbb{C}^n$ with C^2 boundary is pseudoconvex if and only if the Levi form $L_{\partial\Omega}$ is semipositive at every point of $\partial\Omega$.*

Proof. Set $\delta(z)=d(z,\complement\Omega)$, $z\in\overline{\Omega}$. Then $\rho=-\delta$ is C^2 near $\partial\Omega$ and satisfies (7.9). If Ω is pseudoconvex, the plurisubharmonicity of $-\log(-\rho)$ means that for all $z\in\Omega$ near $\partial\Omega$ and all $\xi\in\mathbb{C}^n$ one has

$$\sum_{1\le j,k\le n}\Big(\frac{1}{|\rho|}\,\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}+\frac{1}{\rho^2}\,\frac{\partial\rho}{\partial z_j}\,\frac{\partial\rho}{\partial\overline{z}_k}\Big)\xi_j\overline{\xi}_k\ge0.$$

Hence $\sum\,(\partial^2\rho/\partial z_j\partial\overline{z}_k)\xi_j\overline{\xi}_k\ge0$ if $\sum\,(\partial\rho/\partial z_j)\xi_j=0$, and an easy argument shows that this is also true at the limit on $\partial\Omega$.

Conversely, if Ω is not pseudoconvex, Th. 7.2 and 7.7 show that $-\log\delta$ is not plurisubharmonic in any neighborhood of $\partial\Omega$. Hence there exists $\xi\in\mathbb{C}^n$ such that

$$c=\Big(\frac{\partial^2}{\partial t\partial\overline{t}}\,\log\delta(z+t\xi)\Big)_{|t=0}>0$$

for some z in the neighborhood of $\partial\Omega$ where $\delta\in C^2$. By Taylor's formula, we have

$$\log\delta(z+t\xi)=\log\delta(z)+\mathrm{Re}(at+bt^2)+c|t|^2+\mathrm{o}(|t|^2)$$

with $a,b\in\mathbb{C}$. Now, choose $z_0\in\partial\Omega$ such that $\delta(z)=|z-z_0|$ and set

$$h(t)=z+t\xi+e^{at+bt^2}(z_0-z),\qquad t\in\mathbb{C}.$$

Then we get $h(0)=z_0$ and

$$\begin{aligned}\delta(h(t))&\ge\delta(z+t\xi)-\delta(z)\,\big|e^{at+bt^2}\big|\\&\ge\delta(z)\,\big|e^{at+bt^2}\big|\,\big(e^{c|t|^2/2}-1\big)\ge\delta(z)\,c|t|^2/3\end{aligned}$$

when $|t|$ is sufficiently small. Since $\delta(h(0))=\delta(z_0)=0$, we obtain at $t=0$:

$$\frac{\partial}{\partial t}\,\delta(h(t)) = \sum \frac{\partial \delta}{\partial z_j}(z_0)\,h_j'(0) = 0,$$

$$\frac{\partial^2}{\partial t\partial \overline{t}}\,\delta(h(t)) = \sum \frac{\partial^2 \delta}{\partial z_j \partial \overline{z}_k}(z_0)\,h_j'(0)\overline{h_k'(0)} > 0,$$

hence $h'(0) \in {}^hT_{\partial\Omega,z_0}$ and $L_{\partial\Omega,z_0}(h'(0)) < 0$. □

(7.13) Definition. *The boundary $\partial\Omega$ is said to be weakly (resp. strongly) pseudoconvex if $L_{\partial\Omega}$ is semipositive (resp. positive definite) on $\partial\Omega$. The boundary is said to be Levi flat if $L_{\partial\Omega} \equiv 0$.*

(7.14) Remark. Lemma 7.11 shows that $\partial\Omega$ is Levi flat if and only if the subbundle ${}^hT_{\partial\Omega} \subset T_{\partial\Omega}$ is integrable (i.e. stable under the Lie bracket). Assume that $\partial\Omega$ is of class C^k, $k \geq 2$. Then ${}^hT_{\partial\Omega}$ is of class C^{k-1}. By Frobenius' theorem, the integrability condition implies that ${}^hT_{\partial\Omega}$ is the tangent bundle to a C^k foliation of $\partial\Omega$ whose leaves have real dimension $2n-2$. But the leaves themselves must be complex analytic since ${}^hT_{\partial\Omega}$ is a complex vector space (cf. Lemma 7.15 below). Therefore $\partial\Omega$ is Levi flat if and only if it is foliated by complex analytic hypersurfaces.

(7.15) Lemma. *Let Y be a C^1-submanifold of a complex analytic manifold X. If the tangent space $T_{Y,x}$ is a complex subspace of $T_{X,x}$ at every point $x \in Y$, then Y is complex analytic.*

Proof. Let $x_0 \in Y$. Select holomorphic coordinates $(z_1, \ldots, z_n)$ on X centered at x_0 such that T_{Y,x_0} is spanned by $\partial/\partial z_1, \ldots, \partial/\partial z_p$. Then there exists a neighborhood $U = U' \times U''$ of x_0 such that $Y \cap U$ is a graph

$$z'' = h(z'), \qquad z' = (z_1, \ldots, z_p) \in U', \quad z'' = (z_{p+1}, \ldots, z_n)$$

with $h \in C^1(U')$ and $dh(0) = 0$. The differential of h at z' is the composite of the projection of $\mathbb{C}^p \times \{0\}$ on $T_{Y,(z',h(z'))}$ along $\{0\} \times \mathbb{C}^{n-p}$ and of the second projection $\mathbb{C}^n \to \mathbb{C}^{n-p}$. Hence $dh(z')$ is $\mathbb{C}$-linear at every point and h is holomorphic. □

§8. Exercises

8.1. Let $\Omega \subset \mathbb{C}^n$ be an open set such that

$$z \in \Omega, \quad \lambda \in \mathbb{C}, \quad |\lambda| \leq 1 \Longrightarrow \lambda z \in \Omega.$$

Show that Ω is a union of polydisks of center 0 (with arbitrary linear changes of coordinates) and infer that the space of polynomials $\mathbb{C}[z_1, \ldots, z_n]$ is dense in $\mathcal{O}(\Omega)$ for the topology of uniform convergence on compact subsets and in $\mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$

for the topology of uniform convergence on $\overline{\Omega}$.
Hint: consider the Taylor expansion of a function $f \in \mathcal{O}(\Omega)$ at the origin, writing it as a series of homogeneous polynomials. To deal with the case of $\overline{\Omega}$, first apply a dilation to f.

8.2. Let $B \subset \mathbb{C}^n$ be the unit euclidean ball, $S = \partial B$ and $f \in \mathcal{O}(B) \cap C^0(\overline{B})$. Our goal is to check the following Cauchy formula:

$$f(w) = \frac{1}{\sigma_{2n-1}} \int_S \frac{f(z)}{(1 - \langle w, z \rangle)^n} \, d\sigma(z).$$

a) By means of a unitary transformation and Exercise 8.1, reduce the question to the case when $w = (w_1, 0, \ldots, 0)$ and $f(z)$ is a monomial z^α.

b) Show that the integral $\int_B z^\alpha \overline{z}_1^k \, d\lambda(z)$ vanishes unless $\alpha = (k, 0, \ldots, 0)$. Compute the value of the remaining integral by the Fubini theorem, as well as the integrals $\int_S z^\alpha \overline{z}_1^k \, d\sigma(z)$.

c) Prove the formula by a suitable power series expansion.

8.3. A current $T \in \mathcal{D}'_p(M)$ is said to be *normal* if both T and dT are of order zero, i.e. have measure coefficients.

a) If T is normal and has support contained in a C^1 submanifold $Y \subset M$, show that there exists a normal current Θ on Y such that $T = j_\star \Theta$, where $j : Y \longrightarrow M$ is the inclusion.
Hint: if $x_1 = \ldots = x_q = 0$ are equations of Y in a coordinate system $(x_1, \ldots, x_n)$, observe that $x_j T = x_j dT = 0$ for $1 \leq j \leq q$ and infer that $dx_1 \wedge \ldots \wedge dx_q$ can be factorized in all terms of T.

b) What happens if $p > \dim Y$?

c) Are a) and b) valid when the normality assumption is dropped ?

8.4. Let $T = \sum_{1 \leq j \leq n} T_j d\overline{z}_j$ be a closed current of bidegree $(0,1)$ with compact support in $\mathbb{C}^n$ such that $d''T = 0$.

a) Show that the partial convolution $S = (1/\pi z_1) \star_1 T_1$ is a solution of the equation $d''S = T$.

b) Let $K = \operatorname{Supp} T$. If $n \geq 2$, show that S has support in the compact set $\widetilde{K}$ equal to the union of K and of all bounded components of $\mathbb{C}^n \smallsetminus K$.
Hint: observe that S is holomorphic on $\mathbb{C}^n \smallsetminus K$ and that S vanishes for $|z_2| + \ldots + |z_n|$ large.

8.5. Alternative proof of the Dolbeault-Grothendieck lemma. Let $v = \sum_{|J|=q} v_J d\overline{z}_J$, $q \geq 1$, be a smooth form of bidegree $(0,q)$ on a polydisk $\Omega = D(0,R) \subset \mathbb{C}^n$, such that $d''v = 0$, and let $\omega = D(0,r) \subset\subset \omega$. Let k be the smallest integer such that the monomials $d\overline{z}_J$ appearing in v only involve $d\overline{z}_1, \ldots, d\overline{z}_k$. Prove by induction on k that the equation $d''u = v$ can be solved on ω.
Hint: set $v = f \wedge d\overline{z}_k + g$ where f, g only involve $d\overline{z}_1, \ldots, d\overline{z}_{k-1}$. Then consider $v - d''F$ where

$$F = \sum_{|J|=q-1} F_J d\overline{z}_J, \qquad F_J(z) = (\psi(z_k) f_J) \star_k \left(\frac{1}{\pi z_k} \right),$$

where $\star_k$ denotes the partial convolution with respect to z_k, $\psi(z_k)$ is a cut-off function equal to 1 on $D(0, r_k + \varepsilon)$ and $f = \sum_{|J|=q-1} f_J d\overline{z}_J$.

8.6. Construct locally bounded non continuous subharmonic functions on $\mathbb{C}$.
Hint: consider e^u where $u(z) = \sum_{j\ge 1} 2^{-j}\log|z-1/j|$.

8.7. Let ω be an open subset of $\mathbb{R}^n$, $n \ge 2$, and u a subharmonic function which is not locally $-\infty$.

a) For every open set $\omega \subset\subset \Omega$, show that there is a positive measure μ with support in $\overline{\omega}$ and a harmonic function h on ω such that $u = N\star\mu + h$ on ω.
b) Use this representation to prove the following properties: $u \in L^p_{\rm loc}$ for all $p < n/(n-2)$ and $\partial u/\partial x_j \in L^p_{\rm loc}$ for all $p < n/(n-1)$.

8.8. Show that a connected open set $\Omega \subset \mathbb{R}^n$ is convex if and only if Ω has a locally convex exhaustion function φ.
Hint: to show the sufficiency, take a path $\gamma : [0,1] \to \Omega$ joining two arbitrary points $a, b \in \Omega$ and consider the restriction of φ to $[a,\gamma(t_0)]\cap\Omega$ where t_0 is the supremum of all t such that $[a,\gamma(u)] \subset \Omega$ for $u \in [0,t]$.

8.9. Let $r_1, r_2 \in\,]1,+\infty[$. Consider the compact set

$$K = \{|z_1| \le r_1\,,\ |z_2| \le 1\} \cup \{|z_1| \le 1\,,\ |z_2| \le r_2\} \subset \mathbb{C}^2.$$

Show that the holomorphic hull of K in $\mathbb{C}^2$ is

$$\widehat{K} = \{|z_1| \le r_1\,,\ |z_2| \le r_2\,,\ |z_1|^{1/\log r_1}|z_2|^{1/\log r_2} \le e\}.$$

Hint: to show that $\widehat{K}$ is contained in this set, consider all holomorphic monomials $f(z_1,z_2) = z_1^{\alpha_1}z_2^{\alpha_2}$. To show the converse inclusion, apply the maximum principle to the domain $|z_1| \le r_1$, $|z_2| \le r_2$ on suitably chosen Riemann surfaces $z_1^{\alpha_1}z_2^{\alpha_2} = \lambda$.

8.10. Compute the rank of the Levi form of the ellipsoid $|z_1|^2 + |z_3|^4 + |z_3|^6 < 1$ at every point of the boundary.

8.11. Let X be a complex manifold and let $u(z) = \sum_{j\in\mathbb{N}} |f_j|^2$, $f_j \in \mathcal{O}(X)$, be a series converging uniformly on every compact subset of X. Prove that the limit is real analytic and that the series remains uniformly convergent by taking derivatives term by term.
Hint: since the problem is local, take $X = B(0,r)$, a ball in $\mathbb{C}^n$. Let $g_j(z) = \overline{g_j(\overline{z})}$ be the conjugate function of f_j and let $U(z,w) = \sum_{j\in\mathbb{N}} f_j(z)g_j(w)$ on $X\times X$. Using the Cauchy-Schwarz inequality, show that this series of holomorphic functions is uniformly convergent on every compact subset of $X \times X$.

8.12. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with C^2 boundary.

a) Let $a \in \partial\Omega$ be a given point. Let e_n be the outward normal vector to $T_{\partial\Omega,a}$, $(e_1,\ldots,e_{n-1})$ an orthonormal basis of ${}^hT_a(\partial\Omega)$ in which the Levi form is diagonal and $(z_1,\ldots,z_n)$ the associated linear coordinates centered at a. Show that there is a neighborhood V of a such that $\partial\Omega\cap V$ is the graph $\mathrm{Re}\,z_n = -\varphi(z_1,\ldots,z_{n-1},\mathrm{Im}\,z_n)$ of a function φ such that $\varphi(z) = O(|z|^2)$ and the matrix $\partial^2\varphi/\partial z_j\partial\overline{z}_k(0)$, $1 \le j,k \le n-1$ is diagonal.
b) Show that there exist local analytic coordinates $w_1 = z_1, \ldots, w_{n-1} = z_{n-1}$, $w_n = z_n + \sum c_{jk}z_jz_k$ on a neighborhood V' of $a = 0$ such that

$$\Omega\cap V' = V'\cap\{\mathrm{Re}\,w_n + \sum_{1\le j\le n}\lambda_j|w_j|^2 + \mathrm{o}(|w|^2) < 0\},\qquad \lambda_j \in \mathbb{R}$$

and that λ_n can be assigned to any given value by a suitable choice of the coordinates.
Hint: Consider the Taylor expansion of order 2 of the defining function $\rho(z) = (\mathrm{Re}\, z_n + \varphi(z))(1 + \mathrm{Re} \sum c_j z_j)$ where $c_j \in \mathbb{C}$ are chosen in a suitable way.

c) Prove that $\partial\Omega$ is strongly pseudoconvex at a if and only if there is a neighborhood U of a and a biholomorphism Φ of U onto some open set of $\mathbb{C}^n$ such that $\Phi(\Omega \cap U)$ is strongly convex.

d) Assume that the Levi form of $\partial\Omega$ is not semipositive. Show that all holomorphic functions $f \in \mathcal{O}(\Omega)$ extend to some (fixed) neighborhood of a.
Hint: assume for example $\lambda_1 < 0$. For $\varepsilon > 0$ small, show that Ω contains the Hartogs figure

$$\begin{aligned}&\{\varepsilon/2 < |w_1| < \varepsilon\} \times \{|w_j| < \varepsilon^2\}_{1<j<n} \times \{|w_n| < \varepsilon^{3/2}\,,\ \mathrm{Re}\, w_n < \varepsilon^3\} \ \cup \\ &\qquad \{|w_1| < \varepsilon\} \times \{|w_j| < \varepsilon^2\}_{1<j<n} \times \{|w_n| < \varepsilon^{3/2}\,,\ \mathrm{Re}\, w_n < -\varepsilon^2\}.\end{aligned}$$

8.13. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with C^2 boundary and $\rho \in C^2(\Omega, \mathbb{R})$ such that $\rho < 0$ on Ω, $\rho = 0$ and $d\rho \neq 0$ on $\partial\Omega$. Let $f \in C^1(\partial\Omega, \mathbb{C})$ be a function satisfying the *tangential Cauchy-Riemann equations*

$$\xi'' \cdot f = 0, \qquad \forall \xi \in {}^hT_{\partial\Omega}, \qquad \xi'' = \frac{1}{2}(\xi + \mathrm{i}J\xi).$$

a) Let f_0 be a C^1 extension of f to $\overline{\Omega}$. Show that $d''f_0 \wedge d''\rho = 0$ on $\partial\Omega$ and infer that $v = 1\!\!1_\Omega d''f_0$ is a d''-closed current on $\mathbb{C}^n$.

b) Show that the solution u of $d''u = v$ provided by Cor. 3.27 is continuous and that f admits an extension $\widetilde{f} \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$ if $\partial\Omega$ is connected.

8.14. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^2 boundary and let $\delta(z) = d(z, \complement\Omega)$ be the euclidean distance to the boundary.

a) Use the plurisubharmonicity of $-\log\delta$ to prove the following fact: for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\frac{-H\delta_z(\xi)}{\delta(z)} + \varepsilon\frac{|d'\delta_z.\xi|^2}{|\delta(z)|^2} + C_\varepsilon|\xi|^2 \geq 0$$

for $\xi \in \mathbb{C}^n$ and z near $\partial\Omega$.

b) Set $\psi(z) = -\log\delta(z) + K|z|^2$. Show that for K large and α small the function

$$\rho(z) = -\exp\Big(-\alpha\psi(z)\Big) = -\Big(e^{-K|z|^2}\delta(z)\Big)^\alpha$$

is plurisubharmonic.

c) Prove the existence of a plurisubharmonic exhaustion function $u : \Omega \to [-1, 0[$ of class C^2 such that $|u(z)|$ has the same order of magnitude as $\delta(z)^\alpha$ when z tends to $\partial\Omega$.
Hint: consult (Diederich-Fornaess 1976).

8.15. Let $\Omega = \omega + \mathrm{i}\mathbb{R}^n$ be a connected tube in $\mathbb{C}^n$ of base ω.

a) Assume first that $n = 2$. Let $T \subset \mathbb{R}^2$ be the triangle $x_1 \ge 0$, $x_2 \ge 0$, $x_1 + x_2 \le 1$, and assume that the two edges $[0,1] \times \{0\}$ and $\{0\} \times [0,1]$ are contained in ω. Show that every holomorphic function $f \in \mathcal{O}(\Omega)$ extends to a neighborhood of $T + \mathrm{i}\mathbb{R}^2$.
Hint: let $\pi : \mathbb{C}^2 \longrightarrow \mathbb{R}^2$ be the projection on the real part and M_ε the intersection of $\pi^{-1}((1+\varepsilon)T)$ with the Riemann surface $z_1 + z_2 - \frac{\varepsilon}{2}(z_1^2 + z_2^2) = 1$ (a non degenerate affine conic). Show that M_ε is compact and that

$$\pi(\partial M_\varepsilon) \subset ([0, 1+\varepsilon] \times \{0\}) \cup (\{0\} \times [0, 1+\varepsilon]) \subset \omega,$$
$$\pi([0,1] \cdot M_\varepsilon) \supset T$$

for ε small. Use the Cauchy formula along ∂M_ε (in some parametrization of the conic) to obtain an extension of f to $[0,1] \cdot M_\varepsilon + \mathrm{i}\mathbb{R}^n$.

b) In general, show that every $f \in \mathcal{O}(\Omega)$ extends to the convex hull $\widehat{\Omega}$.
Hint: given $a, b \in \omega$, consider a polygonal line joining a and b and apply a) inductively to obtain an extension along $[a,b] + \mathrm{i}\mathbb{R}^n$.

8.16. For each integer $\nu \ge 1$, consider the algebraic variety

$$X_\nu = \Big\{(z, w, t) \in \mathbb{C}^3 \,;\, wt = p_\nu(z)\Big\}, \qquad p_\nu(z) = \prod_{1 \le k \le \nu} (z - 1/k),$$

and the map $j_\nu : X_\nu \to X_{\nu+1}$ such that

$$j_\nu(z, w, t) = \left(z,\, w,\, t\Big(z - \frac{1}{\nu+1}\Big)\right).$$

a) Show that X_ν is a Stein manifold, and that j_ν is an embedding of X_ν onto an open subset of $X_{\nu+1}$.

b) Define $X = \lim(X_\nu, j_\nu)$, and let $\pi_\nu : X_\nu \to \mathbb{C}^2$ be the projection to the first two coordinates. Since $\pi_{\nu+1} \circ j_\nu = \pi_\nu$, there exists a holomorphic map $\pi : X \to \mathbb{C}^2$, $\pi = \lim \pi_\nu$. Show that

$$\mathbb{C}^2 \smallsetminus \pi(X) = \Big\{(z, 0) \in \mathbb{C}^2 \,;\, z \ne 1/\nu,\ \forall \nu \in \mathbb{N},\ \nu \ge 1\Big\},$$

and especially, that $(0,0) \notin \pi(X)$.

c) Consider the compact set

$$K = \pi^{-1}\Big(\{(z, w) \in \mathbb{C}^2 \,;\, |z| \le 1,\ |w| = 1\}\Big).$$

By looking at points of the forms $(1/\nu, w, 0)$, $|w| = 1$, show that $\pi^{-1}(1/\nu, 1/\nu) \in \widehat{K}_{\mathcal{O}(X)}$. Conclude from this that X is not holomorphically convex (this example is due to Fornaess 1977).

8.17. Let X be a complex manifold, and let $\pi : \widetilde{X} \to X$ be a holomorphic unramified covering of X (X and $\widetilde{X}$ are assumed to be connected).

a) Let g be a complete riemannian metric on X, and let $\widetilde{d}$ be the geodesic distance on $\widetilde{X}$ associated to $\widetilde{g} = \pi^\star g$ (see VIII-2.3 for definitions). Show that $\widetilde{g}$ is complete and that $\delta_0(x) := \widetilde{d}(x, x_0)$ is a continuous exhaustion function on $\widetilde{X}$, for any given point $x_0 \in \widetilde{X}$.

b) Let (U_α) be a locally finite covering of X by open balls contained in coordinate open sets, such that all intersections $U_\alpha \cap U_\beta$ are diffeomorphic to convex open sets (see Lemma IV-6.9). Let θ_α be a partition of unity subordinate to the covering (U_α), and let $\delta_{\varepsilon_\alpha}$ be the convolution of δ_0 with a regularizing kernel $\rho_{\varepsilon_\alpha}$ on each piece of $\pi^{-1}(U_\alpha)$ which is mapped biholomorphically onto U_α. Finally, set $\delta = \sum(\theta_\alpha \circ \pi)\delta_{\varepsilon_\alpha}$. Show that if (ε_α) is a collection of sufficiently small positive numbers, then δ is a smooth exhaustion function on $\widetilde{X}$.

c) Using the fact that δ_0 is 1-Lipschitz with respect to $\widetilde{d}$, show that derivatives $\partial^{|\nu|}\delta(x)/\partial x^\nu$ of a given order with respect to coordinates in U_α are uniformly bounded in all components of $\pi^{-1}(U_\alpha)$, at least when x lies in the compact subset $\operatorname{Supp}\theta_\alpha$. Conclude from this that there exists a positive hermitian form with continuous coefficients on X such that $H\delta \geq -\pi^\star\gamma$ on $\widetilde{X}$.

d) If X is strongly pseudoconvex, show that $\widetilde{X}$ is also strongly pseudoconvex.
Hint: let ψ be a smooth strictly plurisubharmonic exhaustion function on X. Show that there exists a smooth convex increasing function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\delta + \chi \circ \psi$ is strictly plurisubharmonic.

Chapter II.
Coherent Sheaves and Analytic Spaces

The chapter starts with rather general and abstract concepts concerning sheaves and ringed spaces. Introduced in the decade 1950-1960 by Leray, Cartan, Serre and Grothendieck, sheaves and ringed spaces have since been recognized as the adequate tools to handle algebraic varieties and analytic spaces in a unified framework. We then concentrate ourselves on the theory of complex analytic functions. The second section is devoted to a proof of the Weierstrass preparation theorem, which is nothing but a division algorithm for holomorphic functions. It is used to derive algebraic properties of the ring $\mathcal{O}_n$ of germs of holomorphic functions in $\mathbb{C}^n$. Coherent analytic sheaves are then introduced and the fundamental coherence theorem of Oka is proved. Basic properties of analytic sets are investigated in detail: local parametrization theorem, Hilbert's Nullstellensatz, coherence of the ideal sheaf of an analytic set, analyticity of the singular set. The formalism of complex spaces is then developed and gives a natural setting for the proof of more global properties (decomposition into global irreducible components, maximum principle). After a few definitions concerning cycles, divisors and meromorphic functions, we investigate the important notion of normal space and establish the Oka normalization theorem. Next, the Remmert-Stein extension theorem and the Remmert proper mapping theorem on images of analytic sets are proved by means of semi-continuity results on the rank of morphisms. As an application, we give a proof of Chow's theorem asserting that every analytic subset of $\mathbb{P}^n$ is algebraic. Finally, the concept of analytic scheme with nilpotent elements is introduced as a generalization of complex spaces, and we discuss the concepts of bimeromorphic maps, modifications and blowing-up.

§1. Presheaves and Sheaves

§1.A. Main Definitions

Sheaves have become a very important tool in analytic or algebraic geometry as well as in algebraic topology. They are especially useful when one wants to relate global properties of an object to its local properties (the latter being usually easier to establish). We first introduce the axioms of presheaves and sheaves in full generality and give some basic examples.

(1.1) Definition. *Let X be a topological space. A presheaf $\mathcal{A}$ on X consists of the following data:*

a) *a collection of non empty sets $\mathcal{A}(U)$ associated with every open set $U \subset X$,*

b) *a collection of maps $\rho_{U,V} : \mathcal{A}(V) \longrightarrow \mathcal{A}(U)$ defined whenever $U \subset V$ and satisfying the transitivity property*

c) $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$ *for* $U \subset V \subset W$, $\rho_{U,U} = \mathrm{Id}_U$ *for every* U.

The set $\mathcal{A}(U)$ is called the set of sections of the presheaf $\mathcal{A}$ over U.

Most often, the presheaf $\mathcal{A}$ is supposed to carry an additional algebraic structure. For instance:

(1.2) Definition. *A presheaf $\mathcal{A}$ is said to be a presheaf of abelian groups (resp. rings, R-modules, algebras) if all sets $\mathcal{A}(U)$ are abelian groups (resp. rings, R-modules, algebras) and if the maps $\rho_{U,V}$ are morphisms of these algebraic structures. In this case, we always assume that $\mathcal{A}(\emptyset) = \{0\}$.*

(1.3) Example. If we assign to each open set $U \subset X$ the set $\mathcal{C}(U)$ of all real valued continuous functions on U and let $\rho_{U,V}$ be the obvious restriction morphism $\mathcal{C}(V) \to \mathcal{C}(U)$, then $\mathcal{C}$ is a presheaf of rings on X. Similarly if X is a differentiable (resp. complex analytic) manifold, there are well defined presheaves of rings $\mathcal{C}^k$ of functions of class C^k (resp. $\mathcal{O}$) of holomorphic functions) on X. Because of these examples, the maps $\rho_{U,V}$ in Def. 1.1 are often viewed intuitively as "restriction homomorphisms", although the sets $\mathcal{A}(U)$ are not necessarily sets of functions defined over U. For the simplicity of notation we often just write $\rho_{U,V}(f) = f_{\restriction U}$ whenever $f \in \mathcal{A}(V)$, $V \supset U$. □

For the above presheaves $\mathcal{C}$, $\mathcal{C}^k$, $\mathcal{O}$, the properties of functions under consideration are purely local. As a consequence, these presheaves satisfy the following additional *gluing axioms*, where (U_α) and $U = \bigcup U_\alpha$ are arbitrary open subsets of X:

(1.4′) If $F_\alpha \in \mathcal{A}(U_\alpha)$ are such that $\rho_{U_\alpha \cap U_\beta, U_\alpha}(F_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(F_\beta)$ for all α, β, there exists $F \in \mathcal{A}(U)$ such that $\rho_{U_\alpha, U}(F) = F_\alpha$;

(1.4″) If $F, G \in \mathcal{A}(U)$ and $\rho_{U_\alpha, U}(F) = \rho_{U_\alpha, U}(G)$ for all α, then $F = G$;

in other words, local sections over the sets U_α can be glued together if they coincide in the intersections and the resulting section on U is uniquely defined. Not all presheaves satisfy (1.4′) and (1.4″):

(1.5) Example. Let E be an arbitrary set with a distinguished element 0 (e.g. an abelian group, a R-module, …). The *constant presheaf* E_X on X is defined to be $E_X(U) = E$ for all $\emptyset \neq U \subset X$ and $E_X(\emptyset) = \{0\}$, with restriction maps $\rho_{U,V} = \mathrm{Id}_E$ if $\emptyset \neq U \subset V$ and $\rho_{U,V} = 0$ if $U = \emptyset$. Then axiom (1.4′) is not satisfied if U is the union of two disjoint open sets U_1, U_2 and E contains a non zero element.

(1.6) Definition. *A presheaf $\mathcal{A}$ is said to be a sheaf if it satisfies the gluing axioms* (1.4′) *and* (1.4″).

If $\mathcal{A}$, $\mathcal{B}$ are presheaves of abelian groups (or of some other algebraic structure) on the same space X, a presheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is a collection of morphisms $\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$ commuting with the restriction morphisms, i.e. such that for each pair $U \subset V$ there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}(V) & \xrightarrow{\varphi_V} & \mathcal{B}(V) \\ \rho^{\mathcal{B}}_{U,V}\big\downarrow & & \big\downarrow\rho^{\mathcal{A}}_{U,V} \\ \mathcal{A}(U) & \xrightarrow{\varphi_U} & \mathcal{B}(U). \end{array}$$

We say that $\mathcal{A}$ is a subpresheaf of $\mathcal{B}$ in the case where $\varphi_U : \mathcal{A}(U) \subset \mathcal{B}(U)$ is the inclusion morphism; the commutation property then means that $\rho^{\mathcal{B}}_{U,V}(\mathcal{A}(V)) \subset \mathcal{A}(U)$ for all U, V, and that $\rho^{\mathcal{A}}_{U,V}$ coincides with $\rho^{\mathcal{B}}_{U,V}$ on $\mathcal{A}(V)$. If $\mathcal{A}$ is a subpresheaf of a presheaf $\mathcal{B}$ of abelian groups, there is a presheaf quotient $\mathcal{C} = \mathcal{B}/\mathcal{A}$ defined by $\mathcal{C}(U) = \mathcal{B}(U)/\mathcal{A}(U)$. In a similar way, one defines the presheaf kernel (resp. presheaf image, presheaf cokernel) of a presheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ to be the presheaves

$$U \mapsto \operatorname{Ker}\varphi_U, \qquad U \mapsto \operatorname{Im}\varphi_U, \qquad U \mapsto \operatorname{Coker}\varphi_U.$$

The direct sum $\mathcal{A} \oplus \mathcal{B}$ of presheaves of abelian groups $\mathcal{A}$, $\mathcal{B}$ is the presheaf $U \mapsto \mathcal{A}(U) \oplus \mathcal{B}(U)$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ of presheaves of R-modules is $U \mapsto \mathcal{A}(U) \otimes_R \mathcal{B}(U)$, etc ...

(1.7) Remark. The reader should take care of the fact that the presheaf quotient of a sheaf by a subsheaf is not necessarily a sheaf. To give a specific example, let $X = S^1$ be the unit circle in $\mathbb{R}^2$, let $\mathcal{C}$ be the sheaf of continuous complex valued functions and $\mathcal{Z}$ the subsheaf of integral valued continuous functions (i.e. locally constant functions to $\mathbb{Z}$). The exponential map

$$\varphi = \exp(2\pi\mathrm{i}\bullet) : \mathcal{C} \longrightarrow \mathcal{C}^\star$$

is a morphism from $\mathcal{C}$ to the sheaf $\mathcal{C}^\star$ of invertible continuous functions, and the kernel of φ is precisely $\mathcal{Z}$. However φ_U is surjective for all $U \neq X$ but maps $\mathcal{C}(X)$ onto the multiplicative subgroup of continuous functions of $\mathcal{C}^\star(X)$ of degree 0. Therefore the quotient presheaf $\mathcal{C}/\mathcal{Z}$ is not isomorphic with $\mathcal{C}^\star$, although their groups of sections are the same for all $U \neq X$. Since $\mathcal{C}^\star$ is a sheaf, we see that $\mathcal{C}/\mathcal{Z}$ does not satisfy property (1.4′). □

In order to overcome the difficulty appearing in Example 1.7, it is necessary to introduce a suitable process by which we can produce a sheaf from a presheaf. For this, it is convenient to introduce a slightly modified viewpoint for sheaves.

(1.8) Definition. *If $\mathcal{A}$ is a presheaf, we define the set $\widetilde{\mathcal{A}}_x$ of germs of $\mathcal{A}$ at a point $x \in X$ to be the abstract inductive limit*

$$\widetilde{\mathcal{A}}_x = \varinjlim_{U \ni x} \big(\mathcal{A}(U), \rho_{U,V}\big).$$

More explicitely, $\widetilde{\mathcal{A}}_x$ is the set of equivalence classes of elements in the disjoint union $\coprod_{U \ni x} \mathcal{A}(U)$ taken over all open neighborhoods U of x, with two elements $F_1 \in \mathcal{A}(U_1)$, $F_2 \in \mathcal{A}(U_2)$ being equivalent, $F_1 \sim F_2$, if and only if there is a neighborhood $V \subset U_1, U_2$ such that $F_{1\restriction V} = F_{2\restriction V}$, i.e., $\rho_{VU_1}(F_1) = \rho_{VU_2}(F_2)$. The germ of an element $F \in \mathcal{A}(U)$ at a point $x \in U$ will be denoted by F_x.

Let $\mathcal{A}$ be an arbitrary presheaf. The disjoint union $\widetilde{\mathcal{A}} = \coprod_{x \in X} \widetilde{\mathcal{A}}_x$ can be equipped with a natural topology as follows: for every $F \in \mathcal{A}(U)$, we set

$$\Omega_{F,U} = \big\{F_x \ ; \ x \in U\big\}$$

and choose the $\Omega_{F,U}$ to be a basis of the topology of $\widetilde{\mathcal{A}}$; note that this family is stable by intersection: $\Omega_{F,U} \cap \Omega_{G,V} = \Omega_{H,W}$ where W is the (open) set of points $x \in U \cap V$ at which $F_x = G_x$ and $H = \rho_{W,U}(F)$. The obvious projection map $\pi : \widetilde{\mathcal{A}} \to X$ which sends $\widetilde{\mathcal{A}}_x$ to $\{x\}$ is then a local homeomorphism (it is actually a homeomorphism from $\Omega_{F,U}$ onto U). This leads in a natural way to the following definition:

(1.9) Definition. *Let X and $\mathcal{S}$ be topological spaces (not necessarily Hausdorff), and let $\pi : \mathcal{S} \longrightarrow X$ be a mapping such that*

a) *π maps $\mathcal{S}$ onto X ;*

b) *π is a local homeomorphism, that is, every point in $\mathcal{S}$ has an open neighborhood which is mapped homeomorphically by π onto an open subset of X.*

Then $\mathcal{S}$ is called a sheaf-space on X and π is called the projection of $\mathcal{S}$ on X. If $x \in X$, then $\mathcal{S}_x = \pi^{-1}(x)$ is called the stalk of $\mathcal{S}$ at x.

If Y is a subset of X, we denote by $\Gamma(Y, \mathcal{S})$ the set of sections of $\mathcal{S}$ on Y, i.e. the set of continuous functions $F : Y \to \mathcal{S}$ such that $\pi \circ F = \mathrm{Id}_Y$. It is clear that the presheaf defined by the collection of sets $\mathcal{S}'(U) := \Gamma(U, \mathcal{S})$ for all open sets $U \subset X$ together with the restriction maps $\rho_{U,V}$ satisfies axioms (1.4′) and (1.4″), hence $\mathcal{S}'$ is a sheaf. The set of germs of $\mathcal{S}'$ at x is in one-to-one correspondence with the stalk $\mathcal{S}_x = \pi^{-1}(x)$, thanks to the local homeomorphism assumption 1.9 b). This shows that one can associate in a natural way a sheaf $\mathcal{S}'$ to every sheaf-space $\mathcal{S}$, and that the sheaf-space $(\mathcal{S}')^{\sim}$ can be considered to be identical to the original sheaf-space $\mathcal{S}$. Since the assignment $\mathcal{S} \mapsto \mathcal{S}'$ from sheaf-spaces to sheaves is an equivalence of categories, we will usually omit the prime sign in the notation of $\mathcal{S}'$ and thus use the same symbols for a sheaf-space and its associated sheaf of sections; in a corresponding way, we write $\Gamma(U, \mathcal{S}) = \mathcal{S}(U)$ when U is an open set.

Conversely, given a presheaf $\mathcal{A}$ on X, we have an associated sheaf-space $\widetilde{\mathcal{A}}$ and an obvious presheaf morphism

$$(1.10) \quad \mathcal{A}(U) \longrightarrow \widetilde{\mathcal{A}}'(U) = \Gamma(U, \widetilde{\mathcal{A}}), \qquad F \longmapsto \widetilde{F} = (U \ni x \mapsto F_x).$$

This morphism is clearly injective if and only if $\mathcal{A}$ satisfies axiom $(1.4'')$, and it is not difficult to see that $(1.4')$ and $(1.4'')$ together imply surjectivity. Therefore $\mathcal{A} \to \widetilde{\mathcal{A}}'$ is an isomorphism if and only if $\mathcal{A}$ is a sheaf. According to the equivalence of categories between sheaves and sheaf-spaces mentioned above, we will use from now on the same symbol $\widetilde{\mathcal{A}}$ for the sheaf-space and its associated sheaf $\widetilde{\mathcal{A}}'$; one says that $\widetilde{\mathcal{A}}$ is the *sheaf associated with the presheaf* $\mathcal{A}$. If $\mathcal{A}$ itself is a sheaf, we will again identify $\widetilde{\mathcal{A}}$ and $\mathcal{A}$, but we will of course keep the notational difference for a presheaf $\mathcal{A}$ which is not a sheaf.

(1.11) Example. The sheaf associated to the constant presheaf of stalk E over X is the sheaf of locally constant functions $X \to E$. This sheaf will be denoted merely by E_X or E if there is no risk of confusion with the corresponding presheaf. In Example 1.7, we have $\mathcal{Z} = \mathbb{Z}_X$ and the sheaf $(\mathcal{C}/\mathbb{Z}_X)^{\sim}$ associated with the quotient presheaf $\mathcal{C}/\mathbb{Z}_X$ is isomorphic to $\mathcal{C}^\star$ via the exponential map. □

In the sequel, we usually work in the category of sheaves rather than in the category of presheaves themselves. For instance, the quotient $\mathcal{B}/\mathcal{A}$ of a sheaf $\mathcal{B}$ by a subsheaf $\mathcal{A}$ generally refers to the sheaf associated with the quotient presheaf: its stalks are equal to $\mathcal{B}_x/\mathcal{A}_x$, but a section G of $\mathcal{B}/\mathcal{A}$ over an open set U need not necessarily come from a global section of $\mathcal{B}(U)$; what can be only said is that there is a covering (U_α) of U and local sections $F_\alpha \in \mathcal{B}(U_\alpha)$ representing $G_{\restriction U_\alpha}$ such that $(F_\beta - F_\alpha)_{\restriction U_\alpha \cap U_\beta}$ belongs to $\mathcal{A}(U_\alpha \cap U_\beta)$. A sheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be injective (resp. surjective) if the germ morphism $\varphi_x : \mathcal{A}_x \to \mathcal{B}_x$ is injective (resp. surjective) for every $x \in X$. Let us note again that a surjective sheaf morphism φ does not necessarily give rise to surjective morphisms $\varphi_U : \mathcal{A}(U) \to \mathcal{B}(U)$.

§1.B. Direct and Inverse Images of Sheaves

Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. If $\mathcal{A}$ is a presheaf on X, the *direct image* $f_\star\mathcal{A}$ is the presheaf on Y defined by

$$(1.12) \quad f_\star\mathcal{A}(U) = \mathcal{A}\big(f^{-1}(U)\big)$$

for all open sets $U \subset Y$. When $\mathcal{A}$ is a sheaf, it is clear that $f_\star\mathcal{A}$ also satisfies axioms $(1.4')$ and $(1.4'')$, thus $f_\star\mathcal{A}$ is a sheaf. Its stalks are given by

$$(1.13) \quad (f_\star\mathcal{A})_y = \varinjlim_{V \ni y} \mathcal{A}\big(f^{-1}(V)\big)$$

where V runs over all open neighborhoods of $y \in Y$.

Now, let $\mathcal{B}$ be a sheaf on Y, viewed as a sheaf-space with projection map $\pi : \mathcal{B} \to Y$. We define the *inverse image* $f^{-1}\mathcal{B}$ by

$$(1.14) \quad f^{-1}\mathcal{B} = \mathcal{B} \times_Y X = \big\{(s,x) \in \mathcal{B} \times X\,;\ \pi(s) = f(x)\big\}$$

with the topology induced by the product topology on $\mathcal{B} \times X$. It is then easy to see that the projection $\pi' = \mathrm{pr}_2 : f^{-1}\mathcal{B} \to X$ is a local homeomorphism, therefore $f^{-1}\mathcal{B}$ is a sheaf on X. By construction, the stalks of $f^{-1}\mathcal{B}$ are

$$(1.15) \quad (f^{-1}\mathcal{B})_x = \mathcal{B}_{f(x)},$$

and the sections $\sigma \in f^{-1}\mathcal{B}(U)$ can be considered as continuous mappings $s : U \to \mathcal{B}$ such that $\pi \circ \sigma = f$. In particular, any section $s \in \mathcal{B}(V)$ on an open set $V \subset Y$ has a *pull-back*

$$(1.16) \quad f^\star s = s \circ f \quad \in \quad f^{-1}\mathcal{B}\big(f^{-1}(V)\big).$$

There are always natural sheaf morphisms

$$(1.17) \quad f^{-1}f_\star\mathcal{A} \longrightarrow \mathcal{A}, \qquad \mathcal{B} \longrightarrow f_\star f^{-1}\mathcal{B}$$

defined as follows. A germ in $(f^{-1}f_\star\mathcal{A})_x = (f_\star\mathcal{A})_{f(x)}$ is defined by a local section $s \in (f_\star\mathcal{A})(V) = \mathcal{A}(f^{-1}(V))$ for some neighborhood V of $f(x)$; this section can be mapped to the germ $s_x \in \mathcal{A}_x$. In the opposite direction, the pull-back $f^\star s$ of a section $s \in \mathcal{B}(V)$ can be seen by (1.16) as a section of $f_\star f^{-1}\mathcal{B}(V)$. It is not difficult to see that these natural morphisms are not isomorphisms in general. For instance, if f is a finite covering map with q sheets and if we take $\mathcal{A} = E_X$, $\mathcal{B} = E_Y$ to be constant sheaves, then $f_\star E_X \simeq E_Y^q$ and $f^{-1}E_Y = E_X$, thus $f^{-1}f_\star E_X \simeq E_X^q$ and $f_\star f^{-1}E_Y \simeq E_Y^q$.

§1.C. Ringed Spaces

Many natural geometric structures considered in analytic or algebraic geometry can be described in a convenient way as topological spaces equipped with a suitable "structure sheaf" which, most often, is a sheaf of commutative rings. For instance, a lot of properties of C^k differentiable (resp. real analytic, complex analytic) manifolds can be described in terms of their sheaf of rings $\mathcal{C}_X^k$ of differentiable functions (resp. $\mathcal{C}_X^\omega$ of real analytic functions, $\mathcal{O}_X$ of holomorphic functions). We first recall a few standard definitions concerning rings, referring to textbooks on algebra for more details (see e.g. Lang 1965).

(1.18) Some definitions and conventions about rings. *All our rings R are supposed implicitly to have a unit element 1_R (if $R = \{0\}$, we agree that $1_R = 0_R$), and a ring morphism $R \to R'$ is supposed to map 1_R to $1_{R'}$. In the subsequent definitions, we assume that all rings under consideration are commutative.*

a) *An ideal* $I \subset R$ *is said to be prime if* $xy \in I$ *implies* $x \in I$ *or* $y \in I$, *i.e., if the quotient ring* R/I *is entire.*

b) *An ideal* $I \subset R$ *is said to be maximal if* $I \neq R$ *and there are no ideals* J *such that* $I \subsetneq J \subsetneq R$ (*equivalently, if the quotient ring* R/I *is a field*).

c) *The ring* R *is said to be a local ring if* R *has a unique maximal ideal* $\mathfrak{m}$ (*equivalently, if* R *has an ideal* $\mathfrak{m}$ *such that all elements of* $R \smallsetminus \mathfrak{m}$ *are invertible*). *Its residual field is defined to be the quotient field* $R/\mathfrak{m}$.

d) *The ring* R *is said to be Noetherian if every ideal* $I \subset R$ *is finitely generated* (*equivalently, if every increasing sequence of ideals* $I_1 \subset I_2 \subset \ldots$ *is stationary*).

e) *The radical* $\sqrt{I}$ *of an ideal* I *is the set of all elements* $x \in R$ *such that some power* x^m, $m \in \mathbb{N}^\star$, *lies in in* I. *Then* $\sqrt{I}$ *is again an ideal of* R.

f) *The nilradical* $N(R) = \sqrt{\{0\}}$ *is the ideal of nilpotent elements of* R. *The ring* R *is said to be reduced if* $N(R) = \{0\}$. *Otherwise, its reduction is defined to be the reduced ring* $R/N(R)$.

We now introduce the general notion of a ringed space.

(1.19) Definition. *A ringed space is a pair* $(X, \mathcal{R}_X)$ *consisting of a topological space* X *and of a sheaf of rings* $\mathcal{R}_X$ *on* X, *called the structure sheaf. A morphism*

$$F : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$$

of ringed spaces is a pair $(f, F^\star)$ *where* $f : X \to Y$ *is a continuous map and*

$$F^\star \;:\; f^{-1}\mathcal{R}_Y \to \mathcal{R}_X, \qquad F^\star_x \;:\; \mathcal{R}_{Y,f(x)} \to \mathcal{R}_{X,x}$$

a homomorphism of sheaves of rings on X, *called the comorphism of* F.

If $F : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ and $G : (Y, \mathcal{R}_Y) \to (Z, \mathcal{R}_Z)$ are morphisms of ringed spaces, the composite $G \circ F$ is the pair consisting of the map $g \circ f : X \to Z$ and of the comorphism $(G \circ F)^\star = F^\star \circ f^{-1}G^\star$:

$$(1.20)\quad \begin{array}{llllll} F^\star \circ f^{-1}G^\star & : & f^{-1}g^{-1}\mathcal{R}_Z & \xrightarrow{f^{-1}G^\star} f^{-1}\mathcal{R}_Y & \xrightarrow{F^\star} \mathcal{R}_X, \\ F^\star_x \circ G^\star_{f(x)} & : & \mathcal{R}_{Z,g\circ f(x)} & \longrightarrow \mathcal{R}_{Y,f(x)} & \longrightarrow \mathcal{R}_{X,x}. \end{array}$$

We say of course that F is an isomorphism of ringed spaces if there exists G such that $G \circ F = \mathrm{Id}_X$ and $F \circ G = \mathrm{Id}_Y$.

If $(X, \mathcal{R}_X)$ is a ringed space, the nilradical of $\mathcal{R}_X$ defines an ideal subsheaf $\mathcal{N}_X$ of $\mathcal{R}_X$, and the identity map $\mathrm{Id}_X : X \to X$ together with the ring homomorphism $\mathcal{R}_X \to \mathcal{R}_X/\mathcal{N}_X$ defines a ringed space morphism

$$(1.21)\quad (X, \mathcal{R}_X/\mathcal{N}_X) \to (X, \mathcal{R}_X)$$

called the *reduction morphism.* Quite often, the letter X by itself is used to denote the ringed space $(X, \mathcal{R}_X)$; we then denote by $X_{\mathrm{red}} = (X, \mathcal{R}_X/\mathcal{N}_X)$ its reduction. The ringed space X is said to be *reduced* if $\mathcal{N}_X = 0$, in which case the reduction morphism $X_{\mathrm{red}} \to X$ is an isomorphism. In all examples considered later on in this book, the structure sheaf $\mathcal{R}_X$ will be a sheaf of *local rings* over some field k. The relevant definition is as follows.

(1.22) Definition.

a) *A locally ringed space is a ringed space $(X, \mathcal{R}_X)$ such that all stalks $\mathcal{R}_{X,x}$ are local rings. The maximal ideal of $\mathcal{R}_{X,x}$ will be denoted by $\mathfrak{m}_{X,x}$. A morphism $F = (f, F^\star) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ of locally ringed spaces is a morphism of ringed spaces such that $F^\star_x(\mathfrak{m}_{Y,f(x)}) \subset \mathfrak{m}_{X,x}$ at any point $x \in X$ (i.e., $F^\star_x$ is a "local" homomorphism of rings).*

b) *A locally ringed space over a field k is a locally ringed space $(X, \mathcal{R}_X)$ such that all rings $\mathcal{R}_{X,x}$ are local k-algebras with residual field $\mathcal{R}_{X,x}/\mathfrak{m}_{X,x} \simeq k$. A morphism F between such spaces is supposed to have its comorphism defined by local k-homomorphisms $F^\star_x : \mathcal{R}_{Y,f(x)} \to \mathcal{R}_{X,x}$.*

If $(X, \mathcal{R}_X)$ is a locally ringed space over k, we can associate to each section $s \in \mathcal{R}_X(U)$ a function

$$\overline{s} : U \to k, \qquad x \mapsto \overline{s}(x) \in k = \mathcal{R}_{X,x}/\mathfrak{m}_{X,x},$$

and we get a sheaf morphism $\mathcal{R}_X \to \overline{\mathcal{R}}_X$ onto a subsheaf of rings $\overline{\mathcal{R}}_X$ of the sheaf of functions from X to k. We clearly have a factorization

$$\mathcal{R}_X \to \mathcal{R}_X/\mathcal{N}_X \to \overline{\mathcal{R}}_X,$$

and thus a corresponding factorization of ringed space morphisms (with Id_X as the underlying set theoretic map)

$$X_{\text{st-red}} \to X_{\mathrm{red}} \to X$$

where $X_{\text{st-red}} = (X, \overline{R}_X)$ is called the *strong reduction* of $(X, \mathcal{R}_X)$. It is easy to see that $X_{\text{st-red}}$ is actually a reduced locally ringed space over k. We say that X is strongly reduced if $\mathcal{R}_X \to \overline{\mathcal{R}}_X$ is an isomorphism, that is, if $\mathcal{R}_X$ can be identified with a subsheaf of the sheaf of functions $X \to k$ (in our applications to the theory of algebraic or analytic schemes, the concepts of reduction and strong reduction will actually be the same ; in general, these notions differ, see Exercise ??.??). It is important to observe that reduction (resp. strong reduction) is a fonctorial process:
if $F = (f, F^\star) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ is a morphism of ringed spaces (resp. of locally ringed spaces over k), there are natural reductions

$$F_{\mathrm{red}} = (f, F^\star_{\mathrm{red}}) : X_{\mathrm{red}} \to Y_{\mathrm{red}}, \quad F^\star_{\mathrm{red}} : \mathcal{R}_{Y,f(x)}/\mathcal{N}_{Y,f(x)} \to \mathcal{R}_{X,x}/\mathcal{N}_{X,x},$$
$$F_{\text{st-red}} = (f, f^\star) : X_{\text{st-red}} \to Y_{\text{st-red}}, \quad f^\star : \overline{\mathcal{R}}_{Y,f(x)} \to \overline{\mathcal{R}}_{X,x}, \quad \overline{s} \mapsto \overline{s} \circ f$$

where $f^\star$ is the usual pull-back comorphism associated with f. Therefore, if $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ are strongly reduced, the morphism F is completely determined by the underlying set-theoretic map f. Our first basic examples of (strongly reduced) ringed spaces are the various types of manifolds already defined in Chapter I. The language of ringed spaces provides an equivalent but more elegant and more intrinsic definition.

(1.23) Definition. *Let X be a Hausdorff separable topological space. One can define the category of C^k, $k \in \mathbb{N} \cup \{\infty, \omega\}$, differentiable manifolds (resp. complex analytic manifolds) to be the category of reduced locally ringed spaces $(X, \mathcal{R}_X)$ over $\mathbb{R}$ (resp. over $\mathbb{C}$), such that every point $x \in X$ has a neighborhood U on which the restriction $(U, \mathcal{R}_{X\restriction U})$ is isomorphic to a ringed space $(\Omega, \mathcal{C}^k_\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set and $\mathcal{C}^k_\Omega$ is the sheaf of C^k differentiable functions (resp. $(\Omega, \mathcal{O}_\Omega)$, where $\Omega \subset \mathbb{C}^n$ is an open subset, and $\mathcal{O}_\Omega$ is the sheaf of holomorphic functions on Ω).*

We say that the ringed spaces $(\Omega, \mathcal{C}^k_\Omega)$ and $(\Omega, \mathcal{O}_\Omega)$ are the *models* of the category of differentiable (resp. complex analytic) manifolds, and that a general object $(X, \mathcal{R}_X)$ in the category is *locally isomorphic* to one of the given model spaces. It is easy to see that the corresponding ringed spaces morphisms are nothing but the usual concepts of differentiable and holomorphic maps.

§1.D. Algebraic Varieties over a Field

As a second illustration of the notion of ringed space, we present here a brief introduction to the formalism of algebraic varieties, referring to (Hartshorne 1977) or (EGA 1967) for a much more detailed exposition. Our hope is that the reader who already has some background of analytic or algebraic geometry will find some hints of the strong interconnections between both theories. Beginners are invited to skip this section and proceed directly to the theory of complex analytic sheaves in §,2. All rings or algebras occurring in this section are supposed to be commutative rings with unit.

§1.D.1. Affine Algebraic Sets. Let k be an algebraically closed field of any characteristic. An *affine algebraic set* is a subset $X \subset k^N$ of the affine space k^N defined by an arbitrary collection $S \subset k[T_1, \ldots, T_N]$ of polynomials, that is,

$$X = V(S) = \big\{(z_1, \ldots, z_N) \in k^N \,;\, P(z_1, \ldots, z_N) = 0, \, \forall P \in S\big\}.$$

Of course, if $J \subset k[T_1, \ldots, T_N]$ is the ideal generated by S, then $V(S) = V(J)$. As $k[T_1, \ldots, T_N]$ is Noetherian, J is generated by finitely many elements $(P_1, \ldots, P_m)$, thus $X = V(\{P_1, \ldots, P_m\})$ is always defined by finitely many equations. Conversely, for any subset $Y \subset k^N$, we consider the ideal $I(Y)$ of $k[T_1, \ldots, T_N]$, defined by

$$I(Y) = \{P \in k[T_1, \ldots, T_N]\,;\, P(z) = 0,\ \forall z \in Y\}.$$

Of course, if $Y \subset k^N$ is an algebraic set, we have $V(I(Y)) = Y$. In the opposite direction, we have the following fundamental result.

(1.24) Hilbert's Nullstellensatz (see Lang 1965). *If $J \subset k[T_1, \ldots, T_N]$ is an ideal, then $I(V(J)) = \sqrt{J}$.*

If $X = V(J) \subset k^N$ is an affine algebraic set, we define the (reduced) ring $\mathcal{O}(X)$ of algebraic functions on X to be the set of all functions $X \to k$ which are restrictions of polynomials, i.e.,

$$(1.25)\quad \mathcal{O}(X) = k[T_1, \ldots, T_N]/I(X) = k[T_1, \ldots, T_N]/\sqrt{J}.$$

This is clearly a reduced k-algebra. An (algebraic) morphism of affine algebraic sets $X = V(J) \subset k^N$, $Y = V(J') \subset k^{N'}$ is a map $f : Y \to X$ which is the restriction of a polynomial map $k^{N'} tok^N$. We then get a k-algebra homomomorphism

$$f^\star : \mathcal{O}(X) \to \mathcal{O}(Y), \qquad s \mapsto s \circ f,$$

called the *comorphism* of f. In this way, we have defined a contravariant fonctor

$$(1.26)\quad X \mapsto \mathcal{O}(X), \qquad f \mapsto f^\star$$

from the category of affine algebraic sets to the category of finitely generated reduced k-algebras.

We are going to show the existence of a natural fonctor going in the opposite direction. In fact, let us start with an arbitrary finitely generated algebra A (not necessarily reduced at this moment). For any choice of generators $(g_1, \ldots, g_N)$ of A we get a surjective morphism of the polynomial ring $k[T_1, \ldots, T_N]$ onto A,

$$k[T_1, \ldots, T_N] \to A, \qquad T_j \mapsto g_j,$$

and thus $A \simeq k[T_1, \ldots, T_N]/J$ with the ideal J being the kernel of this morphism. It is well-known that every maximal ideal $\mathfrak{m}$ of A has codimension 1 in A (see Lang 1965), so that $\mathfrak{m}$ gives rise to a k-algebra homomorphism $A \to A/\mathfrak{m} = k$. We thus get a bijection

$$\mathrm{Hom}_{\mathrm{alg}}(A, k) \to \mathrm{Spm}(A), \qquad u \mapsto \mathrm{Ker}\, u$$

between the set of k-algebra homomorphisms and the set $\mathrm{Spm}(A)$ of maximal ideals of A. In fact, if $A = k[T_1, \ldots, T_N]/J$, an element $\varphi \in \mathrm{Hom}_{\mathrm{alg}}(A, k)$ is completely determined by the values $z_j = \varphi(T_j \bmod J)$, and the corresponding algebra homomorphism $k[T_1, \ldots, T_N] \to k$, $P \mapsto P(z_1, \ldots, z_N)$ can be factorized mod J if and only if $z = (z_1, \ldots, z_N) \in k^N$ satisfies the equations

$$P(z_1, \ldots, z_N) = 0, \qquad \forall P \in J.$$

We infer from this that

$$\mathrm{Spm}(A) \simeq V(J) = \big\{(z_1, \ldots, z_N) \in k^N \,;\, P(z_1, \ldots, z_N) = 0, \,\forall P \in J\big\}$$

can be identified with the *affine algebraic set* $V(J) \subset k^N$. If we are given an algebra homomorphism $\Phi : A \to B$ of finitely generated k-algebras we get a corresponding map $\mathrm{Spm}(\Phi) : \mathrm{Spm}(B) \to \mathrm{Spm}(A)$ described either as

$$\begin{aligned}&\mathrm{Spm}(B) \to \mathrm{Spm}(A), \quad \mathfrak{m} \mapsto \Phi^{-1}(\mathfrak{m}) \quad \text{or}\\ &\mathrm{Hom}_{\mathrm{alg}}(B, k) \to \mathrm{Hom}_{\mathrm{alg}}(A, k), \quad v \mapsto v \circ \Phi.\end{aligned}$$

If $B = k[T'_1, \ldots, T'_{N'}]/J'$ and $\mathrm{Spm}(B) = V(J') \subset k^{N'}$, it is easy to see that $\mathrm{Spm}(\Phi) : \mathrm{Spm}(B) \to \mathrm{Spm}(A)$ is the restriction of the polynomial map

$$f : k^{N'} \to k^N, \qquad w \mapsto f(w) = (P_1(w), \ldots, P_N(w)),$$

where $P_j \in k[T'_1, \ldots, T'_{N'}]$ are polynomials such that $P_j = \Phi(T_j) \bmod J'$ in B. We have in this way defined a contravariant fonctor

$$(1.27) \quad A \mapsto \mathrm{Spm}(A), \qquad \Phi \mapsto \mathrm{Spm}(\Phi)$$

from the category of finitely generated k-algebras to the category of affine algebraic sets.

Since $A = k[T_1, \ldots, T_N]/J$ and its reduction $A/N(A) = k[T_1, \ldots, T_N]/\sqrt{J}$ give rise to the same algebraic set

$$V(J) = \mathrm{Spm}(A) = \mathrm{Spm}(A/N(A)) = V(\sqrt{J}),$$

we see that the category of affine algebraic sets is actually equivalent to the subcategory of *reduced* finitely generated k-algebras.

(1.28) Example. The simplest example of an affine algebraic set is the affine space

$$k^N = \mathrm{Spm}(k[T_1, \ldots, T_N]),$$

in particular $\mathrm{Spm}(k) = k^0$ is just one point. We agree that $\mathrm{Spm}(\{0\}) = \emptyset$ (observe that $V(J) = \emptyset$ when J is the unit ideal in $k[T_1, \ldots, T_N]$).

§1.D.2. Zariski Topology and Affine Algebraic Schemes. Let A be a finitely generated algebra and $X = \mathrm{Spm}(A)$. To each ideal $\mathfrak{a} \subset A$ we associate the zero variety $V(\mathfrak{a}) \subset X$ which consists of all elements $\mathfrak{m} \in X = \mathrm{Spm}(A)$ such that $\mathfrak{m} \supset \mathfrak{a}$; if

$$A \simeq k[T_1, \ldots, T_N]/J \quad \text{and} \quad X \simeq V(J) \subset k^N,$$

then $V(\mathfrak{a})$ can be identified with the zero variety $V(J_{\mathfrak{a}}) \subset X$ of the inverse image $J_{\mathfrak{a}}$ of $\mathfrak{a}$ in $k[T_1, \ldots, T_N]$. For any family $(\mathfrak{a}_\alpha)$ of ideals in A we have

$$V(\sum \mathfrak{a}_\alpha) = \bigcap V(\mathfrak{a}_\alpha), \qquad V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) = V(\mathfrak{a}_1\mathfrak{a}_2),$$

hence there exists a unique topology on X such that the closed sets consist precisely of all algebraic subsets $(V(\mathfrak{a}))_{\mathfrak{a}\subset A}$ of X. This topology is called the Zariski topology. The Zariski topology is almost never Hausdorff (for example, if $X = k$ is the affine line, the open sets are $\emptyset$ and complements of finite sets, thus any two nonempty open sets have nonempty intersection). However, X is a *Noetherian space*, that is, a topological space in which every decreasing sequence of closed sets is stationary; an equivalent definition is to require that every open set is quasi-compact (from any open covering of an open set, one can extract a finite covering).

We now come to the concept of affine open subsets. For $s \in A$, the open set $D(s) = X \smallsetminus V(s)$ can be given the structure of an affine algebraic variety. In fact, if $A = k[T_1, \ldots, T_N]/J$ and s is represented by a polynomial in $k[T_1, \ldots, T_N]$, the localized ring $A[1/s]$ can be written as $A[1/s] = k[T_1, \ldots, T_N, T_{N+1}]/J_s$ where $J_s = J[T_{N+1}] + (sT_{N+1} - 1)$, thus

$$V(J_s) = \{(z, w) \in V(J) \times k\,;\; s(z)\,w = 1\} \simeq V(I) \smallsetminus s^{-1}(0)$$

and $D(s)$ can be identified with $\mathrm{Spm}(A[1/s])$. We have $D(s_1) \cap D(s_2) = D(s_1 s_2)$, and the sets $(D(s))_{s\in A}$ are easily seen to be a basis of the Zariski topology on X. The open sets $D(s)$ are called *affine open sets*. Since the open sets $D(s)$ containing a given point $x \in X$ form a basis of neighborhoods, one can define a sheaf space $\mathcal{O}_X$ such that the ring of germs $\mathcal{O}_{X,x}$ is the inductive limit

$$\mathcal{O}_{X,x} = \varinjlim_{D(s)\ni x} A[1/s] = \{\text{fractions } p/q\,;\; p, q \in A,\; q(x) \neq 0\}.$$

This is a local ring with maximal ideal

$$\mathfrak{m}_{X,x} = \{p/q\,;\; p, q \in A,\; p(x) = 0,\; q(x) \neq 0\},$$

and residual field $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k$. In this way, we get a ringed space $(X, \mathcal{O}_X)$ over k. It is easy to see that $\Gamma(X, \mathcal{O}_X)$ coincides with the finitely generated k-algebra A. In fact, from the definition of $\mathcal{O}_X$, a global section is obtained by gluing together local sections p_j/s_j on affine open sets $D(s_j)$ with $\bigcup D(s_j) = X$, $1 \le j \le m$. This means that the ideal $\mathfrak{a} = (s_1, \ldots, s_m) \subset A$ has an empty zero variety $V(\mathfrak{a})$, thus $\mathfrak{a} = A$ and there are elements $u_j \in A$ with $\sum u_j s_j = 1$. The compatibility condition $p_j/s_j = p_k/s_k$ implies that these elements are induced by

$$\sum u_j p_j \Big/ \sum u_j s_j = \sum u_j p_j \in A,$$

as desired. More generally, since the open sets $D(s)$ are affine, we get

$$\Gamma(D(s), \mathcal{O}_X) = A[1/s].$$

It is easy to see that the ringed space $(X, \mathcal{O}_X)$ is reduced if and only if A itself is reduced; in this case, X is even strongly reduced as Hilbert's Nullstellensatz shows. Otherwise, the reduction X_{red} can obtained from the reduced algebra $A_{\mathrm{red}} = A/N(A)$.

Ringed spaces $(X, \mathcal{O}_X)$ as above are called *affine algebraic schemes* over k (although substantially different from the usual definition, our definition can be shown to be equivalent in this special situation; compare with (Hartshorne 1977); see also Exercise ??.??). The category of affine algebraic schemes is equivalent to the category of finitely generated k-algebras (with the arrows reversed).

1.D.3. Algebraic Schemes. Algebraic schemes over k are defined to be ringed spaces over k which are locally isomorphic to affine algebraic schemes, modulo an ad hoc separation condition.

(1.29) Definition. *An algebraic scheme over k is a locally ringed space $(X, \mathcal{O}_X)$ over k such that*

a) *X has a finite covering by open sets U_α such that $(U_\alpha, \mathcal{O}_{X\restriction U_\alpha})$ is isomorphic as a ringed space to an affine algebraic scheme* $(\mathrm{Spm}(A_\alpha), \mathcal{O}_{\mathrm{Spm}(A_\alpha)})$.

b) *X satisfies the algebraic separation axiom, namely the diagonal Δ_X of $X \times X$ is closed for the Zariski topology.*

A morphism of algebraic schemes is just a morphism of the underlying locally ringed spaces. An (*abstract*) *algebraic variety is the same as a reduced algebraic scheme.*

In the above definition, some words of explanation are needed for b), since the product $X \times Y$ of algebraic schemes over k *is not* the ringed space theoretic product, i.e., the product topological space equipped with the structure sheaf $\mathrm{pr}_1^\star \mathcal{O}_X \otimes_k \mathrm{pr}_2^\star \mathcal{O}_Y$. Instead, we define the product of two affine algebraic schemes $X = \mathrm{Spm}(A)$ and $Y = \mathrm{Spm}(B)$ to be $X \times Y = \mathrm{Spm}(A \otimes_k B)$, equipped with the Zariski topology and the structural sheaf associated with $A \otimes_k B$. Notice that the Zariski topology on $X \times Y$ *is not the product topology* of the Zariski topologies on X, Y, as the example $k^2 = k \times k$ shows; also, the rational function $1/(1 - z_1 - z_2) \in \mathcal{O}_{k^2,(0,0)}$ is not in $\mathcal{O}_{k,0} \otimes_k \mathcal{O}_{k,0}$. In general, if X, Y are written as $X = \bigcup U_\alpha$ and $Y = \bigcup V_\beta$ with affine open sets U_α, V_β, we define $X \times Y$ to be the union of all open affine charts $U_\alpha \times V_\beta$ with their associated structure sheaves of affine algebraic varieties, the open sets of $X \times Y$ being all unions of open sets in the various charts $U_\alpha \times V_\beta$. The separation axiom b) is introduced for the sake of excluding pathological examples such as an affine line $k \amalg \{0'\}$ with the origin changed into a double point.

1.D.4. Subschemes. If $(X, \mathcal{O}_X)$ is an affine algebraic scheme and $A = \Gamma(X, \mathcal{O}_X)$ is the associated algebra, we say that $(Y, \mathcal{O}_Y)$ is a *subscheme* of $(X, \mathcal{O}_X)$ if there is an ideal $\mathfrak{a}$ of A such that $Y \hookrightarrow X$ is the morphism defined by the algebra morphism $A \to A/\mathfrak{a}$ as its comorphism. As $\mathrm{Spm}(A/\mathfrak{a}) \to \mathrm{Spm}(A)$ has for image the set $V(\mathfrak{a})$ of maximal ideals $\mathfrak{m}$ of A containing $\mathfrak{a}$, we see that $Y = V(\mathfrak{a})$ as a set; let us introduce the ideal subsheaf $\mathcal{J} = \mathfrak{a}\mathcal{O}_X \subset \mathcal{O}_X$. Since the structural sheaf $\mathcal{O}_Y$ is obtained by taken localizations $A/\mathfrak{a}[1/s]$, it is easy to see that $\mathcal{O}_Y$ coincides with the quotient sheaf $\mathcal{O}_X/\mathcal{J}$ restricted to Y. Since $\mathfrak{a}$ has finitely many generators, the ideal sheaf $\mathcal{J}$ is locally finitely generated (see § 2 below). This leads to the following definition.

(1.30) Definition. *If $(X, \mathcal{O}_X)$ is an algebraic scheme, a (closed) subscheme is an algebraic scheme $(Y, \mathcal{O}_Y)$ such that Y is a Zariski closed subset of X, and there is a locally finitely generated ideal subsheaf $\mathcal{J} \subset \mathcal{O}_X$ such that $Y = V(\mathcal{J})$ and $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{J})_{\restriction Y}$.*

If $(Y, \mathcal{O}_Y)$, $(Z, \mathcal{O}_Z)$ are subschemes of $(X, \mathcal{O}_X)$ defined by ideal subsheaves $\mathcal{J}$, $\mathcal{J}' \subset \mathcal{O}_X$, there are corresponding subschemes $Y \cap Z$ and $Y \cup Z$ defined as ringed spaces

$$(Y \cap Z, \mathcal{O}_X/(\mathcal{J} + \mathcal{J}')), \qquad (Y \cup Z, \mathcal{O}_X/\mathcal{J}\mathcal{J}').$$

§1.D.5. Projective Algebraic Varieties. A very important subcategory of the category of algebraic varieties is provided by *projective algebraic varieties*. Let $\mathbb{P}^N_k$ be the projective N-space, that is, the set $k^{N+1} \smallsetminus \{0\}/k^\star$ of equivalence classes of $(N+1)$-tuples $(z_0, \ldots, z_N) \in k^{N+1} \smallsetminus \{0\}$ under the equivalence relation given by $(z_0, \ldots, z_N) \sim \lambda(z_0, \ldots, z_N)$, $\lambda \in k^\star$. The corresponding element of $\mathbb{P}^N_k$ will be denoted $[z_0 : z_1 : \ldots : z_N]$. It is clear that $\mathbb{P}^k_N$ can be covered by the $(N+1)$ affine charts U_α, $0 \le \alpha \le N$, such that

$$U_\alpha = \big\{[z_0 : z_1 : \ldots : z_N] \in \mathbb{P}^N_k \ \ z_\alpha \neq 0\big\}.$$

The set U_α can be identified with the affine N-space k^N by the map

$$U_\alpha \to k^N, \qquad [z_0 : z_1 : \ldots : z_N] \mapsto \Big(\frac{z_0}{z_\alpha}, \frac{z_1}{z_\alpha}, \ldots, \frac{z_{\alpha-1}}{z_\alpha}, \frac{z_{\alpha+1}}{z_\alpha}, \ldots, \frac{z_N}{z_\alpha}\Big).$$

With this identification, $\mathcal{O}(U_\alpha)$ is the algebra of homogeneous rational functions of degree 0 in $z_0, \ldots, z_N$ which have just a power of z_α in their denominator. It is easy to see that the structure sheaves $\mathcal{O}_{U_\alpha}$ and $\mathcal{O}_{U_\beta}$ coincide in the intersections $U_\alpha \cap U_\beta$; they can be glued together to define an algebraic variety structure $(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N})$, such that $\mathcal{O}_{\mathbb{P}^N,[z]}$ consists of all homogeneous rational functions p/q of degree 0 (i.e., $\deg p = \deg q$), such that $q(z) \neq 0$.

(1.30) Definition. *An algebraic scheme or variety $(X, \mathcal{O}_X)$ is said to be projective if it is isomorphic to a closed subscheme of some projective space $(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N})$.*

We now indicate a standard way of constructing projective schemes. Let S be a collection of homogeneous polynomials $P_j \in k[z_0, \ldots, z_N]$, of degree $d_j \in \mathbb{N}$. We define an associated *projective algebraic set*

$$\widetilde{V}(S) = \big\{[z_0 : \ldots : z_N] \in \mathbb{P}^N_k\,;\, P(z) = 0,\ \forall P \in S\big\}.$$

Let J be the *homogeneous ideal* of $k[z_0, \ldots, z_N]$ generated by S (recall that an ideal J is said to be homogeneous if $J = \bigoplus J_m$ is the direct sum of its homogeneous components, or equivalently, if J is generated by homogeneous elements). We have an associated graded algebra

$$B = k[z_0, \ldots, z_N]/J = \bigoplus B_m, \qquad B_m = k[z_0, \ldots, z_N]_m/J_m$$

such that B is generated by B_1 and B_m is a finite dimensional vector space over k for each k. This is enough to construct the desired scheme structure on $\widetilde{V}(J) := \bigcap \widetilde{V}(J_m)$, as we see in the next subsection.

1.D.6. Projective Scheme Associated with a Graded Algebra. Let us start with a reduced graded k-algebra

$$B = \bigoplus_{m\in\mathbb{N}} B_m$$

such that B is generated by B_0 and B_1 as an algebra, and B_0, B_1 are finite dimensional vector spaces over k (it then follows that B is finitely generated and that all B_m are finite dimensional vector spaces). Given $s \in B_m$, $m > 0$, we define a k-algebra A_s to be the ring of all fractions of homogeneous degree 0 with a power of s as their denominator, i.e.,

$$(1.31)\quad A_s = \big\{p/s^d\,;\, p \in B_{dm},\, d \in \mathbb{N}\big\}.$$

Since A_s is generated by $\frac{1}{s}B_1^m$ over B_0, A_s is a finitely generated algebra. We define $U_s = \mathrm{Spm}(A_s)$ to be the associated affine algebraic variety. For $s \in B_m$ and $s' \in B_{m'}$, we clearly have algebra homomorphisms

$$A_s \to A_{ss'}, \qquad A_{s'} \to A_{ss'},$$

since $A_{ss'}$ is the algebra of all 0-homogeneous fractions with powers of s and s' in the denominator. As $A_{ss'}$ is the same as the localized ring $A_s[s^{m'}/s'^m]$, we see that $U_{ss'}$ can be identified with an affine open set in U_s, and we thus get canonical injections

$$U_{ss'} \hookrightarrow U_s, \qquad U_{ss'} \hookrightarrow U_{s'}.$$

(1.32) Definition. *If $B = \bigoplus_{m\in\mathbb{N}} B_m$ is a reduced graded algebra generated by its finite dimensional vector subspaces B_0 and B_1, we associate an algebraic scheme $(X, \mathcal{O}_X) = \mathrm{Proj}(B)$ as follows. To each finitely generated algebra $A_s = \big\{p/s^d\,;\, p \in B_{dm},\, d \in \mathbb{N}\big\}$ we associate an affine algebraic variety*

$U_s = \mathrm{Spm}(A_s)$. *We let X be the union of all open charts U_s with the identifications $U_s \cap U_{s'} = U_{ss'}$; then the collection (U_s) is a basis of the topology of X, and $\mathcal{O}_X$ is the unique sheaf of local k-algebras such that $\Gamma(U_s, \mathcal{O}_X) = A_s$ for each U_s.*

The following proposition shows that only finitely many open charts are actually needed to describe X (as required in Def. 1.29 a)).

(1.33) Lemma. *If $s_0, \ldots, s_N$ is a basis of B_1, then* $\mathrm{Proj}(B) = \bigcup_{0 \le j \le N} U_{s_j}$.

Proof. In fact, if $x \in X$ is contained in a chart U_s for some $s \in B_m$, then $U_s = \mathrm{Spm}(A_s) \neq \emptyset$, and therefore $A_s \neq \{0\}$. As A_s is generated by $\frac{1}{s} B_1^m$, we can find a fraction $f = s_{j_1} \ldots s_{j_m}/s$ representing an element $f \in \mathcal{O}(U_s)$ such that $f(x) \neq 0$. Then $x \in U_s \smallsetminus f^{-1}(0)$, and $U_s \smallsetminus f^{-1}(0) = \mathrm{Spm}(A_s[1/f]) = U_s \cap U_{s_{j_1}} \cap \ldots \cap U_{s_{j_m}}$. In particular $x \in U_{s_{j_1}}$. □

(1.34) Example. One can consider the *projective space* $\mathbb{P}_k^N$ to be the algebraic scheme

$$\mathbb{P}_k^N = \mathrm{Proj}(k[T_0, \ldots, T_N]).$$

The Proj construction is fonctorial in the following sense: if we have a graded homomorphism $\Phi : B \to B'$ (i.e. an algebra homomorphism such that $\Phi(B_m) \subset B'_m$, then there are corresponding morphisms $A_s \to A'_{\Phi(s)}$, $U'_{\Phi(s)} \to U_s$, and we thus find a scheme morphism

$$F : \mathrm{Proj}(B') \to \mathrm{Proj}(B).$$

Also, since $p/s^d = ps^l/s^{d+l}$, the algebras A_s depend only on components B_m of large degree, and we have $A_s = A_{s^l}$. It follows easily that there is a canonical isomorphism

$$\mathrm{Proj}(B) \simeq \mathrm{Proj}\Big(\bigoplus_m B_{lm}\Big).$$

Similarly, we may if we wish change a finite number of components B_m without affecting $\mathrm{Proj}(B)$. In particular, we may alway assume that $B_0 = k\,1_B$. By selecting finitely many generators $g_0, \ldots, g_N$ in B_1, we then find a surjective graded homomorphism $k[T_0, \ldots, T_N] \to B$, thus $B \simeq k[T_0, \ldots, T_N]/J$ for some graded ideal $J \subset B$. The algebra homomorphism $k[T_0, \ldots, T_N] \to B$ therefore yields a scheme embedding $\mathrm{Proj}(B) \to \mathbb{P}^N$ onto $V(J)$.

We will not pursue further the study of algebraic varieties from this point of view; in fact we are mostly interested in the case $k = \mathbb{C}$, and algebraic varieties over $\mathbb{C}$ are a special case of the more general concept of complex analytic space.

§2. The Local Ring of Germs of Analytic Functions

§2.A. The Weierstrass Preparation Theorem

Our first goal is to establish a basic factorization and division theorem for analytic functions of several variables, which is essentially due to Weierstrass. We follow here a simple proof given by C.L. Siegel, based on a clever use of the Cauchy formula. Let g be a holomorphic function defined on a neighborhood of 0 in $\mathbb{C}^n$, $g \not\equiv 0$. There exists a dense set of vectors $v \in \mathbb{C}^n \smallsetminus \{0\}$ such that the function $\mathbb{C} \ni t \longmapsto g(tv)$ is not identically zero. In fact the Taylor series of g at the origin can be written

$$g(tv) = \sum_{k=0}^{+\infty} \frac{1}{k!} t^k g^{(k)}(v)$$

where $g^{(k)}$ is a homogeneous polynomial of degree k on $\mathbb{C}^n$ and $g^{(k_0)} \not\equiv 0$ for some index k_0. Thus it suffices to select v such that $g^{(k_0)}(v) \neq 0$. After a change of coordinates, we may assume that $v = (0, \ldots, 0, 1)$. Let s be the vanishing order of $z_n \longmapsto g(0, \ldots, 0, z_n)$ at $z_n = 0$. There exists $r_n > 0$ such that $g(0, \ldots, 0, z_n) \neq 0$ when $0 < |z_n| \leq r_n$. By continuity of g and compactness of the circle $|z_n| = r_n$, there exists $r' > 0$ and $\varepsilon > 0$ such that

$$g(z', z_n) \neq 0 \quad \text{for } z' \in \mathbb{C}^{n-1}, \quad |z'| \leq r', \quad r_n - \varepsilon \leq |z_n| \leq r_n + \varepsilon.$$

For every integer $k \in \mathbb{N}$, let us consider the integral

$$S_k(z') = \frac{1}{2\pi \mathrm{i}} \int_{|z_n| = r_n} \frac{1}{g(z', z_n)} \frac{\partial g}{\partial z_n}(z', z_n)\, z_n^k \, dz_n.$$

Then S_k is holomorphic in a neighborhood of $|z'| \leq r'$. Rouché's theorem shows that $S_0(z')$ is the number of roots z_n of $g(z', z_n) = 0$ in the disk $|z_n| < r_n$, thus by continuity $S_0(z')$ must be a constant s. Let us denote by $w_1(z'), \ldots, w_s(z')$ these roots, counted with multiplicity. By definition of r_n, we have $w_1(0) = \ldots = w_s(0) = 0$, and by the choice of r', ε we have $|w_j(z')| < r_n - \varepsilon$ for $|z'| \leq r'$. The Cauchy residue formula yields

$$S_k(z') = \sum_{j=1}^{s} w_j(z')^k.$$

Newton's formula shows that the elementary symmetric function $c_k(z')$ of degree k in $w_1(z'), \ldots, w_s(z')$ is a polynomial in $S_1(z'), \ldots, S_k(z')$. Hence $c_k(z')$ is holomorphic in a neighborhood of $|z'| \leq r'$. Let us set

$$P(z', z_n) = z_n^s - c_1(z') z_n^{s-1} + \cdots + (-1)^s c_s(z') = \prod_{j=1}^{s} \big(z_n - w_j(z')\big).$$

For $|z'| \leq r'$, the quotient $f = g/P$ (resp. $f = P/g$) is holomorphic in z_n on the disk $|z_n| < r_n + \varepsilon$, because g and P have the same zeros with the same multiplicities, and $f(z', z_n)$ is holomorphic in z' for $r_n - \varepsilon \leq |z_n| \leq r_n + \varepsilon$. Therefore

$$f(z', z_n) = \frac{1}{2\pi \mathrm{i}} \int_{|w_n| = r_n + \varepsilon} \frac{f(z', w_n)\, dw_n}{w_n - z_n}$$

is holomorphic in z on a neighborhood of the closed polydisk $\overline{\Delta}(r', r_n) = \{|z'| \leq r'\} \times \{|z_n| \leq r_n\}$. Thus g/P is invertible and we obtain:

(2.1) Weierstrass preparation theorem. *Let g be holomorphic on a neighborhood of 0 in $\mathbb{C}^n$, such that $g(0, z_n)/z_n^s$ has a not zero finite limit at $z_n = 0$. With the above choice of r' and r_n, one can write $g(z) = u(z)P(z', z_n)$ where u is an invertible holomorphic function in a neighborhood of the polydisk $\overline{\Delta}(r', r_n)$, and P is a Weierstrass polynomial in z_n, that is, a polynomial of the form*

$$P(z', z_n) = z_n^s + a_1(z') z_n^{s-1} + \cdots + a_s(z'), \qquad a_k(0) = 0,$$

with holomorphic coefficients $a_k(z')$ on a neighborhood of $|z'| \leq r'$ in $\mathbb{C}^{n-1}$.

(2.2) Remark. If g vanishes at order m at 0 and $v \in \mathbb{C}^n \smallsetminus \{0\}$ is selected such that $g^{(m)}(v) \neq 0$, then $s = m$ and P must also vanish at order m at 0. In that case, the coefficients $a_k(z')$ are such that $a_k(z') = O(|z'|^k)$, $1 \leq k \leq s$.

(2.3) Weierstrass division theorem. *Every bounded holomorphic function f on $\Delta = \Delta(r', r_n)$ can be represented in the form*

$$(2.4) \qquad f(z) = g(z)q(z) + R(z', z_n),$$

where q and R are analytic in Δ, $R(z', z_n)$ is a polynomial of degree $\leq s - 1$ in z_n, and

$$(2.5) \qquad \sup_\Delta |q| \leq C \sup_\Delta |f|, \qquad \sup_\Delta |R| \leq C \sup_\Delta |f|$$

for some constant $C \geq 0$ independent of f. The representation (2.4) *is unique.*

Proof (Siegel) It is sufficient to prove the result when $g(z) = P(z', z_n)$ is a Weierstrass polynomial.

Let us first prove the uniqueness. If $f = Pq_1 + R_1 = Pq_2 + R_2$, then

$$P(q_2 - q_1) + (R_2 - R_1) = 0.$$

It follows that the s roots z_n of $P(z', \bullet) = 0$ are zeros of $R_2 - R_1$. Since $\deg_{z_n}(R_2 - R_1) \leq s - 1$, we must have $R_2 - R_1 \equiv 0$, thus $q_2 - q_1 \equiv 0$.

In order to prove the existence of (q, R), we set

$$q(z', z_n) = \lim_{\varepsilon\to 0+} \frac{1}{2\pi \mathrm{i}} \int_{|w_n|=r_n-\varepsilon} \frac{f(z', w_n)}{P(z', w_n)(w_n - z_n)}\, dw_n, \qquad z \in \Delta\ ;$$

observe that the integral does not depend on ε when $\varepsilon < r_n - |z_n|$ is small enough. Then q is holomorphic on Δ. The function $R = f - Pq$ is also holomorphic on Δ and

$$R(z) = \lim_{\varepsilon\to 0+} \frac{1}{2\pi \mathrm{i}} \int_{|w_n|=r_n-\varepsilon} \frac{f(z', w_n)}{P(z', w_n)} \Big[\frac{P(z', w_n) - P(z', z_n)}{(w_n - z_n)}\Big]\, dw_n.$$

The expression in brackets has the form

$$\big[(w_n^s - z_n^s) + \sum_{j=1}^{s} a_j(z')(w_n^{s-j} - z_n^{s-j})\big]/(w_n - z_n)$$

hence is a polynomial in z_n of degree $\le s-1$ with coefficients that are holomorphic functions of z'. Thus we have the asserted decomposition $f = Pq + R$ and

$$\sup_{\Delta} |R| \le C_1 \sup_{\Delta} |f|$$

where C_1 depends on bounds for the $a_j(z')$ and on $\mu = \min |P(z', z_n)|$ on the compact set $\{|z'| \le r'\} \times \{|z_n| = r_n\}$. By the maximum principle applied to $q = (f - R)/P$ on each disk $\{z'\} \times \{|z_n| < r_n - \varepsilon\}$, we easily get

$$\sup_{\Delta} |q| \le \mu^{-1}(1 + C_1) \sup_{\Delta} |f|. \qquad \square$$

§2.B. Algebraic Properties of the Ring $\mathcal{O}_n$

We give here important applications of the Weierstrass preparation theorem to the study of the ring of germs of holomorphic functions in $\mathbb{C}^n$.

(2.6) Notation. *We let $\mathcal{O}_n$ be the ring of germs of holomorphic functions on $\mathbb{C}^n$ at 0. Alternatively, $\mathcal{O}_n$ can be identified with the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ of convergent power series in $z_1, \ldots, z_n$.*

(2.7) Theorem. *The ring $\mathcal{O}_n$ is Noetherian, i.e. every ideal $\mathcal{I}$ of $\mathcal{O}_n$ is finitely generated.*

Proof. By induction on n. For $n = 1$, $\mathcal{O}_n$ is principal: every ideal $\mathcal{I} \ne \{0\}$ is generated by z^s, where s is the minimum of the vanishing orders at 0 of the non zero elements of $\mathcal{I}$. Let $n \ge 2$ and $\mathcal{I} \subset \mathcal{O}_n$, $\mathcal{I} \ne \{0\}$. After a change of variables, we may assume that $\mathcal{I}$ contains a Weierstrass polynomial $P(z', z_n)$. For every $f \in \mathcal{I}$, the Weierstrass division theorem yields

$$f(z) = P(z', z_n)q(z) + R(z', z_n), \qquad R(z', z_n) = \sum_{k=0}^{s-1} c_k(z')\, z_n^k,$$

and we have $R \in \mathcal{I}$. Let us consider the set $\mathcal{M}$ of coefficients $(c_0, \ldots, c_{s-1})$ in $\mathcal{O}_{n-1}^{\oplus s}$ corresponding to the polynomials $R(z', z_n)$ which belong to $\mathcal{I}$. Then $\mathcal{M}$ is a $\mathcal{O}_{n-1}$-submodule of $\mathcal{O}_{n-1}^{\oplus s}$. By the induction hypothesis $\mathcal{O}_{n-1}$ is Noetherian; furthermore, every submodule of a finitely generated module over a Noetherian ring is finitely generated (Lang 1965, Chapter VI). Therefore $\mathcal{M}$ is finitely generated, and $\mathcal{I}$ is generated by P and by polynomials $R_1, \ldots, R_N$ associated with a finite set of generators of $\mathcal{M}$. $\square$

Before going further, we need two lemmas which relate the algebraic properties of $\mathcal{O}_n$ to those of the polynomial ring $\mathcal{O}_{n-1}[z_n]$.

(2.8) Lemma. *Let $P, F \in \mathcal{O}_{n-1}[z_n]$ where P is a Weierstrass polynomial. If P divides F in $\mathcal{O}_n$, then P divides F in $\mathcal{O}_{n-1}[z_n]$.*

Proof. Assume that $F(z', z_n) = P(z', z_n)h(z)$, $h \in \mathcal{O}_n$. The standard division algorithm of F by P in $\mathcal{O}_{n-1}[z_n]$ yields

$$F = PQ + R, \qquad Q, R \in \mathcal{O}_{n-1}[z_n], \quad \deg R < \deg P.$$

The uniqueness part of Th. 2.3 implies $h(z) = Q(z', z_n)$ and $R \equiv 0$. $\square$

(2.9) Lemma. *Let $P(z', z_n)$ be a Weierstrass polynomial.*

a) *If $P = P_1 \ldots P_N$ with $P_j \in \mathcal{O}_{n-1}[z_n]$, then, up to invertible elements of $\mathcal{O}_{n-1}$, all P_j are Weierstrass polynomials.*

b) *$P(z', z_n)$ is irreducible in $\mathcal{O}_n$ if and only if it is irreducible in $\mathcal{O}_{n-1}[z_n]$.*

Proof. a) Assume that $P = P_1 \ldots P_N$ with polynomials $P_j \in \mathcal{O}_{n-1}[z_n]$ of respective degrees s_j, $\sum_{1 \le j \le N} s_j = s$. The product of the leading coefficients of $P_1, \ldots, P_N$ in $\mathcal{O}_{n-1}$ is equal to 1; after normalizing these polynomials, we may assume that $P_1, \ldots, P_N$ are unitary and $s_j > 0$ for all j. Then

$$P(0, z_n) = z_n^s = P_1(0, z_n) \ldots P_N(0, z_n),$$

hence $P_j(0, z_n) = z_n^{s_j}$ and therefore P_j is a Weierstrass polynomial.

b) Set $s = \deg P$ and $P(0, z_n) = z_n^s$. Assume that P is reducible in $\mathcal{O}_n$, with $P(z', z_n) = g_1(z)g_2(z)$ for non invertible elements $g_1, g_2 \in \mathcal{O}_n$. Then $g_1(0, z_n)$ and $g_2(0, z_n)$ have vanishing orders $s_1, s_2 > 0$ with $s_1 + s_2 = s$, and

$$g_j = u_j P_j, \qquad \deg P_j = s_j, \quad j = 1, 2,$$

where P_j is a Weierstrass polynomial and $u_j \in \mathcal{O}_n$ is invertible. Therefore $P_1 P_2 = uP$ for an invertible germ $u \in \mathcal{O}_n$. Lemma 2.8 shows that P divides $P_1 P_2$ in $\mathcal{O}_{n-1}[z_n]$; since P_1, P_2 are unitary and $s = s_1 + s_2$, we get $P = P_1 P_2$, hence P is reducible in $\mathcal{O}_{n-1}[z_n]$. The converse implication is obvious from a). $\square$

(2.10) Theorem. *$\mathcal{O}_n$ is a factorial ring, i.e. $\mathcal{O}_n$ is entire and:*

a) *every non zero germ $f \in \mathcal{O}_n$ admits a factorization $f = f_1 \ldots f_N$ in irreducible elements;*

b) *the factorization is unique up to invertible elements.*

Proof. The existence part a) follows from Lemma 2.9 if we take f to be a Weierstrass polynomial and $f = f_1 \ldots f_N$ be a decomposition of maximal length N into polynomials of positive degree. In order to prove the uniqueness, it is sufficient to verify the following statement:

b′) *If g is an irreducible element that divides a product $f_1 f_2$, then g divides either f_1 or f_2.*

By Th. 2.1, we may assume that f_1, f_2, g are Weierstrass polynomials in z_n. Then g is irreducible and divides $f_1 f_2$ in $\mathcal{O}_{n-1}[z_n]$ thanks to Lemmas 2.8 and 2.9 b). By induction on n, we may assume that $\mathcal{O}_{n-1}$ is factorial. The standard Gauss lemma (Lang 1965, Chapter V) says that the polynomial ring $A[T]$ is factorial if the ring A is factorial. Hence $\mathcal{O}_{n-1}[z_n]$ is factorial by induction and thus g must divide f_1 or f_2 in $\mathcal{O}_{n-1}[z_n]$. $\square$

(2.11) Theorem. *If $f, g \in \mathcal{O}_n$ are relatively prime, then the germs f_z, g_z at every point $z \in \mathbb{C}^n$ near 0 are again relatively prime.*

Proof. One may assume that $f = P$, $g = Q$ are Weierstrass polynomials. Let us recall that unitary polynomials $P, Q \in \mathcal{A}[X]$ ($\mathcal{A}$ = a factorial ring) are relatively prime if and only if their resultant $R \in \mathcal{A}$ is non zero. Then the resultant $R(z') \in \mathcal{O}_{n-1}$ of $P(z', z_n)$ and $Q(z', z_n)$ has a non zero germ at 0. Therefore the germ $R_{z'}$ at points $z' \in \mathbb{C}^{n-1}$ near 0 is also non zero. $\square$

§3. Coherent Sheaves

§3.1. Locally Free Sheaves and Vector Bundles

Section 9 will greatly develope this philosophy. Before introducing the more general notion of a coherent sheaf, we discuss the notion of locally free sheaves over a sheaf a ring. All rings occurring in the sequel are supposed to be commutative with unit (the non commutative case is also of considerable interest, e.g. in view of the theory of $\mathcal{D}$-modules, but this subject is beyond the scope of the present book).

(3.1) Definition. *Let $\mathcal{A}$ be a sheaf of rings on a topological space X and let $\mathcal{S}$ a sheaf of modules over $\mathcal{A}$ (or briefly a $\mathcal{A}$-module). Then $\mathcal{S}$ is said to be locally* *free of rank r over $\mathcal{A}$, if $\mathcal{S}$ is locally isomorphic to $\mathcal{A}^{\oplus r}$ on a neighborhood of every point, i.e. for every $x_0 \in X$ one can find a neighborhood Ω and sections $F_1, \ldots, F_r \in \mathcal{S}(\Omega)$ such that the sheaf homomorphism*

$$F : \mathcal{A}^{\oplus r}_{\restriction \Omega} \longrightarrow \mathcal{S}_{\restriction \Omega}, \qquad \mathcal{A}^{\oplus r}_x \ni (w_1, \ldots, w_r) \longmapsto \sum_{1 \le j \le r} w_j F_{j,x} \in \mathcal{S}_x$$

is an isomorphism.

By definition, if $\mathcal{S}$ is locally free, there is a covering $(U_\alpha)_{\alpha \in I}$ by open sets on which $\mathcal{S}$ admits free generators $F^1_\alpha, \ldots, F^r_\alpha \in \mathcal{S}(U_\alpha)$. Because the generators can be uniquely expressed in terms of any other system of independent generators, there is for each pair (α, β) a $r \times r$ matrix

$$G_{\alpha\beta} = (G^{jk}_{\alpha\beta})_{1 \le j,k \le r}, \qquad G^{jk}_{\alpha\beta} \in \mathcal{A}(U_\alpha \cap U_\beta),$$

such that

$$F^k_\beta = \sum_{1 \le j \le r} F^j_\alpha G^{jk}_{\alpha\beta} \qquad \text{on} \quad U_\alpha \cap U_\beta.$$

In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A}^{\oplus r}_{\restriction U_\alpha \cap U_\beta} & \xrightarrow{F_\alpha} & \mathcal{S}_{\restriction U_\alpha \cap U_\beta} \\ G_{\alpha\beta} \Big\uparrow & & \Big\| \\ \mathcal{A}^{\oplus r}_{\restriction U_\alpha \cap U_\beta} & \xrightarrow[F_\beta]{} & \mathcal{S}_{\restriction U_\alpha \cap U_\beta} \end{array}$$

It follows easily from the equality $G_{\alpha\beta} = F^{-1}_\alpha \circ F_\beta$ that the *transition matrices* $G_{\alpha\beta}$ are invertible matrices satisfying the transition relation

$$(3.2) \quad G_{\alpha\gamma} = G_{\alpha\beta} G_{\beta\gamma} \qquad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma$$

for all indices $\alpha, \beta, \gamma \in I$. In particular $G_{\alpha\alpha} = \mathrm{Id}$ on U_α and $G^{-1}_{\alpha\beta} = G_{\beta\alpha}$ on $U_\alpha \cap U_\beta$.

Conversely, if we are given a system of invertible $r \times r$ matrices $G_{\alpha\beta}$ with coefficients in $\mathcal{A}(U_\alpha \cap U_\beta)$ satisfying the transition relation (3.2), we can define a locally free sheaf $\mathcal{S}$ of rank r over $\mathcal{A}$ by taking $\mathcal{S} \simeq \mathcal{A}^{\oplus r}$ over each U_α, the identification over $U_\alpha \cap U_\beta$ being given by the isomorphism $G_{\alpha\beta}$. A section H of $\mathcal{S}$ over an open set $\Omega \subset X$ can just be seen as a collection of sections $H_\alpha = (H^1_\alpha, \ldots, H^r_\alpha)$ of $\mathcal{A}^{\oplus r}(\Omega \cap U_\alpha)$ satisfying the transition relations $H_\alpha = G_{\alpha\beta} H_\beta$ over $\Omega \cap U_\alpha \cap U_\beta$.

The notion of locally free sheaf is closely related to another essential notion of differential geometry, namely the notion of vector bundle (resp. topological, differentiable, holomorphic ..., vector bundle). To describe the relation between these notions, we assume that the sheaf of rings $\mathcal{A}$ is a subsheaf of the sheaf $\mathcal{C}_{\mathbb{K}}$ of continous functions on X with values in the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, containing the sheaf of locally constant functions $X \to \mathbb{K}$. Then, for each $x \in X$, there is an evaluation map

$$\mathcal{A}_x \to \mathbb{K}, \qquad w \mapsto w(x)$$

whose kernel is a maximal ideal $\mathfrak{m}_x$ of $\mathcal{A}_x$, and $\mathcal{A}_x/\mathfrak{m}_x = \mathbb{K}$. Let $\mathcal{S}$ be a locally free sheaf of rank r over $\mathcal{A}$. To each $x \in X$, we can associate a $\mathbb{K}$-vector space $E_x = \mathcal{S}_x/\mathfrak{m}_x\mathcal{S}_x$: since $\mathcal{S}_x \simeq \mathcal{A}_x^{\oplus r}$, we have $E_x \simeq (\mathcal{A}_x/\mathfrak{m}_x)^{\oplus r} = \mathbb{K}^r$. The set $E = \coprod_{x \in X} E_x$ is equipped with a natural projection

$$\pi : E \to X, \qquad \xi \in E_x \mapsto \pi(\xi) := x,$$

and the fibers $E_x = \pi^{-1}(x)$ have a structure of r-dimensional $\mathbb{K}$-vector space: such a structure E is called a $\mathbb{K}$-*vector bundle of rank* r over X. Every section $s \in \mathcal{S}(U)$ gives rise to a *section* of E over U by setting $s(x) = s_x \bmod \mathfrak{m}_x$. We obtain a function (still denoted by the same symbol) $s : U \to E$ such that $s(x) \in E_x$ for every $x \in U$, i.e. $\pi \circ s = \mathrm{Id}_U$. It is clear that $\mathcal{S}(U)$ can be considered as a $\mathcal{A}(U)$-submodule of the $\mathbb{K}$-vector space of functions $U \to E$ mapping a point $x \in U$ to an element in the fiber E_x. Thus we get a subsheaf of the sheaf of E-valued sections, which is in a natural way a $\mathcal{A}$-module isomorphic to $\mathcal{S}$. This subsheaf will be denoted by $\mathcal{A}(E)$ and will be called the *sheaf of* $\mathcal{A}$-*sections* of E. If we are given a $\mathbb{K}$-vector bundle E over X and a subsheaf $\mathcal{S} = \mathcal{A}(E)$ of the sheaf of all sections of E which is in a natural way a locally free $\mathcal{A}$-module of rank r, we say that E (or more precisely the pair $(E, \mathcal{A}(E))$) is a $\mathcal{A}$-vector bundle of rank r over X.

(3.3) Example. In case $\mathcal{A} = \mathcal{C}_{X,\mathbb{K}}$ is the sheaf of all $\mathbb{K}$-valued continuous functions on X, we say that E is a *topological* vector bundle over X. When X is a manifold and $\mathcal{A} = \mathcal{C}^p_{X,\mathbb{K}}$, we say that E is a C^p-*differentiable* vector bundle; finally, when X is complex analytic and $\mathcal{A} = \mathcal{O}_X$, we say that E is a *holomorphic* vector bundle.

Let us introduce still a little more notation. Since $\mathcal{A}(E)$ is a locally free sheaf of rank r over any open set U_α in a suitable covering of X, a choice of generators $(F^1_\alpha, \ldots, F^r_\alpha)$ for $\mathcal{A}(E)_{\restriction U_\alpha}$ yields corresponding generators $(e^1_\alpha(x), \ldots, e^r_\alpha(x))$ of the fibers E_x over $\mathbb{K}$. Such a system of generators is called a $\mathcal{A}$-*admissible frame* of E over U_α. There is a corresponding isomorphism

$$(3.4) \quad \theta_\alpha : E_{\restriction U_\alpha} := \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{K}^r$$

which to each $\xi \in E_x$ associates the pair $(x, (\xi^1_\alpha, \ldots, \xi^r_\alpha)) \in U_\alpha \times \mathbb{K}^r$ composed of x and of the components $(\xi^j_\alpha)_{1 \le j \le r}$ of ξ in the basis $(e^1_\alpha(x), \ldots, e^r_\alpha(x))$ of E_x. The bundle E is said to be *trivial* if it is of the form $X \times \mathbb{K}^r$, which is the same as saying that $\mathcal{A}(E) = \mathcal{A}^{\oplus r}$. For this reason, the isomorphisms

θ_α are called *trivializations* of E over U_α. The corresponding *transition automorphisms* are

$$\theta_{\alpha\beta} := \theta_\alpha \circ \theta_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{K}^r \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{K}^r,$$
$$\theta_{\alpha\beta}(x,\xi) = (x, g_{\alpha\beta}(x)\cdot\xi), \qquad (x,\xi) \in (U_\alpha \cap U_\beta) \times \mathbb{K}^r,$$

where $(g_{\alpha\beta}) \in \mathrm{GL}_r(\mathcal{A})(U_\alpha \cap U_\beta)$ are the transition matrices already described (except that they are just seen as matrices with coefficients in $\mathbb{K}$ rather than with coefficients in a sheaf). Conversely, if we are given a collection of matrices $g_{\alpha\beta} = (g_{\alpha\beta}^{jk}) \in \mathrm{GL}_r(\mathcal{A})(U_\alpha \cap U_\beta)$ satisfying the transition relation

$$g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma} \qquad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma,$$

we can define a $\mathcal{A}$-vector bundle

$$E = \Big(\coprod_{\alpha\in I} U_\alpha \times \mathbb{K}^r\Big)/\sim$$

by gluing the charts $U_\alpha \times \mathbb{K}^r$ via the identification $(x_\alpha, \xi_\alpha) \sim (x_\beta, \xi_\beta)$ if and only if $x_\alpha = x_\beta = x \in U_\alpha \cap U_\beta$ and $\xi_\alpha = g_{\alpha\beta}(x)\cdot\xi_\beta$.

(3.5) Example. When X is a real differentiable manifold, an interesting example of real vector bundle is the *tangent bundle* T_X ; if $\tau_\alpha : U_\alpha \to \mathbb{R}^n$ is a collection of coordinate charts on X, then $\theta_\alpha = \pi \times d\tau_\alpha : T_{X\restriction U_\alpha} \to U_\alpha \times \mathbb{R}^m$ define trivializations of T_X and the transition matrices are given by $g_{\alpha\beta}(x) = d\tau_{\alpha\beta}(x^\beta)$ where $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta^{-1}$ and $x^\beta = \tau_\beta(x)$. The dual $T_X^\star$ of T_X is called the *cotangent bundle* of X. If X is complex analytic, then T_X has the structure of a holomorphic vector bundle.

We now briefly discuss the concept of sheaf and bundle morphisms. If $\mathcal{S}$ and $\mathcal{S}'$ are sheaves of $\mathcal{A}$-modules over a topological space X, then by a morphism $\varphi : \mathcal{S} \to \mathcal{S}'$ we just mean a $\mathcal{A}$-linear sheaf morphism. If $\mathcal{S} = \mathcal{A}(E)$ and $\mathcal{S}' = \mathcal{A}(E')$ are locally free sheaves, this is the same as a $\mathcal{A}$-linear bundle morphism, that is, a fiber preserving $\mathbb{K}$-linear morphism $\varphi(x) : E_x \to E'_x$ such that the matrix representing φ in any local $\mathcal{A}$-admissible frames of E and E' has coefficients in $\mathcal{A}$.

(3.6) Proposition. *Suppose that $\mathcal{A}$ is a sheaf of local rings, i.e. that a section of $\mathcal{A}$ is invertible in $\mathcal{A}$ if and only if it never takes the zero value in $\mathbb{K}$. Let $\varphi : \mathcal{S} \to \mathcal{S}'$ be a $\mathcal{A}$-morphism of locally free $\mathcal{A}$-modules of rank r, r'. If the rank of the $r' \times r$ matrix $\varphi(x) \in M_{r'r}(\mathbb{K})$ is constant for all $x \in X$, then* $\mathrm{Ker}\,\varphi$ *and* $\mathrm{Im}\,\varphi$ *are locally free subsheaves of $\mathcal{S}$, $\mathcal{S}'$ respectively, and* $\mathrm{Coker}\,\varphi = \mathcal{S}'/\mathrm{Im}\,\varphi$ *is locally free.*

Proof. This is just a consequence of elementary linear algebra, once we know that non zero determinants with coefficients in $\mathcal{A}$ can be inverted. □

Note that all three sheaves $\mathcal{C}_{X,\mathbb{K}}$, $\mathcal{C}^p_{X,\mathbb{K}}$, $\mathcal{O}_X$ are sheaves of local rings, so Prop. 3.6 applies to these cases. However, even if we work in the holomorphic category ($\mathcal{A} = \mathcal{O}_X$), a difficulty immediately appears that the kernel or cokernel of an arbitrary morphism of locally free sheaves is in general not locally free.

(3.7) Examples.

a) Take $X = \mathbb{C}$, let $\mathcal{S} = \mathcal{S}' = \mathcal{O}$ be the trivial sheaf, and let $\varphi : \mathcal{O} \to \mathcal{O}$ be the morphism $u(z) \mapsto z\,u(z)$. It is immediately seen that φ is injective as a sheaf morphism ($\mathcal{O}$ being an entire ring), and that $\operatorname{Coker}\varphi$ is the *skyscraper sheaf* $\mathbb{C}_0$ of stalk $\mathbb{C}$ at $z = 0$, having zero stalks at all other points $z \neq 0$. Thus $\operatorname{Coker}\varphi$ is not a locally free sheaf, although φ is everywhere injective (note however that the corresponding morphism $\varphi : E \to E'$, $(z, \xi) \mapsto (z, z\xi)$ of trivial rank 1 vector bundles $E = E' = \mathbb{C} \times \mathbb{C}$ is *not injective* on the zero fiber E_0).

b) Take $X = \mathbb{C}^3$, $\mathcal{S} = \mathcal{O}^{\oplus 3}$, $\mathcal{S}' = \mathcal{O}$ and

$$\varphi : \mathcal{O}^{\oplus 3} \to \mathcal{O}, \qquad (u_1, u_2, u_3) \mapsto \sum_{1 \leq j \leq 3} z_j u_j(z_1, z_2, z_3).$$

Since φ yields a surjective bundle morphism on $\mathbb{C}^3 \smallsetminus \{0\}$, one easily sees that $\operatorname{Ker}\varphi$ is locally free of rank 2 over $\mathbb{C}^3 \smallsetminus \{0\}$. However, by looking at the Taylor expansion of the u_j's at 0, it is not difficult to check that $\operatorname{Ker}\varphi$ is the $\mathcal{O}$-submodule of $\mathcal{O}^{\oplus 3}$ generated by the three sections $(-z_2, z_1, 0)$, $(-z_3, 0, z_1)$ and $(0, z_3, -z_2)$, and that any two of these three sections cannot generate the 0-stalk $(\operatorname{Ker}\varphi)_0$. Hence $\operatorname{Ker}\varphi$ is not locally free.

Since the category of locally free $\mathcal{O}$-modules is not stable by taking kernels or cokernels, one is led to introduce a more general category which will be stable under these operations. This leads to the notion of *coherent sheaves.*

§3.2. Notion of Coherence

The notion of coherence again deals with sheaves of modules over a sheaf of rings. It is a semi-local property which says roughly that the sheaf of modules locally has a finite presentation in terms of generators and relations. We describe here some general properties of this notion, before concentrating ourselves on the case of coherent $\mathcal{O}_X$-modules.

(3.8) Definition. *Let $\mathcal{A}$ be a sheaf of rings on a topological space X and $\mathcal{S}$ a sheaf of modules over $\mathcal{A}$ (or briefly a $\mathcal{A}$-module). Then $\mathcal{S}$ is said to be locally finitely generated if for every point $x_0 \in X$ one can find a neighborhood Ω and sections $F_1, \ldots, F_q \in \mathcal{S}(\Omega)$ such that for every $x \in \Omega$ the stalk $\mathcal{S}_x$ is generated by the germs $F_{1,x}, \ldots, F_{q,x}$ as an $\mathcal{A}_x$-module.*

(3.9) Lemma. *Let $\mathcal{S}$ be a locally finitely generated sheaf of $\mathcal{A}$-modules on X and $G_1, \ldots, G_N$ sections in $\mathcal{S}(U)$ such that $G_{1,x_0}, \ldots, G_{N,x_0}$ generate $\mathcal{S}_{x_0}$ at $x_0 \in U$. Then $G_{1,x}, \ldots, G_{N,x}$ generate $\mathcal{S}_x$ for x near x_0.*

Proof. Take $F_1, \ldots, F_q$ as in Def. 3.8. As $G_1, \ldots, G_N$ generate $\mathcal{S}_{x_0}$, one can find a neighborhood $\Omega' \subset \Omega$ of x_0 and $H_{jk} \in \mathcal{A}(\Omega')$ such that $F_j = \sum H_{jk} G_k$ on Ω'. Thus $G_{1,x}, \ldots, G_{N,x}$ generate $\mathcal{S}_x$ for all $x \in \Omega'$. □

§**3.2.1.** Definition of Coherent Sheaves. If U is an open subset of X, we denote by $\mathcal{S}_{\restriction U}$ the restriction of $\mathcal{S}$ to U, i.e. the union of all stalks $\mathcal{S}_x$ for $x \in U$. If $F_1, \ldots, F_q \in \mathcal{S}(U)$, the kernel of the sheaf homomorphism $F : \mathcal{A}^{\oplus q}_{\restriction U} \longrightarrow \mathcal{S}_{\restriction U}$ defined by

$$(3.10) \qquad \mathcal{A}_x^{\oplus q} \ni (g^1, \ldots, g^q) \longmapsto \sum_{1 \le j \le q} g^j F_{j,x} \in \mathcal{S}_x, \qquad x \in U$$

is a subsheaf $\mathcal{R}(F_1, \ldots, F_q)$ of $\mathcal{A}^{\oplus q}_{\restriction U}$, called the *sheaf of relations* between $F_1, \ldots, F_q$.

(3.11) Definition. *A sheaf $\mathcal{S}$ of $\mathcal{A}$-modules on X is said to be coherent if:*

a) $\mathcal{S}$ *is locally finitely generated* ;

b) *for any open subset U of X and any $F_1, \ldots, F_q \in \mathcal{S}(U)$, the sheaf of relations $\mathcal{R}(F_1, \ldots, F_q)$ is locally finitely generated.*

Assumption a) means that every point $x \in X$ has a neighborhood Ω such that there is a surjective sheaf morphism $F : \mathcal{A}^{\oplus q}_{\restriction \Omega} \longrightarrow \mathcal{S}_{\restriction \Omega}$, and assumption b) implies that the kernel of F is locally finitely generated. Thus, after shrinking Ω, we see that $\mathcal{S}$ admits over Ω a finite presentation under the form of an exact sequence

$$(3.12) \qquad \mathcal{A}^{\oplus p}_{\restriction \Omega} \xrightarrow{G} \mathcal{A}^{\oplus q}_{\restriction \Omega} \xrightarrow{F} \mathcal{S}_{\restriction \Omega} \longrightarrow 0,$$

where G is given by a $q \times p$ matrix (G_{jk}) of sections of $\mathcal{A}(\Omega)$ whose columns $(G_{j1}), \ldots, (G_{jp})$ are generators of $\mathcal{R}(F_1, \ldots, F_q)$.

It is clear that every locally finitely generated subsheaf of a coherent sheaf is coherent. From this we easily infer:

(3.13) Theorem. *Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a $\mathcal{A}$-morphism of coherent sheaves. Then* $\operatorname{Im} \varphi$ *and* $\ker \varphi$ *are coherent.*

Proof. Clearly $\operatorname{Im} \varphi$ is a locally finitely generated subsheaf of $\mathcal{G}$, so it is coherent. Let $x_0 \in X$, let $F_1, \ldots, F_q \in \mathcal{F}(\Omega)$ be generators of $\mathcal{F}$ on a neighborhood Ω of x_0, and $G_1, \ldots, G_r \in \mathcal{A}(\Omega')^{\oplus q}$ be generators of $\mathcal{R}\big(\varphi(F_1), \ldots, \varphi(F_q)\big)$ on a neighborhood $\Omega' \subset \Omega$ of x_0. Then $\ker \varphi$ is generated over Ω' by the sections

$$H_j = \sum_{k=1}^{q} G_j^k F_k \in \mathcal{F}(\Omega'), \qquad 1 \le j \le r. \qquad \square$$

(3.14) Theorem. *Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S} \longrightarrow \mathcal{G} \longrightarrow 0$ be an exact sequence of $\mathcal{A}$-modules. If two of the sheaves $\mathcal{F}, \mathcal{S}, \mathcal{G}$ are coherent, then all three are coherent.*

Proof. If $\mathcal{S}$ and $\mathcal{G}$ are coherent, then $\mathcal{F} = \ker(\mathcal{S} \to \mathcal{G})$ is coherent by Th. 3.13. If $\mathcal{S}$ and $\mathcal{F}$ are coherent, then $\mathcal{G}$ is locally finitely generated; to prove the coherence, let $G_1, \ldots, G_q \in \mathcal{G}(U)$ and $x_0 \in U$. Then there is a neighborhood Ω of x_0 and sections $\tilde{G}_1, \ldots, \tilde{G}_q \in \mathcal{S}(\Omega)$ which are mapped to $G_1, \ldots, G_q$ on Ω. After shrinking Ω, we may assume also that $\mathcal{F}_{\restriction \Omega}$ is generated by sections $F_1, \ldots, F_p \in \mathcal{F}(\Omega)$. Then $\mathcal{R}(G_1, \ldots, G_q)$ is the projection on the last q-components of $\mathcal{R}(F_1, \ldots, F_p, \tilde{G}_1, \ldots, \tilde{G}_q) \subset \mathcal{A}^{p+q}$, which is finitely generated near x_0 by the coherence of $\mathcal{S}$. Hence $\mathcal{R}(G_1, \ldots, G_q)$ is finitely generated near x_0 and $\mathcal{G}$ is coherent.

Finally, assume that $\mathcal{F}$ and $\mathcal{G}$ are coherent. Let $x_0 \in X$ be any point, let $F_1, \ldots, F_p \in \mathcal{F}(\Omega)$ and $G_1, \ldots, G_q \in \mathcal{G}(\Omega)$ be generators of $\mathcal{F}$, $\mathcal{G}$ on a neighborhood Ω of x_0. There is a neighborhood Ω' of x_0 such that $G_1, \ldots, G_q$ admit liftings $\tilde{G}_1, \ldots, \tilde{G}_q \in \mathcal{S}(\Omega')$. Then $(F_1, \ldots, F_q, \tilde{G}_1, \ldots, \tilde{G}_q)$ generate $\mathcal{S}_{\restriction \Omega'}$, so $\mathcal{S}$ is locally finitely generated. Now, let $S_1, \ldots, S_q$ be arbitrary sections in $\mathcal{S}(U)$ and $\overline{S}_1, \ldots, \overline{S}_q$ their images in $\mathcal{G}(U)$. For any $x_0 \in U$, the sheaf of relations $\mathcal{R}(\overline{S}_1, \ldots, \overline{S}_q)$ is generated by sections $P_1, \ldots, P_s \in \mathcal{A}(\Omega)^{\oplus q}$ on a small neighborhood Ω of x_0. Set $P_j = (P_j^k)_{1 \le k \le q}$. Then $H_j = P_j^1 S_1 + \ldots + P_j^q S_q$, $1 \le j \le s$, are mapped to 0 in $\mathcal{G}$ so they can be seen as sections of $\mathcal{F}$. The coherence of $\mathcal{F}$ shows that $\mathcal{R}(H_1, \ldots, H_s)$ has generators $Q_1, \ldots, Q_t \in \mathcal{A}(\Omega')^s$ on a small neighborhood $\Omega' \subset \Omega$ of x_0. Then $\mathcal{R}(S_1, \ldots, S_q)$ is generated over Ω' by $R_j = \sum Q_j^k P_k \in \mathcal{A}(\Omega')$, $1 \le j \le t$, and $\mathcal{S}$ is coherent. $\square$

(3.15) Corollary. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent subsheaves of a coherent analytic sheaf $\mathcal{S}$, the intersection $\mathcal{F} \cap \mathcal{G}$ is a coherent sheaf.*

Proof. Indeed, the intersection sheaf $\mathcal{F} \cap \mathcal{G}$ is the kernel of the composite morphism $\mathcal{F} \hookrightarrow \mathcal{S} \longrightarrow \mathcal{S}/\mathcal{G}$, and $\mathcal{S}/\mathcal{G}$ is coherent. $\square$

§**3.2.2. Coherent Sheaf of Rings.** A sheaf of rings $\mathcal{A}$ is said to be coherent if it is coherent as a module over itself. By Def. 3.11, this means that for any open set $U \subset X$ and any sections $F_j \in \mathcal{A}(U)$, the sheaf of relations $\mathcal{R}(F_1, \ldots, F_q)$ is finitely generated. The above results then imply that all free modules $\mathcal{A}^{\oplus p}$ are coherent. As a consequence:

(3.16) Theorem. *If $\mathcal{A}$ is a coherent sheaf of rings, any locally finitely generated subsheaf of $\mathcal{A}^{\oplus p}$ is coherent. In particular, if $\mathcal{S}$ is a coherent $\mathcal{A}$-module* *and $F_1, \ldots, F_q \in \mathcal{S}(U)$, the sheaf of relations $\mathcal{R}(F_1, \ldots, F_q) \subset \mathcal{A}^{\oplus q}$ is also coherent.*

Let $\mathcal{S}$ be a coherent sheaf of modules over a coherent sheaf of ring $\mathcal{A}$. By an iteration of construction (3.12), we see that for every integer $m \geq 0$ and every point $x \in X$ there is a neighborhood Ω of x on which there is an exact sequence of sheaves

$$(3.17) \quad \mathcal{A}_{\restriction\Omega}^{\oplus p_m} \xrightarrow{F_m} \mathcal{A}_{\restriction\Omega}^{\oplus p_{m-1}} \longrightarrow \cdots \longrightarrow \mathcal{A}_{\restriction\Omega}^{\oplus p_1} \xrightarrow{F_1} \mathcal{A}_{\restriction\Omega}^{\oplus p_0} \xrightarrow{F_0} \mathcal{S}_{\restriction\Omega} \longrightarrow 0,$$

where F_j is given by a $p_{j-1} \times p_j$ matrix of sections in $\mathcal{A}(\Omega)$.

§3.3. Analytic Sheaves and the Oka Theorem

Many properties of holomorphic functions which will be considered in this book can be expressed in terms of sheaves. Among them, analytic sheaves play a central role. The Oka theorem (Oka 1950) asserting the coherence of the sheaf of holomorphic functions can be seen as a far-reaching deepening of the noetherian property seen in Sect. 1. The theory of analytic sheaves could not be presented without it.

(3.18) Definition. *Let M be a n-dimensional complex analytic manifold and let $\mathcal{O}_M$ be the sheaf of germs of analytic functions on M. An analytic sheaf over M is by definition a sheaf $\mathcal{S}$ of modules over $\mathcal{O}_M$.*

(3.19) Coherence theorem of Oka. *The sheaf of rings $\mathcal{O}_M$ is coherent for any complex manifold M.*

Let $F_1, \ldots, F_q \in \mathcal{O}(U)$. Since $\mathcal{O}_{M,x}$ is Noetherian, we already know that every stalk $\mathcal{R}(F_1, \ldots, F_q)_x \subset \mathcal{O}_{M,x}^{\oplus q}$ is finitely generated, but the important new fact expressed by the theorem is that the sheaf of relations is locally finitely generated, namely that the "same" generators can be chosen to generate each stalk in a neighborhood of a given point.

Proof. By induction on $n = \dim_{\mathbb{C}} M$. For $n = 0$, the stalks $\mathcal{O}_{M,x}$ are equal to $\mathbb{C}$ and the result is trivial. Assume now that $n \geq 1$ and that the result has already been proved in dimension $n-1$. Let U be an open set of M and $F_1, \ldots, F_q \in \mathcal{O}_M(U)$. To show that $\mathcal{R}(F_1, \ldots, F_q)$ is locally finitely generated, we may assume that $U = \Delta = \Delta' \times \Delta_n$ is a polydisk in $\mathbb{C}^n$ centered at $x_0 = 0$; after a change of coordinates and multiplication of $F_1, \ldots, F_q$ by invertible functions, we may also suppose that $F_1, \ldots, F_q$ are Weierstrass polynomials in z_n with coefficients in $\mathcal{O}(\Delta')$. We need a lemma.

(3.20) Lemma. *If $x = (x', x_n) \in \Delta$, the $\mathcal{O}_{\Delta,x}$-module $\mathcal{R}(F_1, \ldots, F_q)_x$ is generated by those of its elements whose components are germs of analytic* *polynomials in $\mathcal{O}_{\Delta',x'}[z_n]$ with a degree in z_n at most equal to μ, the maximum of the degrees of $F_1, \ldots, F_q$.*

Proof. Assume for example that F_q is of the maximum degree μ. By the Weierstrass preparation Th. 1.1 and Lemma 1.9 applied at x, we can write $F_{q,x} = f'f''$ where $f', f'' \in \mathcal{O}_{\Delta',x'}[z_n]$, f' is a Weierstrass polynomial in $z_n - x_n$ and $f''(x) \neq 0$. Let μ' and μ'' denote the degrees of f' and f'' with respect to z_n, so $\mu' + \mu'' = \mu$. Now, take $(g^1, \ldots, g^q) \in \mathcal{R}(F_1, \ldots, F_q)_x$. The Weierstrass division theorem gives

$$g^j = F_{q,x} t^j + r^j, \qquad j = 1, \ldots, q-1,$$

where $t^j \in \mathcal{O}_{\Delta,x}$ and $r^j \in \mathcal{O}_{\Delta',x'}[z_n]$ is a polynomial of degree $< \mu'$. For $j = q$, define $r^q = g^q + \sum_{1\le j\le q-1} t^j F_{j,x}$. We can write

$$(3.21)\quad (g^1, \ldots, g^q) = \sum_{1\le j\le q} t^j (0, \ldots, F_q, \ldots, 0, -F_j)_x + (r^1, \ldots, r^q)$$

where F_q is in the j-th position in the q-tuples of the summation. Since these q-tuples are in $\mathcal{R}(F_1, \ldots, F_q)_x$, we have $(r^1, \ldots, r^q) \in \mathcal{R}(F_1, \ldots, F_q)_x$, thus

$$\sum_{1\le j\le q-1} F_{j,x} r^j + f'f'' r^q = 0.$$

As the sum is a polynomial in z_n of degree $< \mu + \mu'$, it follows from Lemma 1.9 that $f'' r^q$ is a polynomial in z_n of degree $< \mu$. Now we have

$$(r^1, \ldots, r^q) = 1/f''(f'' r^1, \ldots, f'' r^q)$$

where $f'' r^j$ is of degree $< \mu' + \mu'' = \mu$. In combination with (3.21) this proves the lemma. □

Proof of Theorem 3.19 (end) If $g = (g^1, \ldots, g^q)$ is one of the polynomials of $\mathcal{R}(F_1, \ldots, F_q)_x$ described in Lemma 3.20, we can write

$$g^j = \sum_{0\le k\le \mu} u^{jk} z_n^k, \qquad u^{jk} \in \mathcal{O}_{\Delta',x'}.$$

The condition for $(g^1, \ldots, g^q)$ to belong to $\mathcal{R}(F_1, \ldots, F_q)_x$ therefore consists of $2\mu + 1$ linear conditions for the germ $u = (u^{jk})$ with coefficients in $\mathcal{O}(\Delta')$. By the induction hypothesis, $\mathcal{O}_{\Delta'}$ is coherent and Th. 3.16 shows that the corresponding modules of relations are generated over $\mathcal{O}_{\Delta',x'}$, for x' in a neighborhood Ω' of 0, by finitely many $(q \times \mu)$-tuples $U_1, \ldots, U_N \in \mathcal{O}(\Omega')^{q\mu}$. By Lemma 3.20, $\mathcal{R}(F_1, \ldots, F_q)_x$ is generated at every point $x \in \Omega = \Omega' \times \Delta_n$ by the germs of the corresponding polynomials

$$G_l(z) = \Big(\sum_{0\le k\le \mu} U_l^{jk}(z') z_n^k \Big)_{1\le j\le q}, \qquad z \in \Omega, \quad 1 \le l \le N. \qquad □$$

(3.22) Strong Noetherian property. *Let $\mathcal{F}$ be a coherent analytic sheaf on a complex manifold M and let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ be an increasing sequence of coherent subsheaves of $\mathcal{F}$. Then the sequence $(\mathcal{F}_k)$ is stationary on every compact subset of M.*

Proof. Since $\mathcal{F}$ is locally a quotient of a free module $\mathcal{O}_M^{\oplus q}$, we can pull back the sequence to $\mathcal{O}_M^{\oplus q}$ and thus suppose $\mathcal{F} = \mathcal{O}_M$ (by easy reductions similar to those in the proof of Th. 3.14). Suppose M connected and $\mathcal{F}_{k_0} \neq \{0\}$ for some index k_0 (otherwise, there is nothing to prove). By the analytic continuation theorem, we easily see that $\mathcal{F}_{k_0,x} \neq \{0\}$ for every $x \in M$. We can thus find a non zero Weierstrass polynomial $P \in \mathcal{F}_{k_0}(V)$, $\deg_{z_n} P(z', z_n) = \mu$, in a coordinate neighborhood $V = \Delta' \times \Delta_n$ of any point $x \in M$. A division by P shows that for $k \geq k_0$ and $x \in V$, all stalks $\mathcal{F}_{k,x}$ are generated by P_x and by polynomials of degree $< \mu$ in z_n with coefficients in $\mathcal{O}_{\Delta',x'}$. Therefore, we can apply induction on n to the coherent $\mathcal{O}_{\Delta'}$-module

$$\mathcal{F}' = \mathcal{F} \cap \big\{Q \in \mathcal{O}_{\Delta'}[z_n]\,;\, \deg Q \leq \mu\big\}$$

and its increasing sequence of coherent subsheaves $\mathcal{F}'_k = \mathcal{F}_k \cap \mathcal{F}'$. □

§4. Complex Analytic Sets. Local Properties

§4.1. Definition. Irreducible Components

A complex analytic set is a set which can be defined locally by finitely many holomorphic equations; such a set has in general singular points, because no assumption is made on the differentials of the equations. We are interested both in the description of the singularities and in the study of algebraic properties of holomorphic functions on analytic sets. For a more detailed study than ours, we refer to H. Cartan's seminar (Cartan 1950), to the books of (Gunning-Rossi 1965), (Narasimhan 1966) or the recent book by (Grauert-Remmert 1984).

(4.1) Definition. *Let M be a complex analytic manifold. A subset $A \subset M$ is said to be an analytic subset of M if A is closed and if for every point $x_0 \in A$ there exist a neighborhood U of x_0 and holomorphic functions $g_1, \ldots, g_n$ in $\mathcal{O}(U)$ such that*

$$A \cap U = \{z \in U \ ;\ g_1(z) = \ldots = g_N(z) = 0\}.$$

Then $g_1, \ldots, g_N$ are said to be (*local*) *equations of A in U.*

It is easy to see that a finite union or intersection of analytic sets is analytic: if (g'_j), (g''_k) are equations of A', A'' in the open set U, then the family of all products $(g'_j g''_k)$ and the family $(g'_j) \cup (g''_k)$ define equations of $A' \cup A''$ and $A' \cap A''$ respectively.

(4.2) Remark. Assume that M is connected. The analytic continuation theorem shows that either $A = M$ or A has no interior point. In the latter case, each piece $A \cap U = g^{-1}(0)$ is the set of points where the function $\log |g|^2 = \log(|g_1|^2 + \cdots + |g_N|^2) \in \mathrm{Psh}(U)$ takes the value $-\infty$, hence A is pluripolar. In particular $M \smallsetminus A$ is connected and every function $f \in \mathcal{O}(M \smallsetminus A)$ that is locally bounded near A can be extended to a function $\tilde{f} \in \mathcal{O}(M)$. □

We focus now our attention on local properties of analytic sets. By definition, a germ of set at a point $x \in M$ is an equivalence class of elements in the power set $\mathcal{P}(M)$, with $A \sim B$ if there is an open neighborhood V of x such that $A \cap V = B \cap V$. The germ of a subset $A \subset M$ at x will be denoted by (A, x). We most often consider the case when $A \subset M$ is a analytic set in a neighborhood U of x, and in this case we denote by $\mathcal{I}_{A,x}$ the ideal of germs $f \in \mathcal{O}_{M,x}$ which vanish on (A, x). Conversely, if $\mathcal{J} = (g_1, \ldots, g_N)$ is an ideal of $\mathcal{O}_{M,x}$, we denote by $\big(V(\mathcal{J}), x\big)$ the germ at x of the zero variety $V(\mathcal{J}) = \{z \in U \ ; \ g_1(z) = \ldots = g_N(z) = 0\}$, where U is a neighborhood of x such that $g_j \in \mathcal{O}(U)$. It is easy to check that the germ $(V(cJ), x)$ does not depend on the choice of generators. Moreover, it is clear that

$$(4.3') \quad \text{for every ideal } \mathcal{J} \text{ in the ring } \mathcal{O}_{M,x}, \qquad \mathcal{I}_{V(\mathcal{J}),x} \supset \mathcal{J},$$

$$(4.3'') \quad \text{for every germ of analytic set } (A, x), \qquad \big(V(\mathcal{I}_{A,x}), x\big) = (A, x).$$

(4.4) Definition. *A germ (A, x) is said to be irreducible if it has no decomposition $(A, x) = (A_1, x) \cup (A_2, x)$ with analytic sets $(A_j, x) \neq (A, x)$, $j = 1, 2$.*

(4.5) Proposition. *A germ (A, x) is irreducible if and only if $\mathcal{I}_{A,x}$ is a prime ideal of the ring $\mathcal{O}_{M,x}$.*

Proof. Let us recall that an ideal $\mathcal{J}$ is said to be *prime* if $fg \in \mathcal{J}$ implies $f \in \mathcal{J}$ or $g \in \mathcal{J}$. Assume that (A, x) is irreducible and that $fg \in \mathcal{I}_{A,x}$. As we can write $(A, x) = (A_1, x) \cup (A_2, x)$ with $A_1 = A \cap f^{-1}(0)$ and $A_2 = A \cap g^{-1}(0)$, we must have for example $(A_1, x) = (A, x)$; thus $f \in \mathcal{I}_{A,x}$ and $\mathcal{I}_{A,x}$ is prime. Conversely, if $(A, x) = (A_1, x) \cup (A_2, x)$ with $(A_j, x) \neq (A, x)$, there exist $f \in \mathcal{I}_{A_1,x}$, $g \in \mathcal{I}_{A_2,x}$ such that $f, g \notin \mathcal{I}_{A,x}$. However $fg \in \mathcal{I}_{A,x}$, thus $\mathcal{I}_{A,x}$ is not prime. □

(4.6) Theorem. *Every decreasing sequence of germs of analytic sets (A_k, x) is stationary.*

Proof. In fact, the corresponding sequence of ideals $\mathcal{J}_k = \mathcal{I}_{A_k,x}$ is increasing, thus $\mathcal{J}_k = \mathcal{J}_{k_0}$ for $k \geq k_0$ large enough by the Noetherian property of $\mathcal{O}_{M,x}$. Hence $(A_k, x) = \big(V(\mathcal{J}_k), x\big)$ is constant for $k \geq k_0$. This result has the following straightforward consequence: $\square$

(4.7) Theorem. *Every analytic germ (A, x) has a finite decomposition*

$$(A, x) = \bigcup_{1\leq k\leq N} (A_k, x)$$

where the germs (A_j, x) are irreducible and $(A_j, x) \not\subset (A_k, x)$ for $j \neq k$. The decomposition is unique apart from the ordering.

Proof. If (A, x) can be split in several components, we split repeatedly each component as long as one of them is reducible. The process must stop by Th. 4.6, whence the existence. For the uniqueness, assume that $(A, x) = \bigcup(A'_l, x)$, $1 \leq l \leq N'$, is another decomposition. Since $(A_k, x) = \bigcup_l (A_k \cap A'_l, x)$, we must have $(A_k, x) = (A_k \cap A'_l, x)$ for some $l = l(k)$, i.e. $(A_k, x) \subset (A'_{l(k)}, x)$, and likewise $(A'_{l(k)}, x) \subset (A_j, x)$ for some j. Hence $j = k$ and $(A'_{l(k)}, x) = (A_k, x)$. $\square$

§4.2. Local Structure of a Germ of Analytic Set

We are going to describe the local structure of a germ, both from the holomorphic and topological points of view. By the above decomposition theorem, we may restrict ourselves to the case of irreducible germs Let $\mathcal{J}$ be a prime ideal of $\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n,0}$ and let $A = V(\mathcal{J})$ be its zero variety. We set $\mathcal{J}_k = \mathcal{J} \cap \mathbb{C}\{z_1, \ldots, z_k\}$ for each $k = 0, 1, \ldots, n$.

(4.8) Proposition. *There exist an integer d, a basis $(e_1, \ldots, e_n)$ of $\mathbb{C}^n$ and associated coordinates $(z_1, \ldots, z_n)$ with the following properties: $\mathcal{J}_d = \{0\}$ and for every integer $k = d + 1, \ldots, n$ there is a Weierstrass polynomial $P_k \in \mathcal{J}_k$ of the form*

$$(4.9) \quad P_k(z', z_k) = z_k^{s_k} + \sum_{1\leq j\leq s_k} a_{j,k}(z')\, z_k^{s_k-j}, \qquad a_{j,k}(z') \in \mathcal{O}_{k-1},$$

where $a_{j,k}(z') = O(|z'|^j)$. Moreover, the basis $(e_1, \ldots, e_n)$ can be chosen arbitrarily close to any preassigned basis $(e_1^0, \ldots, e_n^0)$.

Proof. By induction on n. If $\mathcal{J} = \mathcal{J}_n = \{0\}$, then $d = n$ and there is nothing to prove. Otherwise, select a non zero element $g_n \in \mathcal{J}$ and a vector e_n such that $\mathbb{C} \ni w \longmapsto g_n(we_n)$ has minimum vanishing order s_n. This choice excludes at most the algebraic set $g_n^{(s_n)}(v) = 0$, so e_n can be taken arbitrarily close to e_n^0. Let $(\tilde{z}_1, \ldots, \tilde{z}_{n-1}, z_n)$ be the coordinates associated to the basis $(e_1^0, \ldots, e_{n-1}^0, e_n)$. After multiplication by an invertible element, we may assume that g_n is a Weierstrass polynomial

$$P_n(\tilde{z}, z_n) = z_n^{s_n} + \sum_{1 \le j \le s_n} a_{j,n}(\tilde{z})\, z_n^{s_n - j}, \qquad a_{j,n} \in \mathcal{O}_{n-1},$$

and $a_{j,n}(\tilde{z}) = O(|\tilde{z}|^j)$ by Remark 2.2. If $\mathcal{J}_{n-1} = \mathcal{J} \cap \mathbb{C}\{\tilde{z}\} = \{0\}$ then $d = n-1$ and the construction is finished. Otherwise we apply the induction hypothesis to the ideal $\mathcal{J}_{n-1} \subset \mathcal{O}_{n-1}$ in order to find a new basis $(e_1, \ldots, e_{n-1})$ of $\mathrm{Vect}(e_1^0, \ldots, e_{n-1}^0)$, associated coordinates $(z_1, \ldots, z_{n-1})$ and Weierstrass polynomials $P_k \in \mathcal{J}_k$, $d+1 \le k \le n-1$, as stated in the lemma. □

(4.10) Lemma. *If $w \in \mathbb{C}$ is a root of $w^d + a_1 w^{d-1} + \cdots + a_d = 0$, $a_j \in \mathbb{C}$, then $|w| \le 2 \max |a_j|^{1/j}$.*

Proof. Otherwise $|w| > 2|a_j|^{1/j}$ for all $j = 1, \ldots, d$ and the given equation $-1 = a_1/w + \cdots + a_d/w^d$ implies $1 \le 2^{-1} + \cdots + 2^{-d}$, a contradiction. □

(4.11) Corollary. *Set $z' = (z_1, \ldots, z_d)$, $z'' = (z_{d+1}, \ldots, z_n)$, and let Δ' in $\mathbb{C}^d$, Δ'' in $\mathbb{C}^{n-d}$ be polydisks of center 0 and radii $r', r'' > 0$. Then the germ $(A, 0)$ is contained in a cone $|z''| \le C|z'|$, $C =$ constant, and the restriction of the projection map $\mathbb{C}^n \longrightarrow \mathbb{C}^d$, $(z', z'') \longmapsto z'$:*

$$\pi : A \cap (\Delta' \times \Delta'') \longrightarrow \Delta'$$

is proper if r'' is small enough and $r' \le r''/C$.

Proof. The polynomials $P_k(z_1, \ldots, z_{k-1}\,;\, z_k)$ vanish on $(A, 0)$. By Lemma 4.10 and (4.9), every point $z \in A$ sufficiently close to 0 satisfies

$$|z_k| \le C_k(|z_1| + \cdots + |z_{k-1}|), \qquad d+1 \le k \le n,$$

thus $|z''| \le C|z'|$ and the Corollary follows. □

Since $\mathcal{J}_d = \{0\}$, we have an injective ring morphism

$$\text{(4.12)} \qquad \mathcal{O}_d = \mathbb{C}\{z_1, \ldots, z_d\} \hookrightarrow \mathcal{O}_n/\mathcal{J}.$$

(4.13) Proposition. *$\mathcal{O}_n/\mathcal{J}$ is a finite integral extension of $\mathcal{O}_d$.*

Proof. Let $f \in \mathcal{O}_n$. A division by P_n yields $f = P_n q_n + R_n$ with a remainder $R_n \in \mathcal{O}_{n-1}[z_n]$, $\deg_{z_n} R_n < s_n$. Further divisions of the coefficients of R_n by P_{n-1}, P_{n-2} etc … yield

$$R_{k+1} = P_k q_k + R_k, \qquad R_k \in \mathcal{O}_k[z_{k+1}, \ldots, z_n],$$

where $\deg_{z_j} R_k < s_j$ for $j > k$. Hence

$$\text{(4.14)} \qquad f = R_d + \sum_{d+1 \le k \le n} P_k q_k = R_d \mod (P_{d+1}, \ldots, P_n) \subset \mathcal{J}$$

and $\mathcal{O}_n/\mathcal{J}$ is finitely generated as an $\mathcal{O}_d$-module by the family of monomials $z_{d+1}^{\alpha_{d+1}} \dots z_n^{\alpha_n}$ with $\alpha_j < s_j$. □

As $\mathcal{J}$ is prime, $\mathcal{O}_n/\mathcal{J}$ is an entire ring. We denote by $\tilde{f}$ the class of any germ $f \in \mathcal{O}_n$ in $\mathcal{O}_n/\mathcal{J}$, by $\mathcal{M}_A$ and $\mathcal{M}_d$ the quotient fields of $\mathcal{O}_n/\mathcal{J}$ and $\mathcal{O}_d$ respectively. Then $\mathcal{M}_A = \mathcal{M}_d[\tilde{z}_{d+1}, \dots, \tilde{z}_n]$ is a finite algebraic extension of $\mathcal{M}_d$. Let $q = [\mathcal{M}_A{:}\mathcal{M}_d]$ be its degree and let $\sigma_1, \dots, \sigma_q$ be the embeddings of $\mathcal{M}_A$ over $\mathcal{M}_d$ in an algebraic closure $\overline{\mathcal{M}}_A$. Let us recall that a factorial ring is integrally closed in its quotient field (Lang 1965, Chapter IX). Hence every element of $\mathcal{M}_d$ which is integral over $\mathcal{O}_d$ lies in fact in $\mathcal{O}_d$. By the primitive element theorem, there exists a linear form $u(z'') = c_{d+1}z_{d+1} + \dots + c_n z_n$, $c_k \in \mathbb{C}$, such that $\mathcal{M}_A = \mathcal{M}_d[\tilde{u}]$; in fact, u is of degree q if and only if $\sigma_1\tilde{u}, \dots, \sigma_q\tilde{u}$ are all distinct, and this excludes at most a finite number of vector subspaces in the space $\mathbb{C}^{n-d}$ of coefficients $(c_{d+1}, \dots, c_n)$. As $\tilde{u} \in \mathcal{O}_n/\mathcal{J}$ is integral over the integrally closed ring $\mathcal{O}_d$, the unitary irreducible polynomial W_u of $\tilde{u}$ over $\mathcal{M}_d$ has coefficients in $\mathcal{O}_d$:

$$W_u(z' ; T) = T^q + \sum_{1 \le j \le q} a_j(z_1, \dots, z_d)\, T^{q-j}, \qquad a_j \in \mathcal{O}_d.$$

W_u must be a Weierstrass polynomial, otherwise there would exist a factorization $W_u = W'Q$ in $\mathcal{O}_d[T]$ with a Weierstrass polynomial W' of degree $\deg W' < q = \deg \tilde{u}$ and $Q(0) \ne 0$, hence $W'(\tilde{u}) = 0$, a contradiction. In the same way, we see that $\tilde{z}_{d+1}, \dots, \tilde{z}_n$ have irreducible equations $W_k(z' ; \tilde{z}_k) = 0$ where $W_k \in \mathcal{O}_d[T]$ is a Weierstrass polynomial of degree $= \deg \tilde{z}_k \le q$, $d+1 \le k \le n$.

(4.15) Lemma. *Let $\delta(z') \in \mathcal{O}_d$ be the discriminant of $W_u(z' ; T)$. For every element g of $\mathcal{M}_A$ which is integral over $\mathcal{O}_d$ (or equivalently over $\mathcal{O}_n/\mathcal{J}$) we have $\delta g \in \mathcal{O}_d[\tilde{u}]$.*

Proof. We have $\delta(z') = \prod_{j<k}(\sigma_k\tilde{u} - \sigma_j\tilde{u})^2 \not\equiv 0$, and $g \in \mathcal{M}_A = \mathcal{M}_d[\tilde{u}]$ can be written

$$g = \sum_{0 \le j \le q-1} b_j\, \tilde{u}^j, \qquad b_j \in \mathcal{M}_d,$$

where $b_0, \dots, b_{d-1}$ are the solutions of the linear system $\sigma_k g = \sum b_j(\sigma_k\tilde{u})^j$; the determinant (of Van der Monde type) is $\delta^{1/2}$. It follows that $\delta b_j \in \mathcal{M}_d$ are polynomials in $\sigma_k g$ and $\sigma_k\tilde{u}$, thus δb_j is integral over $\mathcal{O}_d$. As $\mathcal{O}_d$ is integrally closed, we must have $\delta b_j \in \mathcal{O}_d$, hence $\delta g \in \mathcal{O}_d[\tilde{u}]$. □

In particular, there exist unique polynomials $B_{d+1}, \dots, B_n \in \mathcal{O}_d[T]$ with $\deg B_k \le q-1$, such that

(4.16) $\quad \delta(z')z_k = B_k(z' ; u(z'')) \quad (\mathrm{mod}\ \mathcal{J}).$

Then we have

$$(4.17)\quad \delta(z')^q W_k\big(z'\,;B_k(z'\,;T)/\delta(z')\big) \in \text{ideal}\;\; W_u(z'\,;T)\,\mathcal{O}_d[T]\;;$$

indeed, the left-hand side is a polynomial in $\mathcal{O}_d[T]$ and admits $T=\tilde{u}$ as a root in $\mathcal{O}_n/\mathcal{J}$ since $B_k(z'\,;\tilde{u})/\delta(z')=\tilde{z}_k$ and $W_k(z'\,;\tilde{z}_k)=0$.

(4.18) Lemma. *Consider the ideal*

$$\mathcal{G}=\big(W_u(z'\,;u(z''))\,,\,\delta(z')z_k-B_k(z'\,;u(z''))\big)\subset\mathcal{J}$$

and set $m=\max\{q,(n-d)(q-1)\}$. For every germ $f\in\mathcal{O}_n$, there exists a unique polynomial $R\in\mathcal{O}_d[T]$, $\deg_T R\le q-1$, such that

$$\delta(z')^m f(z)=R(z'\,;u(z''))\qquad(\mathrm{mod}\ \mathcal{G}).$$

Moreover $f\in\mathcal{J}$ implies $R=0$, hence $\delta^m\mathcal{J}\subset\mathcal{G}$.

Proof. By (4.17) and a substitution of z_k, we find $\delta(z')^qW_k(z'\,;z_k)\in\mathcal{G}$. The analogue of formula (4.14) with W_k in place of P_k yields

$$f=R_d+\sum_{d+1\le k\le n}W_kq_k,\qquad R_d\in\mathcal{O}_d[z_{d+1},\ldots,z_n],$$

with $\deg_{z_k}R_d<\deg W_k\le q$, thus $\delta^m f=\delta^m R_d$ mod $\mathcal{G}$. We may therefore replace f by R_d and assume that $f\in\mathcal{O}_d[z_{d+1},\ldots,z_n]$ is a polynomial of total degree $\le(n-d)(q-1)\le m$. A substitution of z_k by $B_k(z'\,;u(z''))/\delta(z')$ yields $G\in\mathcal{O}_d[T]$ such that

$$\delta(z')^m f(z)=G(z'\,;u(z''))\quad\mathrm{mod}\ \big(\delta(z')z_k-B_k(z'\,;u(z''))\big).$$

Finally, a division $G=W_uQ+R$ gives the required polynomial $R\in\mathcal{O}_d[T]$. The last statement is clear: if $f\in\mathcal{J}$ satisfies $\delta^m(z')f(z)=R(z\,;u(z''))$ mod $\mathcal{G}$ for $\deg_T R<q$, then $R(z'\,;\tilde{u})=0$, and as $\tilde{u}\in\mathcal{O}_n/\mathcal{J}$ is of degree q, we must have $R=0$. The uniqueness of R is proved similarly. □

(4.19) Local parametrization theorem. *Let $\mathcal{J}$ be a prime ideal of $\mathcal{O}_n$ and let $A=V(\mathcal{J})$. Assume that the coordinates*

$$(z'\,;z'')=(z_1,\ldots,z_d\,;z_{d+1},\ldots,z_n)$$

are chosen as above. Then the ring $\mathcal{O}_n/\mathcal{J}$ is a finite integral extension of $\mathcal{O}_d$; let q be the degree of the extension and let $\delta(z')\in\mathcal{O}_d$ be the discriminant of the irreducible polynomial of a primitive element $u(z'')=\sum_{k>d}c_kz_k$. If Δ',Δ'' are polydisks of sufficiently small radii r',r'' and if $r'\le r''/C$ with C large, the projection map $\pi:A\cap(\Delta'\times\Delta'')\longrightarrow\Delta'$ is a ramified covering with q sheets, whose ramification locus is contained in $S=\{z'\in\Delta';\delta(z')=0\}$. This means that:

a) *the open subset* $A_S = A \cap \big((\Delta' \smallsetminus S) \times \Delta''\big)$ *is a smooth d-dimensional manifold, dense in* $A \cap (\Delta' \times \Delta'')$;

b) $\pi : A_S \longrightarrow \Delta' \smallsetminus S$ *is a covering* ;

c) *the fibers* $\pi^{-1}(z')$ *have exactly* q *elements if* $z' \notin S$ *and at most* q *if* $z' \in S$.

Moreover, A_S *is a connected covering of* $\Delta' \smallsetminus S$, *and* $A \cap (\Delta' \times \Delta'')$ *is contained in a cone* $|z''| \le C|z'|$ *(see Fig. 1).*

Fig. 1 Ramified covering from A to $\Delta' \subset \mathbb{C}^p$.

Proof. After a linear change in the coordinates $z_{d+1}, \ldots, z_n$, we may assume $u(z'') = z_{d+1}$, so $W_u = W_{d+1}$ and $B_{d+1}(z'\,; T) = \delta(z')T$. By Lemma 4.18, we have

$$\mathcal{G} = \big(W_{d+1}(z', z_{d+1})\,,\ \delta(z')z_k - B_k(z', z_{d+1})\big)_{k \ge d+2} \subset \mathcal{J}, \qquad \delta^m \mathcal{J} \subset \mathcal{G}.$$

We can thus find a polydisk $\Delta = \Delta' \times \Delta''$ of sufficiently small radii r', r'' such that $V(\mathcal{J}) \subset V(\mathcal{G}) \subset V(\delta^m \mathcal{J})$ in Δ. As $V(\mathcal{J}) = A$ and $V(\delta) \cap \Delta = S \times \Delta''$, this implies

$$A \cap \Delta \subset V(\mathcal{G}) \cap \Delta \subset (A \cap \Delta) \cup (S \times \Delta'').$$

In particular, the set $A_S = A \cap \big((\Delta' \smallsetminus S) \times \Delta''\big)$ lying above $\Delta' \smallsetminus S$ coincides with $V(\mathcal{G}) \cap \big((\Delta' \smallsetminus S) \times \Delta''\big)$, which is the set of points $z \in \Delta$ parametrized by the equations

$$(4.20) \qquad \begin{cases} \delta(z') \ne 0, \quad W_{d+1}(z', z_{d+1}) = 0, \\ z_k = B_k(z', z_{d+1})/\delta(z'), \quad d+2 \le k \le n. \end{cases}$$

As $\delta(z')$ is the resultant of W_{d+1} and $\partial W_{d+1}/\partial T$, we have

$$\partial W_{d+1}/\partial T(z', z_{d+1}) \ne 0 \quad \text{on} \ \ A_S.$$

The implicit function theorem shows that z_{d+1} is locally a holomorphic function of z' on A_S, and the same is true for $z_k = B_k(z', z_{d+1})/\delta(z')$, $k \ge d+2$. Hence A_S is a smooth manifold, and for $r' \le r''/C$ small, the projection map $\pi : A_S \longrightarrow \Delta' \smallsetminus S$ is a proper local diffeomorphism; by (4.20) the fibers $\pi^{-1}(z')$ have at most q points corresponding to some of the q roots w of $W_{d+1}(z'\,; w) = 0$. Since $\Delta' \smallsetminus S$ is connected (Remark 4.2), either $A_S = \emptyset$ or the map π is a covering of constant sheet number $q' \le q$. However, if w is a root of $W_{d+1}(z', w) = 0$ with $z' \in \Delta' \smallsetminus S$ and if we set $z_{d+1} = w$, $z_k = B_k(z', w)/\delta(z')$, $k \ge d+2$, relation (4.17) shows that $W_k(z', z_k) = 0$, in particular $|z_k| = O(|z'|^{1/q})$ by Lemma 4.10. For z' small enough, the q points $z = (z', z'')$ defined in this way lie in Δ, thus $q' = q$. Property b) and the first parts of a) and c) follow. Now, we need the following lemma.

(4.21) Lemma. *If $\mathcal{J} \subset \mathcal{O}_n$ is prime and $A = V(\mathcal{J})$, then $\mathcal{I}_{A,0} = \mathcal{J}$.*

Proof It is obvious that $\mathcal{I}_{A,0} \supset \mathcal{J}$. Now, for any $f \in \mathcal{I}_{A,0}$, Prop. 4.13 implies that $\tilde f$ satisfies in $\mathcal{O}_n/\mathcal{I}$ an irreducible equation

$$f^r + b_1(z')\, f^{r-1} + \cdots + b_r(z') = 0 \quad (\mathrm{mod}\ \mathcal{J}).$$

Then $b_r(z')$ vanishes on $(A, 0)$ and the first part of c) gives $b_r = 0$ on $\Delta' \smallsetminus S$. Hence $\tilde b_r = 0$ and the irreducibility of the equation of $\tilde f$ implies $r = 1$, so $f \in \mathcal{J}$, as desired. □

Proof of Theorem 4.19 (end). It only remains to prove that A_S is connected and dense in $A \cap \Delta$ and that the fibers $\pi^{-1}(z')$, $z' \in S$, have at most q elements. Let $A_{S,1}, \ldots, A_{S,N}$ be the connected components of A_S. Then $\pi : A_{S,j} \longrightarrow \Delta' \smallsetminus S$ is a covering with q_j sheets, $q_1 + \cdots + q_N = q$. For every point $\zeta' \in \Delta' \smallsetminus S$, there exists a neighborhood U of ζ' such that $A_{S,j} \cap \pi^{-1}(U)$ is a disjoint union of graphs $z'' = g_{j,k}(z')$ of analytic functions $g_{j,k} \in \mathcal{O}(U)$, $1 \le k \le q_j$. If $\lambda(z'')$ is an arbitrary linear form in $z_{d+1}, \ldots, z_n$ and $z' \in \Delta' \smallsetminus S$, we set

$$P_{\lambda,j}(z'\,; T) = \prod_{\{z''\,;\,(z',z'') \in A_{S,j}\}} \big(T - \lambda(z'')\big) = \prod_{1 \le k \le k_j} \big(T - \lambda \circ g_{j,k}(z')\big).$$

This defines a polynomial in T with bounded analytic coefficients on $\Delta' \smallsetminus S$. These coefficients have analytic extensions to Δ' (Remark 4.2), thus $P_{\lambda,j} \in \mathcal{O}(\Delta')[T]$. By construction, $P_{\lambda,j}\big(z'\,; \lambda(z'')\big)$ vanishes identically on $A_{S,j}$. Set

$$P_\lambda = \prod_{1 \le j \le N} P_{\lambda,j}, \qquad f(z) = \delta(z')\, P_\lambda\big(z'\,; \lambda(z'')\big)\ ;$$

f vanishes on $A_{S,1} \cup \ldots \cup A_{S,N} \cup (S \times \Delta'') \supset A \cap \Delta$. Lemma 4.21 shows that $\mathcal{I}_{A,0}$ is prime; as $\delta \notin \mathcal{I}_{A,0}$, we get $P_{\lambda,j}\big(z'\,; \lambda(z'')\big) \in \mathcal{I}_{A,0}$ for some j. This is a contradiction if $N \ge 2$ and if λ is chosen in such a way that λ

separates the q points z''_ν in each fiber $\pi^{-1}(z'_\nu)$, for a sequence $z'_\nu \to 0$ in $\Delta' \smallsetminus S$. Hence $N = 1$, A_S is connected, and for every $\lambda \in (\mathbb{C}^{n-d})^\star$ we have $P_\lambda\big(z', \lambda(z'')\big) \in \mathcal{I}_{(A,0)}$. By construction $P_\lambda\big(z', \lambda(z'')\big)$ vanishes on A_S, so it vanishes on $\overline{A}_S$; hence, for every $z' \in S$, the fiber $\overline{A}_S \cap \pi^{-1}(z')$ has at most q elements, otherwise selecting λ which separates $q+1$ of these points would yield $q+1$ roots $\lambda(z'')$ of $P_\lambda(z'\,;T)$, a contradiction. Assume now that A_S is not dense in $A \cap \Delta$ for arbitrarily small polydisks Δ. Then there exists a sequence $A \ni z_\nu = (z'_\nu, z''_\nu) \to 0$ such that $z'_\nu \in S$ and z''_ν is not in $F_\nu := \mathrm{pr}''\big(\overline{A}_S \cap \pi^{-1}(z'_\nu)\big)$. The continuity of the roots of the polynomial $P_\lambda(z'\,;T)$ as $\Delta' \smallsetminus S \ni z' \to z'_\nu$ implies that the set of roots of $P_\lambda(z'_\nu\,;T)$ is $\lambda(F_\nu)$. Select λ such that $\lambda(z''_\nu) \notin \lambda(F_\nu)$ for all ν. Then $P_\lambda\big(z'_\nu\,;\lambda(z''_\nu)\big) \neq 0$ for every ν and $P_\lambda\big(z'\,;\lambda(z'')\big) \notin \mathcal{I}_{A,0}$, a contradiction. □

At this point, it should be observed that many of the above statements completely fail in the case of real analytic sets. In $\mathbb{R}^2$, for example, the prime ideal $\mathcal{J} = (x^5 + y^4)$ defines an irreducible germ of curve $(A,0)$ and there is an injective integral extension of rings $\mathbb{R}\{x\} \hookrightarrow \mathbb{R}\{x,y\}/\mathcal{J}$ of degree 4; however, the projection of $(A,0)$ on the first factor, $(x,y) \mapsto x$, has not a constant sheet number near 0, and this number is not related to the degree of the extension. Also, the prime ideal $\mathcal{J} = (x^2+y^2)$ has an associated variety $V(\mathcal{J})$ reduced to $\{0\}$, hence $\mathcal{I}_{A,0} = (x,y)$ is strictly larger than $\mathcal{J}$, in contrast with Lemma 4.21.

Let us return to the complex situation, which is much better behaved. The result obtained in Lemma 4.21 can then be extended to non prime ideals and we get the following important result:

(4.22) Hilbert's Nullstellensatz. *For every ideal $\mathcal{J} \subset \mathcal{O}_n$*

$$\mathcal{I}_{V(\mathcal{J}),0} = \sqrt{\mathcal{J}},$$

where $\sqrt{\mathcal{J}}$ is the radical of $\mathcal{J}$, i.e. the set of germs $f \in \mathcal{O}_n$ such that some power f^k lies in $\mathcal{J}$.

Proof. Set $B = V(\mathcal{J})$. If $f^k \in \mathcal{J}$, then f^k vanishes on $(B,0)$ and $f \in \mathcal{I}_{B,0}$. Thus $\sqrt{\mathcal{J}} \subset \mathcal{I}_{B,0}$. Conversely, it is well known that $\sqrt{\mathcal{J}}$ is the intersection of all prime ideals $\mathcal{P} \supset \mathcal{J}$ (Lang 1965, Chapter VI). For such an ideal $(B,0) = \big(V(\mathcal{J}),0\big) \supset \big(V(\mathcal{P}),0\big)$, thus $\mathcal{I}_{B,0} \subset \mathcal{I}_{V(\mathcal{P}),0} = \mathcal{P}$ in view of Lemma 4.21. Therefore $\mathcal{I}_{B,0} \subset \bigcap_{\mathcal{P}\supset\mathcal{J}} \mathcal{P} = \sqrt{\mathcal{J}}$ and the Theorem is proved. □

In other words, if a germ $(B,0)$ is defined by an arbitrary ideal $\mathcal{J} \subset \mathcal{O}_n$ and if $f \in \mathcal{O}_n$ vanishes on $(B,0)$, then some power f^k lies in $\mathcal{J}$.

§4.3. Regular and Singular Points. Dimension

The above powerful results enable us to investigate the structure of singularities of an analytic set. We first give a few definitions.

(4.23) Definition. *Let $A \subset M$ be an analytic set and $x \in A$. We say that $x \in A$ is a regular point of A if $A \cap \Omega$ is a $\mathbb{C}$-analytic submanifold of Ω for some neighborhood Ω of x. Otherwise x is said to be singular. The corresponding subsets of A will be denoted respectively A_{reg} and A_{sing}.*

It is clear from the definition that A_{reg} is an open subset of A (thus A_{sing} is closed), and that the connected components of A_{reg} are $\mathbb{C}$-analytic submanifolds of M (non necessarily closed).

(4.24) Proposition. *If (A, x) is irreducible, there exist arbitrarily small neighborhoods Ω of x such that $A_{\mathrm{reg}} \cap \Omega$ is dense and connected in $A \cap \Omega$.*

Proof. Take $\Omega = \Delta$ as in Th. 4.19. Then $A_S \subset A_{\mathrm{reg}} \cap \Omega \subset A \cap \Omega$, where A_S is connected and dense in $A \cap \Omega$; hence $A_{\mathrm{reg}} \cap \Omega$ has the same properties. □

(4.25) Definition. *The dimension of an irreducible germ of analytic set (A, x) is defined by* $\dim(A, x) = \dim(A_{\mathrm{reg}}, x)$*. If (A, x) has several irreducible components (A_l, x), we set*

$$\dim(A, x) = \max\{\dim(A_l, x)\}, \qquad \operatorname{codim}(A, x) = n - \dim(A, x).$$

(4.26) Proposition. *Let $(B, x) \subset (A, x)$ be germs of analytic sets. If (A, x) is irreducible and $(B, x) \neq (A, x)$, then* $\dim(B, x) < \dim(A, x)$ *and $B \cap \Omega$ has empty interior in $A \cap \Omega$ for all sufficiently small neighborhoods Ω of x.*

Proof. We may assume $x = 0$, $(A, 0) \subset (\mathbb{C}^n, 0)$ and $(B, 0)$ irreducible. Then $\mathcal{I}_{A,0} \subset \mathcal{I}_{B,0}$ are prime ideals. When we choose suitable coordinates for the ramified coverings, we may at each step select vectors $e_n, e_{n-1}, \ldots$ that work simultaneously for A and B. If $\dim B = \dim A$, the process stops for both at the same time, i.e. we get ramified coverings

$$\pi : A \cap (\Delta' \times \Delta'') \longrightarrow \Delta', \qquad \pi : B \cap (\Delta' \times \Delta'') \longrightarrow \Delta'$$

with ramification loci S_A, S_B. Then $B \cap \big((\Delta' \smallsetminus (S_A \cup S_B)) \times \Delta''\big)$ is an open subset of the manifold $A_S = A \cap \big((\Delta' \smallsetminus S_A) \times \Delta''\big)$, therefore $B \cap A_S$ is an analytic subset of A_S with non empty interior. The same conclusion would hold if $B \cap \Delta$ had non empty interior in $A \cap \Delta$. As A_S is connected, we get $B \cap A_S = A_S$, and as $B \cap \Delta$ is closed in Δ we infer $B \cap \Delta \supset \overline{A}_S = A \cap \Delta$, hence $(B, 0) = (A, 0)$, in contradiction with the hypothesis. □

(4.27) Example: parametrization of curves. Suppose that $(A,0)$ is an irreducible germ of curve ($\dim(A,0)=1$). If the disk $\Delta' \subset \mathbb{C}$ is chosen so small that $S=\{0\}$, then A_S is a connected covering of $\Delta' \smallsetminus \{0\}$ with q sheets. Hence, there exists a covering isomorphism between π and the standard covering

$$\mathbb{C} \supset \Delta(r) \smallsetminus \{0\} \longrightarrow \Delta(r^q) \smallsetminus \{0\}, \qquad t \longmapsto t^q, \qquad r^q = \text{radius of } \ \Delta',$$

i.e. a map $\gamma : \Delta(r) \smallsetminus \{0\} \longrightarrow A_S$ such that $\pi \circ \gamma(t) = t^q$. This map extends into a bijective holomorphic map $\gamma : \Delta(r) \longrightarrow A \cap \Delta$ with $\gamma(0)=0$. This means that every irreducible germ of curve can be parametrized by a bijective holomorphic map defined on a disk in $\mathbb{C}$ (see also Exercise 10.8).

§4.4. Coherence of Ideal Sheaves

Let A be an analytic set in a complex manifold M. The *sheaf of ideals* $\mathcal{I}_A$ is the subsheaf of $\mathcal{O}_M$ consisting of germs of holomorphic functions on M which vanish on A. Its stalks are the ideals $\mathcal{I}_{A,x}$ already considered; note that $\mathcal{I}_{A,x} = \mathcal{O}_{M,x}$ if $x \notin A$. If $x \in A$, we let $\mathcal{O}_{A,x}$ be the ring of germs of functions on (A,x) which can be extended as germs of holomorphic functions on (M,x). By definition, there is a surjective morphism $\mathcal{O}_{M,x} \longrightarrow \mathcal{O}_{A,x}$ whose kernel is $\mathcal{I}_{A,x}$, thus

$$(4.28) \qquad \mathcal{O}_{A,x} = \mathcal{O}_{M,x}/\mathcal{I}_{A,x}, \qquad \forall x \in A,$$

i.e. $\mathcal{O}_A = (\mathcal{O}_M/\mathcal{I}_A)_{\restriction A}$. Since $\mathcal{I}_{A,x} = \mathcal{O}_{M,x}$ for $x \notin A$, the quotient sheaf $\mathcal{O}_M/\mathcal{I}_A$ is zero on $M \smallsetminus A$.

(4.29) Theorem (Cartan 1950)**.** *For any analytic set $A \subset M$, the sheaf of ideals $\mathcal{I}_A$ is a coherent analytic sheaf.*

Proof. It is sufficient to prove the result when A is an analytic subset in a neighborhood of 0 in $\mathbb{C}^n$. If $(A,0)$ is not irreducible, there exists a neighborhood Ω such that $A \cap \Omega = A_1 \cup \ldots \cup A_N$ where A_k are analytic sets in Ω and $(A_k,0)$ is irreducible. We have $\mathcal{I}_{A\cap\Omega} = \bigcap \mathcal{I}_{A_k}$, so by Cor. 3.15 we may assume that $(A,0)$ is irreducible. Then we can choose coordinates z', z'', polydisks Δ', Δ'' and a primitive element $u(z'') = c_{d+1}z_{d+1} + \cdots + c_n z_n$ such that Th. 4.19 is valid. Since $\delta(z') = \prod_{j<k}(\sigma_k\tilde{u} - \sigma_j\tilde{u})^2$, we see that $\delta(z')$ is in fact a polynomial in the c_j's with coefficients in $\mathcal{O}_d$. The same is true for the coefficients of the polynomials $W_u(z';T)$ and $B_k(z';T)$ which can be expressed in terms of the elementary symmetric functions of the $\sigma_k\tilde{u}$'s. We suppose that Δ' is chosen small enough in order that all coefficients of these $\mathcal{O}_d[c_{d+1},\ldots,c_n]$ polynomials are in $\mathcal{O}(\Delta')$. Let $\delta_\alpha \in \mathcal{O}(\Delta')$ be some non zero coefficient appearing in $\delta^m = \sum \delta_\alpha c^\alpha$. Also, let $G_1,\ldots,G_N \in \mathcal{O}(\Delta')[z'']$ be

the coefficients of all monomials c^α appearing in the expansion of the functions $W_u(z'\,;u(z''))$ or $\delta(z')z_k - B_k(z'\,;u(z''))$. Clearly, $G_1,\ldots,G_N$ vanish on $A\cap\Delta$. We contend that

$$(4.30)\qquad \mathcal{I}_{A,x} = \big\{f\in\mathcal{O}_{M,x}\ ;\ \delta_\alpha f\in(G_{1,x},\ldots,G_{N,x})\big\}.$$

This implies that the sheaf $\mathcal{I}_A$ is the projection on the first factor of the sheaf of relations $\mathcal{R}(\delta_\alpha,G_1,\ldots,G_N)\subset\mathcal{O}_\Delta^{N+1}$, which is coherent by the Oka theorem; Theorem 4.29 then follows.

We first prove that the inclusion $\mathcal{I}_{A,x}\supset\{\ldots\}$ holds in (4.30). In fact, if $\delta_\alpha f\in(G_{1,x},\ldots,G_{N,x})$, then f vanishes on $A\smallsetminus\{\delta_\alpha=0\}$ in some neighborhood of x. Since $(A\cap\Delta)\smallsetminus\{\delta_\alpha=0\}$ is dense in $A\cap\Delta$, we conclude that $f\in\mathcal{I}_{A,x}$.

To prove the other inclusion $\mathcal{I}_{A,x}\subset\{\ldots\}$, we repeat the proof of Lemma 4.18 with a few modifications. Let $x\in\Delta$ be a fixed point. At x, the irreducible polynomials $W_u(z'\,;T)$ and $W_k(z'\,;T)$ of $\tilde u$ and $\tilde z_k$ in $\mathcal{O}_{M,0}/\mathcal{I}_{A,0}$ split into

$$W_u(z'\,;T) = W_{u,x}\big(z'\,;T-u(x'')\big)\,Q_{u,x}\big(z'\,;T-u(x'')\big),$$
$$W_k(z'\,;T) = W_{k,x}(z'\,;T-x_k)\,Q_{k,x}(z'\,;T-x_k),$$

where $W_{u,x}(z'\,;T)$ and $W_{k,x}(z'\,;T)$ are Weierstrass polynomials in T and $Q_{u,x}(x',0)\neq0$, $Q_{k,x}(x',0)\neq0$. For all $z'\in\Delta'$, the roots of $W_u(z'\,;T)$ are the values $u(z'')$ at all points $z\in A\cap\pi^{-1}(z')$. As A is closed, any point $z\in A\cap\pi^{-1}(z')$ with z' near x' has to be in a small neighborhood of one of the points $y\in A\cap\pi^{-1}(x')$. Choose $c_{d+1},\ldots,c_n$ such that the linear form $u(z'')$ separates all points in the fiber $A\cap\pi^{-1}(x')$. Then, for a root $u(z'')$ of $W_{u,x}\big(z'\,;T-u(x'')\big)$, the point z must be in a neighborhood of $y=x$, otherwise $u(z'')$ would be near $u(y'')\neq u(x'')$ and the Weierstrass polynomial $W_{u,x}(z'\,;T)$ would have a root away from 0, in contradiction with (4.10). Conversely, if $z\in A\cap\pi^{-1}(z')$ is near x, then $Q_{u,x}\big(z'\,;u(z'')-u(x'')\big)\neq0$ and $u(z'')$ is a root of $W_{u,x}\big(z'\,;T-u(x'')\big)$. From this, we infer that every polynomial $P(z'\,;T)\in\mathcal{O}_{\Delta',x'}[T]$ such that $P\big(z'\,;u(z'')\big)=0$ on (A,x) is a multiple of $W_{u,x}\big(z'\,;T-u(x'')\big)$, because the roots of the latter polynomial are simple for z' in the dense set $(\Delta'\smallsetminus S,x)$. In particular $\deg P<\deg W_{u,x}$ implies $P=0$ and

$$\delta(z')^q W_{k,x}\big(z'\,;B_k(z'\,;u(z''))/\delta(z')-x_k\big)$$

is a multiple of $W_{u,x}\big(z'\,;T-u(x'')\big)$. If we replace W_u, W_k by $W_{u,x}$, $W_{k,x}$ respectively, the proof of Lemma 4.18 shows that for every $f\in\mathcal{O}_{M,x}$ there is a polynomial $R\in\mathcal{O}_{\Delta',x'}[T]$ of degree $\deg R<\deg W_{u,x}$ such that

$$\delta(z')^m f(z) = R\big(z'\,;u(z'')\big)\qquad\text{modulo the ideal}$$
$$\Big(\,W_{u,x}\big(z'\,;u(z'')-u(x'')\big),\ \delta(z')z_k - B_k\big(z'\,;u(z'')\big)\,\Big),$$

and $f \in \mathcal{I}_{A,x}$ implies $R = 0$. Since $W_{u,x}$ differs from W_u only by an invertible element in $\mathcal{O}_{M,x}$, we conclude that

$$\Big(\sum \delta_\alpha c^\alpha\Big)\mathcal{I}_{A,x} = \delta^m \mathcal{I}_{A,x} \subset (G_{1,x}, \ldots, G_{N,x}).$$

This is true for a dense open set of coefficients $c_{d+1}, \ldots, c_n$, therefore

$$\delta_\alpha \mathcal{I}_{A,x} \subset (G_{1,x}, \ldots, G_{N,x}) \quad \text{for all} \quad \alpha. \qquad \square$$

(4.31) Theorem. *A_{sing} is an analytic subset of A.*

Proof. The statement is local. Assume first that $(A, 0)$ is an irreducible germ in $\mathbb{C}^n$. Let $g_1, \ldots, g_N$ be generators of the sheaf $\mathcal{I}_A$ on a neighborhood Ω of 0. Set $d = \dim A$. In a neighborhood of every point $x \in A_{\mathrm{reg}} \cap \Omega$, A can be defined by holomorphic equations $u_1(z) = \ldots = u_{n-d}(z) = 0$ such that $du_1, \ldots, du_{n-d}$ are linearly independant. As $u_1, \ldots, u_{n-d}$ are generated by $g_1, \ldots, g_N$, one can extract a subfamily $g_{j_1}, \ldots, g_{j_{n-d}}$ that has at least one non zero Jacobian determinant of rank $n-d$ at x. Therefore $A_{\mathrm{sing}} \cap \Omega$ is defined by the equations

$$\det\Big(\frac{\partial g_j}{\partial z_k}\Big)_{\substack{j\in J\\ k\in K}} = 0, \qquad J \subset \{1, \ldots, N\}, \quad K \subset \{1, \ldots, n\}, \quad |J| = |K| = n - d.$$

Assume now that $(A, 0) = \bigcup (A_l, 0)$ with $(A_l, 0)$ irreducible. The germ of an analytic set at a regular point is irreducible, thus every point which belongs simultaneously to at least two components is singular. Hence

$$(A_{\mathrm{sing}}, 0) = \bigcup (A_{l,\mathrm{sing}}, 0) \cup \bigcup_{k\neq l} (A_k \cap A_l, 0),$$

and A_{sing} is analytic. $\square$

Now, we give a characterization of regular points in terms of a simple algebraic property of the ring $\mathcal{O}_{A,x}$.

(4.32) Proposition. *Let (A, x) be a germ of analytic set of dimension d and let $\mathfrak{m}_{A,x} \subset \mathcal{O}_{A,x}$ be the maximal ideal of functions that vanish at x. Then $\mathfrak{m}_{A,x}$ cannot have less than d generators and $\mathfrak{m}_{A,x}$ has d generators if and only if x is a regular point.*

Proof. If $A \subset \mathbb{C}^n$ is a d-dimensional submanifold in a neighborhood of x, there are local coordinates centered at x such that A is given by the equations $z_{d+1} = \ldots = z_n$ near $z = 0$. Then $\mathcal{O}_{A,x} \simeq \mathcal{O}_d$ and $\mathfrak{m}_{A,x}$ is generated by $z_1, \ldots, z_d$. Conversely, assume that $\mathfrak{m}_{A,x}$ has s generators $g_1(z), \ldots, g_s(z)$ in $\mathcal{O}_{A,x} = \mathcal{O}_{\mathbb{C}^n,x}/\mathcal{I}_{A,x}$. Letting $x = 0$ for simplicity, we can write

$$z_j = \sum_{1\le k\le s} u_{jk}(z)g_k(z) + f_j(z), \qquad u_{jk}\in \mathcal{O}_n,\quad f_j\in \mathcal{I}_{A,0},\quad 1\le j\le n.$$

Then we find $dz_j = \sum c_{jk}(0)dg_k(0) + df_j(0)$, so that the rank of the system of differentials $\big(df_j(0)\big)_{1\le j\le n}$ is at least equal to $n-s$. Assume for example that $df_1(0),\ldots,df_{n-s}(0)$ are linearly independent. By the implicit function theorem, the equations $f_1(z)=\ldots=f_{n-s}(z)=0$ define a germ of submanifold of dimension s containing $(A,0)$, thus $s\ge d$ and $(A,0)$ equals this submanifold if $s=d$. □

(4.33) Corollary. *Let $A\subset M$ be an analytic set of pure dimension d and let $B\subset A$ be an analytic subset of codimension $\ge p$ in A. Then, as an $\mathcal{O}_{A,x}$-module, the ideal $\mathcal{I}_{B,x}$ cannot be generated by less than p generators at any point $x\in B$, and by less than $p+1$ generators at any point $x\in B_{\mathrm{reg}}\cap A_{\mathrm{sing}}$.*

Proof. Suppose that $\mathcal{I}_{B,x}$ admits s-generators $(g_1,\ldots,g_s)$ at x. By coherence of $\mathcal{I}_B$ these germs also generate $\mathcal{I}_B$ in a neighborhood of x, so we may assume that x is a regular point of B. Then there are local coordinates $(z_1,\ldots,z_n)$ on M centered at x such that (B,x) is defined by $z_{k+1}=\ldots=z_n=0$, where $k=\dim(B,x)$. Then the maximal ideal $\mathfrak{m}_{B,x}=\mathfrak{m}_{A,x}/\mathcal{I}_{B,x}$ is generated by $z_1,\ldots,z_k$, so that $\mathfrak{m}_{A,x}$ is generated by $(z_1,\ldots,z_k,g_1,\ldots,g_s)$. By Prop. 4.32, we get $k+s\ge d$, thus $s\ge d-k\ge p$, and we have strict inequalities when $x\in A_{\mathrm{sing}}$. □

§5. Complex Spaces

Much in the same way a manifold is constructed by piecing together open patches isomorphic to open sets in a vector space, a complex space is obtained by gluing together open patches isomorphic to analytic subsets. The general concept of analytic morphism (or holomorphic map between analytic sets) is first needed.

§5.1. Morphisms and Comorphisms

Let $A\subset\Omega\subset\mathbb{C}^n$ and $B\subset\Omega'\subset\mathbb{C}^p$ be analytic sets. A morphism from A to B is by definition a map $F:A\longrightarrow B$ such that for every $x\in A$ there is a neighborhood U of x and a holomorphic map $\tilde F:U\longrightarrow\mathbb{C}^p$ such that $\tilde F_{\restriction A\cap U}=F_{\restriction A\cap U}$. Equivalently, such a morphism can be defined as a continuous map $F:A\longrightarrow B$ such that for all $x\in A$ and $g\in\mathcal{O}_{B,F(x)}$ we have $g\circ F\in\mathcal{O}_{A,x}$. The induced ring morphism

$$(5.1)\qquad F_x^\star\ :\ \ \mathcal{O}_{B,F(x)}\ni g\longmapsto g\circ F\in\mathcal{O}_{A,x}$$

is called the *comorphism* of F at point x.

§5.1. Definition of Complex Spaces

(5.2) Definition. *A complex space X is a locally compact Hausdorff space, countable at infinity, together with a sheaf $\mathcal{O}_X$ of continuous functions on X, such that there exists an open covering (U_λ) of X and for each λ a homeomorphism $F_\lambda : U_\lambda \longrightarrow A_\lambda$ onto an analytic set $A_\lambda \subset \Omega_\lambda \subset \mathbb{C}^{n_\lambda}$ such that the comorphism $F_\lambda^\star : \mathcal{O}_{A_\lambda} \longrightarrow \mathcal{O}_{X \restriction U_\lambda}$ is an isomorphism of sheaves of rings. $\mathcal{O}_X$ is called the structure sheaf of X.*

By definition a complex space X is locally isomorphic to an analytic set, so the concepts of holomorphic function on X, of analytic subset, of analytic morphism, etc ... are meaningful. If X is a complex space, Th. 4.31 implies that X_{sing} is an analytic subset of X.

(5.3) Theorem and definition. *For every complex space X, the set X_{reg} is a dense open subset of X, and consists of a disjoint union of connected complex manifolds X'_α. Let X_α be the closure of X'_α in X. Then (X_α) is a locally finite family of analytic subsets of X, and $X = \bigcup X_\alpha$. The sets X_α are called the global irreducible components of X.*

Observe that the germ at a given point of a global irreducible component can be reducible, as shows the example of the cubic curve $\Gamma : y^2 = x^2(1+x)$; the germ $(\Gamma, 0)$ has two analytic branches $y = \pm x\sqrt{1+x}$, however $\Gamma \smallsetminus \{0\}$ is easily seen to be a connected smooth Riemann surface (the real points of γ corresponding to $-1 \le x \le 0$ form a path connecting the two branches). This example shows that the notion of *global irreducible component* is quite different from the notion of local irreducible component introduced in (4.4).

Fig. 2 The irreducible curve $y^2 = x^2(1+x)$ in $\mathbb{C}^2$.

Proof. By definition of X_{reg}, the connected components X'_α are (disjoint) complex manifolds. Let us show that the germ of $X_\alpha = \overline{X}'_\alpha$ at any point $x \in X$ is analytic. We may assume that (X, x) is a germ of analytic set A in an open subset of $\mathbb{C}^n$. Let (A_l, x), $1 \le l \le N$, be the irreducible components of this germ and U a neighborhood of x such that $X \cap U = \bigcup A_l \cap U$. Let $\Omega_l \subset U$ be a neighborhood of x such that $A_{l,\mathrm{reg}} \cap \Omega_l$ is connected and dense in $A_l \cap \Omega_l$ (Prop. 4.24). Then $A'_l := X_{\mathrm{reg}} \cap A_l \cap \Omega_l$ equals $(A_{l,\mathrm{reg}} \cap \Omega_l) \smallsetminus \bigcup_{k \ne l} A_{l,\mathrm{reg}} \cap \Omega_l \cap A_k$. However, $A_{l,\mathrm{reg}} \cap \Omega_l \cap A_k$ is an analytic subset of $A_{l,\mathrm{reg}} \cap \Omega_l$, distinct from $A_{l,\mathrm{reg}} \cap \Omega_l$, otherwise $A_{l,\mathrm{reg}} \cap \Omega_l$ would be contained in A_k, thus $(A_l, x) \subset (A_k, x)$ by density. Remark 4.2 implies that A'_l is connected and dense in $A_{l,\mathrm{reg}} \cap \Omega_l$, hence in $A_l \cap \Omega_l$. Set $\Omega = \bigcap \Omega_l$ and let $(X_\alpha)_{\alpha \in J}$ be the family of global components which meet Ω (i.e. such that $X'_\alpha \cap \Omega \ne \emptyset$). As $X_{\mathrm{reg}} \cap \Omega = \bigcup A'_l \cap \Omega$, each X'_α, $\alpha \in J$, meets at least one set A'_l, and as $A'_l \subset X_{\mathrm{reg}}$ is connected, we have in fact $A'_l \subset X'_\alpha$. It follows that there exists a partition $(L_\alpha)_{\alpha \in J}$ of $\{1, \ldots, N\}$ such that $X'_\alpha \cap \Omega = \bigcup_{l \in L_\alpha} A'_l \cap \Omega$, $\alpha \in J$. Hence J is finite, $\operatorname{card} J \le N$, and

$$X_\alpha \cap \Omega = \overline{X}'_\alpha \cap \Omega = \bigcup_{l \in L_\alpha} \overline{A}'_l \cap \Omega = \bigcup_{l \in L_\alpha} A_l \cap \Omega$$

is analytic for all $\alpha \in J$. □

(5.4) Corollary. *If A, B are analytic subsets in a complex space X, then the closure $\overline{A \smallsetminus B}$ is an analytic subset, consisting of the union of all global irreducible components A_λ of A which are not contained in B.*

Proof. Let $C = \bigcup A_\lambda$ be the union of these components. Since (A_λ) is locally finite, C is analytic. Clearly $A \smallsetminus B = C \smallsetminus B = \bigcup A_\lambda \smallsetminus B$. The regular part A'_λ of each A_λ is a connected manifold and $A'_\lambda \cap B$ is a proper analytic subset (otherwise $A'_\lambda \subset B$ would imply $A_\lambda \subset B$). Thus $A'_\lambda \smallsetminus (A'_\lambda \cap B)$ is dense in A'_λ which is dense in A_λ, so $\overline{A \smallsetminus B} = \bigcup A_\lambda = C$. □

(5.5) Theorem. *For any family (A_λ) of analytic sets in a complex space X, the intersection $A = \bigcap A_\lambda$ is an analytic subset of X. Moreover, the intersection is stationary on every compact subset of X.*

Proof. It is sufficient to prove the last statement, namely that every point $x \in X$ has a neighborhood Ω such that $A \cap \Omega$ is already obtained as a finite intersection. However, since $\mathcal{O}_{X,x}$ is Noetherian, the family of germs of finite intersections has a minimum element (B, x), $B = \bigcap A_{\lambda_j}$, $1 \le j \le N$. Let $\tilde{B}$ be the union of the global irreducible components B_α of B which contain the point x ; clearly $(B, x) = (\tilde{B}, x)$. For any set A_λ in the family, the minimality of B implies $(B, x) \subset (A_\lambda, x)$. Let B'_α be the regular part of any global irreducible component B_α of $\tilde{B}$. Then $B'_\alpha \cap A_\lambda$ is a closed analytic subset of B'_α containing a non empty open subset (the intersection of B'_α with some neighborhood of x), so we must have $B'_\alpha \cap A_\lambda = B'_\alpha$. Hence $B_\alpha = \overline{B}'_\alpha \subset A_\lambda$ for all $B_\alpha \subset \tilde{B}$ and all A_λ, thus $\tilde{B} \subset A = \bigcap A_\lambda$. We infer

$$(B, x) = (\tilde{B}, x) \subset (A, x) \subset (B, x),$$

and the proof is complete. □

As a consequence of these general results, it is not difficult to show that a complex space always admits a (locally finite) stratification such that the strata are smooth manifolds.

(5.6) Proposition. *Let X be a complex space. Then there is a locally stationary increasing sequence of analytic subsets $Y_k \subset X$, $k \in \mathbb{N}$, such that Y_0 is a discrete set and such that $Y_k \smallsetminus Y_{k-1}$ is a smooth k-dimensional complex manifold for $k \geq 1$. Such a sequence is called a stratification of X, and the sets $Y_k \smallsetminus Y_{k-1}$ are called the strata (the strata may of course be empty for some indices $k < \dim X$).*

Proof. Let $\mathcal{F}$ be the family of irreducible analytic subsets $Z \subset X$ which can be obtained through a finite sequence of steps of the following types:

a) Z is an irreducible component of X;

b) Z is an irreducible component of the singular set Z'_{sing} of some member $Z' \in \mathcal{F}$;

c) Z is an irreducible component of some finite intersection of sets $Z_j \in \mathcal{F}$.

Since X has locally finite dimension and since steps b) or c) decrease the dimension of our sets Z, it is clear that $\mathcal{F}$ is a locally finite family of analytic sets in X. Let Y_k be the union of all sets $Z \in \mathcal{F}$ of dimension $\leq k$. It is easily seen that $\bigcup Y_k = X$ and that the irreducible components of $(Y_k)_{\mathrm{sing}}$ are contained in Y_{k-1} (these components are either intersections of components $Z_j \subset Y_k$ or parts of the singular set of some component $Z \subset Y_k$, so there are in either case obtained by step b) or c) above). Hence $Y_k \smallsetminus Y_{k-1}$ is a smooth manifold and it is of course k-dimensional, because the components of Y_k of dimension $< k$ are also contained in Y_{k-1} by definition.

(5.7) Theorem. *Let X be an irreducible complex space. Then every non constant holomorphic function f on X defines an open map $f : X \longrightarrow \mathbb{C}$.*

Proof. We show that the image $f(\Omega)$ of any neighborhood Ω of $x \in X$ contains a neighborhood of $f(x)$. Let (X_l, x) be an irreducible component of the germ (X, x) (embedded in $\mathbb{C}^n$) and $\Delta = \Delta' \times \Delta'' \subset \Omega$ a polydisk such that the projection $\pi : X_l \cap \Delta \longrightarrow \Delta'$ is a ramified covering. The function f is non constant on the dense open manifold X_{reg}, so we may select a complex line $L \subset \Delta'$ through 0, not contained in the ramification locus of π, such that f

is non constant on the one dimensional germ $\pi^{-1}(L)$. Therefore we can find a germ of curve

$$(\mathbb{C},0) \ni t \longmapsto \gamma(t) \in (X,x)$$

such that $f \circ \gamma$ is non constant. This implies that the image of every neighborhood of $0 \in \mathbb{C}$ by $f \circ \gamma$ already contains a neighborhood of $f(x)$. □

(5.8) Corollary. *If X is a compact irreducible analytic space, then every holomorphic function $f \in \mathcal{O}(X)$ is constant.*

In fact, if $f \in \mathcal{O}(X)$ was non constant, $f(X)$ would be compact and also open in $\mathbb{C}$ by Th. 5.7, a contradiction. This result implies immediately the following consequence.

(5.9) Theorem. *Let X be a complex space such that the global holomorphic functions in $\mathcal{O}(X)$ separate the points of X. Then every compact analytic subset A of X is finite.*

Proof. A has a finite number of irreducible components A_λ which are also compact. Every function $f \in \mathcal{O}(X)$ is constant on A_λ, so A_λ must be reduced to a single point. □

§5.2. Coherent Sheaves over Complex Spaces

Let X be a complex space and $\mathcal{O}_X$ its structure sheaf. Locally, X can be identified to an analytic set A in an open set $\Omega \subset \mathbb{C}^n$, and we have $\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{I}_A$. Thus $\mathcal{O}_X$ is coherent over the sheaf of rings $\mathcal{O}_\Omega$. It follows immediately that $\mathcal{O}_X$ is coherent over itself. Let $\mathcal{S}$ be a $\mathcal{O}_X$-module. If $\tilde{\mathcal{S}}$ denotes the extension of $\mathcal{S}_{\restriction A}$ to Ω obtained by setting $\tilde{\mathcal{S}}_x = 0$ for $x \in \Omega \smallsetminus A$, then $\tilde{\mathcal{S}}$ is a $\mathcal{O}_\Omega$-module, and it is easily seen that $\mathcal{S}_{\restriction A}$ is coherent over $\mathcal{O}_{X\restriction A}$ if and only if $\tilde{\mathcal{S}}$ is coherent over $\mathcal{O}_\Omega$. If Y is an analytic subset of X, then Y is locally given by an analytic subset B of A and the sheaf of ideals of Y in $\mathcal{O}_X$ is the quotient $\mathcal{I}_Y = \mathcal{I}_B/\mathcal{I}_A$; hence $\mathcal{I}_Y$ is coherent. Let us mention the following important property of supports.

(5.10) Theorem. *If $\mathcal{S}$ is a coherent $\mathcal{O}_X$-module, the support of $\mathcal{S}$, defined as* $\operatorname{Supp} \mathcal{S} = \{x \in X \,;\, \mathcal{S}_x \neq 0\}$ *is an analytic subset of X.*

Proof. The result is local, thus after extending $\mathcal{S}$ by 0, we may as well assume that X is an open subset $\Omega \subset \mathbb{C}^n$. By (3.12), there is an exact sequence of sheaves

$$\mathcal{O}_U^{\oplus p} \xrightarrow{G} \mathcal{O}_U^{\oplus q} \xrightarrow{F} \mathcal{S}_{\restriction U} \longrightarrow 0$$

in a neighborhood U of any point. If $G : \mathcal{O}_x^{\oplus p} \longrightarrow \mathcal{O}_x^{\oplus q}$ is surjective it is clear that the linear map $G(x) : \mathbb{C}^p \longrightarrow \mathbb{C}^q$ must be surjective; conversely, if $G(x)$ is surjective, there is a q-dimensional subspace $E \subset \mathbb{C}^p$ on which the restriction of $G(x)$ is a bijection onto $\mathbb{C}^q$; then $G_{\restriction E} : \mathcal{O}_U \otimes_{\mathbb{C}} E \longrightarrow \mathcal{O}_U^{\oplus q}$ is bijective near x and G is surjective. The support of $\mathcal{S}_{\restriction U}$ is thus equal to the set of points $x \in U$ such that all minors of $G(x)$ of order q vanish. □

§6. Analytic Cycles and Meromorphic Functions

§6.1. Complete Intersections

Our goal is to study in more details the dimension of a subspace given by a set of equations. The following proposition is our starting point.

(6.1) Proposition. *Let X be a complex space of pure dimension p and A an analytic subset of X with* $\operatorname{codim}_X A \geq 2$. *Then every function $f \in \mathcal{O}(X \smallsetminus A)$ is locally bounded near A.*

Proof. The statement is local on X, so we may assume that X is an irreducible germ of analytic set in $(\mathbb{C}^n, 0)$. Let $(A_k, 0)$ be the irreducible components of $(A, 0)$. By a reasoning analogous to that of Prop. 4.26, we can choose coordinates $(z_1, \ldots, z_n)$ on $\mathbb{C}^n$ such that all projections

$$\pi : z \longmapsto (z_1, \ldots, z_p), \quad p = \dim X,$$
$$\pi_k : z \longmapsto (z_1, \ldots, z_{p_k}), \quad p_k = \dim A_k\,,$$

define ramified coverings $\pi : X \cap \Delta \longrightarrow \Delta'$, $\pi_k : A_k \cap \Delta \longrightarrow \Delta'_k$. By construction $\pi(A_k) \subset \Delta'$ is contained in the set B_k defined by some Weierstrass polynomials in the variables $z_{p_k+1}, \ldots, z_p$ and $\operatorname{codim}_{\Delta'} B_k = p - p_k \geq 2$. Let S be the ramification locus of π and $B = \bigcup B_k$. We have $\pi(A \cap \Delta) \subset B$. For $z' \in \Delta' \smallsetminus (S \cup B)$, we let

$$\sigma_k(z') = \text{elementary symmetric function of degree } k \text{ in } f(z', z''_\alpha),$$

where (z', z''_α) are the q points of X projecting on z'. Then σ_k is holomorphic on $\Delta' \smallsetminus (S \cup B)$ and locally bounded near every point of $S \smallsetminus B$, thus σ_k extends holomorphically to $\Delta' \smallsetminus B$ by Remark 4.2. Since $\operatorname{codim} B \geq 2$, σ_k extends to Δ' by Cor. 1.4.5. Now, f satisfies $f^q - \sigma_1 f^{q-1} + \ldots + (-1)^q \sigma_q = 0$, thus f is locally bounded on $X \cap \Delta$. □

(6.2) Theorem. *Let X be an irreducible complex space and $f \in \mathcal{O}(X)$, $f \not\equiv 0$. Then $f^{-1}(0)$ is empty or of pure dimension* $\dim X - 1$.

Proof. Let $A = f^{-1}(0)$. By Prop. 4.26, we know that $\dim A \leq \dim X - 1$. If A had an irreducible branch A_j of dimension $\leq \dim X - 2$, then in virtue of Prop. 6.1 the function $1/f$ would be bounded in a neighborhood of $A_j \smallsetminus \bigcup_{k\neq j} A_k$, a contradiction. □

(6.3) Corollary. *If $f_1, \ldots, f_p$ are holomorphic functions on an irreducible complex space X, then all irreducible components of $f_1^{-1}(0) \cap \ldots \cap f_p^{-1}(0)$ have codimension $\geq p$.* □

(6.4) Definition. *Let X be a complex space of pure dimension n and A an analytic subset of X of pure dimension. Then A is said to be a local (set theoretic) complete intersection in X if every point of A has a neighborhood Ω such that*

$$A \cap \Omega = \{x \in \Omega\,;\, f_1(x) = \ldots = f_p(x) = 0\}$$

with exactly $p = \operatorname{codim} A$ functions $f_j \in \mathcal{O}(\Omega)$.

(6.5) Remark. As a converse to Th. 6.2, one may ask whether every hypersurface A in X is locally defined by a single equation $f = 0$. In general the answer is negative. A simple counterexample for $\dim X = 3$ is obtained with the singular quadric $X = \{z_1 z_2 + z_3 z_4 = 0\} \subset \mathbb{C}^4$ and the plane $A = \{z_1 = z_3 = 0\} \subset X$. Then A cannot be defined by a single equation $f = 0$ near the origin, otherwise the plane $B = \{z_2 = z_4 = 0\}$ would be such that

$$f^{-1}(0) \cap B = A \cap B = \{0\},$$

in contradiction with Th. 6.2 (also, by Exercise 10.11, we would get the inequality $\operatorname{codim}_X A \cap B \leq 2$). However, the answer is positive when X is a manifold:

(6.6) Theorem. *Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, let (A, x) be an analytic germ of pure dimension $n-1$ and let A_j, $1 \leq j \leq N$, be its irreducible components.*

a) *The ideal of (A, x) is a principal ideal $\mathcal{I}_{A,x} = (g)$ where g is a product of irreducible germs g_j such that $\mathcal{I}_{A_j,x} = (g_j)$.*

b) *For every $f \in \mathcal{O}_{M,x}$ such that $f^{-1}(0) \subset (A, x)$, there is a unique decomposition $f = u g_1^{m_1} \ldots g_N^{m_N}$ where u is an invertible germ and m_j is the order of vanishing of f at any point $z \in A_{j,\mathrm{reg}} \smallsetminus \bigcup_{k\neq j} A_k$.*

Proof. a) In a suitable local coordinate system centered at x, the projection $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ realizes all A_j as ramified coverings

$$\pi : A_j \cap \Delta \longrightarrow \Delta' \subset \mathbb{C}^{n-1}, \qquad \text{ramification locus} = S_j \subset \Delta'.$$

The function

$$g_j(z', z_n) = \prod_{w \in A_j \cap \pi^{-1}(z')} (z_n - w_n), \qquad z' \in \Delta' \smallsetminus S_j$$

extends into a holomorphic function in $\mathcal{O}_{\Delta'}[z_n]$ and is irreducible at x. Set $g = \prod g_j \in \mathcal{I}_{A,x}$. For any $f \in \mathcal{I}_{A,x}$, the Weierstrass division theorem yields $f = gQ + R$ with $R \in \mathcal{O}_{n-1}[z_n]$ and $\deg R < \deg g$. As $R(z', z_n)$ vanishes when z_n is equal to w_n for each point $w \in A \cap \pi^{-1}(z')$, R has exactly $\deg g$ roots when $z' \in \Delta' \smallsetminus \big(\bigcup S_j \cup \bigcup \pi(A_j \cap A_k)\big)$, so $R = 0$. Hence $\mathcal{I}_{A,x} = (g)$ and similarly $\mathcal{I}_{A_j,x} = (g_j)$. Since $\mathcal{I}_{A_j}$ is coherent, g_j is also a generator of $\mathcal{I}_{A_j,z}$ for z near x and we infer that g_j has order 1 at any regular point $z \in A_{j,\mathrm{reg}}$.

b) As $\mathcal{O}_{M,x}$ is factorial, any $f \in \mathcal{O}_{M,x}$ can be written $f = u\, g_1^{m_1} \ldots g_N^{m_N}$ where u is either invertible or a product of irreducible elements distinct from the g_j's. In the latter case the hypersurface $u^{-1}(0)$ cannot be contained in (A, x), otherwise it would be a union of some of the components A_j and u would be divisible by some g_j. This proves b). □

(6.7) Definition. *Let X be an complex space of pure dimension n.*

a) *An analytic q-cycle Z on X is a formal linear combination $\sum \lambda_j A_j$ where (A_j) is a locally finite family of irreducible analytic sets of dimension q in X and $\lambda_j \in \mathbb{Z}$. The support of Z is $|Z| = \bigcup_{\lambda_j \neq 0} A_j$. The group of all q-cycles on X is denoted* $\mathrm{Cycl}^q(X)$. *Effective q-cycles are elements of the subset* $\mathrm{Cycl}^q_+(X)$ *of cycles such that all coefficients λ_j are ≥ 0 ; rational, real cycles are cycles with coefficients $\lambda_j \in \mathbb{Q}$, $\mathbb{R}$.*

b) *An analytic $(n-1)$-cycle is called a (Weil) divisor, and we set*

$$\mathrm{Div}(X) = \mathrm{Cycl}^{n-1}(X).$$

c) *Assume that* $\dim X_{\mathrm{sing}} \leq n - 2$. *If $f \in \mathcal{O}(X)$ does not vanish identically on any irreducible component of X, we associate to f a divisor*

$$\mathrm{div}(f) = \sum m_j A_j \in \mathrm{Div}_+(X)$$

in the following way: the components A_j are the irreducible components of $f^{-1}(0)$ and the coefficient m_j is the vanishing order of f at every regular point in $X_{\mathrm{reg}} \cap A_{j,\mathrm{reg}} \smallsetminus \bigcup_{k \neq j} A_k$. It is clear that we have

$$\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g).$$

d) *A Cartier divisor is a divisor $D = \sum \lambda_j A_j$ that is equal locally to a $\mathbb{Z}$-linear combination of divisors of the form* $\mathrm{div}(f)$.

It is easy to check that the collection of abelian groups $\mathrm{Cycl}^q(U)$ over all open sets $U \subset X$, together with the obvious restriction morphisms, satisfies axioms (1.4) of sheaves; observe however that the restriction of an irreducible component A_j to a smaller open set may subdivide in several components.

Hence we obtain sheaves of abelian groups Cycl^q and $\mathrm{Div} = \mathrm{Cycl}^{n-1}$ on X. The stalk Cycl^q_x is the free abelian group generated by the set of irreducible germs of q-dimensional analytic sets at the point x. These sheaves carry a natural partial ordering determined by the subsheaf of positive elements Cycl^q_+. We define the sup and inf of two analytic cycles $Z = \sum \lambda_j A_j$, $Z' = \sum \mu_j A_j$ by

$$(6.8)\quad \sup\{Z, Z'\} = \sum \sup\{\lambda_j, \mu_j\}\, A_j, \quad \inf\{Z, Z'\} = \sum \inf\{\lambda_j, \mu_j\}\, A_j\,;$$

it is clear that these operations are compatible with restrictions, i.e. they are defined as sheaf operations.

(6.9) Remark. When X is a manifold, Th. 6.6 shows that every effective $\mathbb{Z}$-divisor is locally the divisor of a holomorphic function; thus, for manifolds, the concepts of Weil and Cartier divisors coincide. This is not always the case in general: in Example 6.5, one can show that A is not a Cartier divisor (exercise 10.?).

§6.2. Divisors and Meromorphic Functions

Let X be a complex space. For $x \in X$, let $\mathcal{M}_{X,x}$ be the ring of quotients of $\mathcal{O}_{X,x}$, i.e. the set of formal quotients g/h, $g, h \in \mathcal{O}_{X,x}$, where h is not a zero divisor in $\mathcal{O}_{X,x}$, with the identification $g/h = g'/h'$ if $gh' = g'h$. We consider the disjoint union

$$(6.10)\quad \mathcal{M}_X = \coprod_{x \in X} \mathcal{M}_{X,x}$$

with the topology in which the open sets open sets are unions of sets of the type $\{G_x/H_x\,;\, x \in V\} \subset \mathcal{M}_X$ where V is open in X and $G, H \in \mathcal{O}_X(V)$. Then $\mathcal{M}_X$ is a sheaf over X, and the sections of $\mathcal{M}_X$ over an open set U are called *meromorphic functions* on U. By definition, these sections can be represented locally as quotients of holomorphic functions, but there need not exist such a global representation on U.

A point $x \in X$ is called a *pole* of a meromorphic function f on X if $f_x \notin \mathcal{O}_{X,x}$. Clearly, the set P_f of poles of f is a closed subset of X with empty interior: if $f = g/h$ on U, then $h \not\equiv 0$ on any irreducible component and $P_f \cap U \subset h^{-1}(0)$. For $x \notin P_f$, one can speak of *the value* $f(x)$. If the restriction of f to $X_{\mathrm{reg}} \smallsetminus P_f$ does not vanish identically on any irreducible component of (X, x), then $1/f$ is a meromorphic function in a neighborhood of x ; the set of poles of $1/f$ will be denoted Z_f and called the *zero set* of f. If f vanishes on some connected open subset of $X_{\mathrm{reg}} \smallsetminus P_f$, then f vanishes identically (outside P_f) on the global irreducible component X_α containing this set; we agree that these components X_α are contained in Z_f. For every point x in the complement of $Z_f \cap P_f$, we have either $f_x \in \mathcal{O}_{X,x}$ or $(1/f)_x \in \mathcal{O}_{X,x}$, thus

f defines a holomorphic map $X \smallsetminus (Z_f \cap P_f) \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$ with values in the projective line. In general, no value (finite or infinite) can be assigned to f at a point $x \in Z_f \cap P_f$, as shows the example of the function $f(z) = z_2/z_1$ in $\mathbb{C}^2$. The set $Z_f \cap P_f$ is called the *indeterminacy set* of f.

(6.11) Theorem. *For every meromorphic function f on X, the sets P_f, Z_f and the indeterminacy set $Z_f \cap P_f$ are analytic subsets.*

Proof. Let $\mathcal{J}_x$ be the ideal of germs $u \in \mathcal{O}_{X,x}$ such that $uf_x \in \mathcal{O}_{X,x}$. Let us write $f = g/h$ on a small open set U. Then $\mathcal{J}_{\restriction U}$ appears as the projection on the first factor of the sheaf of relations $\mathcal{R}(g,h) \subset \mathcal{O}_U \times \mathcal{O}_U$, so $\mathcal{J}$ is a coherent sheaf of ideals. Now

$$P_f = \big\{x \in X\,;\, \mathcal{J}_x = \mathcal{O}_{X,x}\big\} = \mathrm{Supp}\ \mathcal{O}_X/\mathcal{J},$$

thus P_f is analytic by Th. 5.10. Similarly, the projection of $\mathcal{R}(g,h)$ on the second factor defines a sheaf of ideals $\mathcal{J}'$ such that $Z_f = \mathrm{Supp}\ \mathcal{O}_X/\mathcal{J}'$. □

When X has pure dimension n and $\dim X_{\mathrm{sing}} \le n-2$, Def. 6.7 c) can be extended to meromorphic functions: if $f = g/h$ locally, we set

$$(6.12)\quad \mathrm{div}(f) = \mathrm{div}(g) - \mathrm{div}(h).$$

By 6.7 c), we immediately see that this definition does not depend on the choice of the local representant g/h. Furthermore, *Cartier divisors* are precisely those divisors which are associated locally to meromorphic functions.

Assume from now on that M is a connected n-dimensional complex manifold. Then, for every point $x \in M$, the ring $\mathcal{O}_{M,x} \simeq \mathcal{O}_n$ is factorial. This property makes the study of meromorphic functions much easier.

(6.13) Theorem. *Let f be a non zero meromorphic function on a manifold M, $\dim_{\mathbb{C}} M = n$. Then the sets Z_f, P_f are purely $(n-1)$-dimensional, and the indeterminacy set $Z_f \cap P_f$ is purely $(n-2)$-dimensional.*

Proof. For every point $a \in M$, the germ f_a can be written g_a/h_a where $g_a, h_a \in \mathcal{O}_{M,a}$ are relatively prime holomorphic germs. By Th. 1.12, the germs g_x, h_x are still relatively prime for x in a neighborhood U of a. Thus the ideal $\mathcal{J}$ associated to f coincides with (h) on U, and we have

$$P_f \cap U = \mathrm{Supp}\ \mathcal{O}_U/(h) = h^{-1}(0), \qquad Z_f \cap U = g^{-1}(0).$$

Th. 6.2 implies our contentions: if g_λ and h_μ are the irreducible components of g, h, then $Z_f \cap P_f = \bigcup g_\lambda^{-1}(0) \cap h_\mu^{-1}(0)$ is $(n-2)$-dimensional. As we will see in the next section, Th. 6.13 does not hold on an arbitrary complex space. □

Let (A_j), resp. (B_j), be the global irreducible components of Z_f, resp. P_f. In a neighborhood V_j of the $(n-1)$-dimensional analytic set

$$A'_j = A_j \smallsetminus \big(P_f \cup \bigcup_{k\neq j} A_k\big)$$

f is holomorphic and $V \cap f^{-1}(0) = A'_j$. As $A'_{j,\mathrm{reg}}$ is connected, we must have $\mathrm{div}(f_{\restriction V_j}) = m_j A'_j$ for some constant multiplicity m_j equal to the vanishing order of f along $A'_{j,\mathrm{reg}}$. Similarly, $1/f$ is holomorphic in a neighborhood W_j of

$$B'_j = B_j \smallsetminus \big(Z_f \cup \bigcup_{k\neq j} B_k\big)$$

and we have $\mathrm{div}(f_{\restriction V}) = -p_j B'_j$ where p_j is the vanishing order of $1/f$ along $B'_{j,\mathrm{reg}}$. At a point $x \in M$ the germs $A_{j,x}$ and $B_{j,x}$ may subdivide in irreducible local components $A_{j,\lambda,x}$ and $B_{j,\lambda,x}$. If $g_{j,\lambda}$ and $h_{j,\lambda}$ are local generators of the corresponding ideals, we may a priori write

$$f_x = u\,g/h \quad \text{where} \quad g = \prod g_{j,\lambda}^{m_{j,\lambda}}, \quad h = \prod h_{j,\lambda}^{p_{j,\lambda}}$$

and where u is invertible. Then necessarily $m_{j,\lambda} = m_j$ and $p_{j,\lambda} = p_j$ for all λ, and we see that the global divisor of f on M is

$$\text{(6.14)} \quad \mathrm{div}(f) = \sum m_j A_j - \sum p_j B_j.$$

Let us denote by $\mathcal{M}^\star$ the multiplicative sheaf of germs of non zero meromorphic functions, and by $\mathcal{O}^\star$ the sheaf of germs of invertible holomorphic functions. Then we have an exact sequence of sheaves

$$\text{(6.15)} \quad 1 \longrightarrow \mathcal{O}^\star \longrightarrow \mathcal{M}^\star \xrightarrow{\mathrm{div}} \mathrm{Div} \longrightarrow 0.$$

Indeed, the surjectivity of div is a consequence of Th. 6.6. Moreover, any meromorphic function that has a positive divisor must be holomorphic by the fact that $\mathcal{O}_n$ is factorial. Hence a meromorphic function f with $\mathrm{div}(f) = 0$ is an invertible holomorphic function.

§7. Normal Spaces and Normalization

§7.1. Weakly Holomorphic Functions

The goal of this section is to show that the singularities of X can be studied by enlarging the structure sheaf $\mathcal{O}_X$ into a sheaf $\widetilde{\mathcal{O}}_X$ of so-called weakly holomorphic functions.

(7.1) Definition. *Let X be a complex space. A weakly holomorphic function f on X is a holomorphic function on X_{reg} such that every point of X_{sing} has a neighborhood V for which f is bounded on $X_{\mathrm{reg}} \cap V$. We denote by $\tilde{\mathcal{O}}_{X,x}$ the ring of germs of weakly holomorphic functions over neighborhoods of x and $\tilde{\mathcal{O}}_X$ the associated sheaf.*

Clearly, $\tilde{\mathcal{O}}_{X,x}$ is a ring containing $\mathcal{O}_{X,x}$. If (X_j, x) are the irreducible components of (X, x), there is a fundamental system of neighborhoods V of x such that $X_{\mathrm{reg}} \cap V$ is a disjoint union of connected open sets

$$X_{j,\mathrm{reg}} \cap V \smallsetminus \bigcup_{k \neq j} X_k \cap X_{j,\mathrm{reg}} \cap V$$

which are dense in $X_{j,\mathrm{reg}} \cap V$. Therefore any bounded holomorphic function on $X_{\mathrm{reg}} \cap V$ extends to each component $X_{j,\mathrm{reg}} \cap V$ and we see that

$$\tilde{\mathcal{O}}_{X,x} = \bigoplus \tilde{\mathcal{O}}_{X_j,x}.$$

The first important fact is that weakly holomorphic functions are always meromorphic and possess "universal denominators".

(7.2) Theorem. *For every point $x \in X$, there is a neighborhood V of x and $h \in \mathcal{O}_X(V)$ such that $h^{-1}(0)$ is nowhere dense in V and $h_y \tilde{\mathcal{O}}_{X,y} \subset \mathcal{O}_{X,y}$ for all $y \in V$; such a function h is called a universal denominator on V. In particular $\tilde{\mathcal{O}}_X$ is contained in the ring $\mathcal{M}_X$ of meromorphic functions.*

Proof. First assume that (X, x) is irreducible and that we have a ramified covering $\pi : X \cap \Delta \longrightarrow \Delta'$ with ramification locus S. We claim that the discriminant $\delta(z')$ of a primitive element $u(z'') = c_{d+1}z_{d+1} + \cdots + c_n z_n$ is a universal denominator on $X \cap \Delta$. To see this, we imitate the proof of Lemma 4.15. Let $f \in \tilde{\mathcal{O}}_{X,y}$, $y \in X \cap \Delta$. Then we solve the equation

$$f(z) = \sum_{0 \leq j \leq q} b_j(z')u(z'')^j$$

in a neighborhood of y. For $z' \in \Delta' \smallsetminus S$, let us denote by (z', z''_α), $1 \leq \alpha \leq q$, the points in the fiber $X \cap \pi^{-1}(z')$. Among these, only q' are close to y, where q' is the sum of the sheet numbers of the irreducible components of (X, y) by the projection π. The other points (z', z''_α), say $q' < \alpha \leq q$, are in neighborhoods of the points of $\pi^{-1}(y') \smallsetminus \{y\}$. We take $\big(b_j(z')\big)$ to be the solution of the linear system

$$\sum_{0 \leq j \leq q} b_j(z')u(z''_\alpha)^j = \begin{cases} f(z', z''_\alpha) & \text{for } \ 1 \leq \alpha \leq q', \\ 0 & \text{for } \ q' < \alpha \leq n. \end{cases}$$

The solutions $b_j(z')$ are holomorphic on $\Delta' \smallsetminus S$ near y'. Since the determinant is $\delta(z')^{1/2}$, we see that δb_j is bounded, thus $\delta b_j \in \mathcal{O}_{\Delta',y'}$ and $\delta_y f \in \mathcal{O}_{X,y}$.

Now, assume that $(X,x) \subset (\mathbb{C}^n, 0)$ has irreducible components (X_j, x). We can find for each j a neighborhood Ω_j of 0 in $\mathbb{C}^n$ and a function $\delta_j \in \mathcal{O}_n(\Omega_j)$ which is a universal denominator on $X_j \cap \Omega_j$. After adding to δ_j a function which is identically zero on (X_j, x) and non zero on (X_k, x), $k \neq j$, we may assume that $\delta_j^{-1}(0) \cap X_k \cap \Omega$ is nowhere dense in $X_k \cap \Omega$ for all j and k and some small $\Omega \subset \bigcap \Omega_j$. Then $\delta = \prod \delta_j$ is a universal denominator on each component $X_j \cap \Omega$. For some possibly smaller Ω, select a function $v_j \in \mathcal{O}_n(\Omega)$ such that v_j vanishes identically on $\bigcup_{k \neq j} X_k \cap \Omega$ and $v_j^{-1}(0)$ is nowhere dense in $X_j \cap \Omega$, and set $h = \delta \sum v_k$. For any germ $f \in \tilde{\mathcal{O}}_{X,y}$, $y \in X \cap \Omega$, there is a germ $g_j \in \mathcal{O}_{\Omega,y}$ with $\delta f = g_j$ on (X_j, y). We have $h = \delta v_j$ on $X_j \cap \Omega$, so $h^{-1}(0)$ is nowhere dense in $X \cap \Omega$ and

$$hf = v_j \delta f = v_j g_j = \sum v_k g_k \quad \text{on each} \quad (X_j, y).$$

Since $\sum v_k g_k \in \mathcal{O}_{\Omega,y}$, we get $h\tilde{\mathcal{O}}_{X,y} \subset \mathcal{O}_{X,y}$. □

(7.3) Theorem. *If (X,x) is irreducible, $\tilde{\mathcal{O}}_{X,x}$ is the integral closure of $\mathcal{O}_{X,x}$ in its quotient field $\mathcal{M}_{X,x}$. Moreover, every germ $f \in \tilde{\mathcal{O}}_{X,x}$ admits a limit*

$$\lim_{X_{\mathrm{reg}} \ni z \to x} f(z).$$

Observe that $\mathcal{O}_{X,x}$ is an entire ring, so the ring of quotients $\mathcal{M}_{X,x}$ is actually a field. A simple illustration of the theorem is obtained with the irreducible germ of curve $X : z_1^3 = z_2^2$ in $(\mathbb{C}^2, 0)$. Then X can be parametrized by $z_1 = t^2$, $z_2 = t^3$, $t \in \mathbb{C}$, and $\mathcal{O}_{X,0} = \mathbb{C}\{z_1, z_2\}/(z_1^3 - z_2^2) = \mathbb{C}\{t^2, t^3\}$ consists of all convergent series $\sum a_n t^n$ with $a_1 = 0$. The function $z_2/z_1 = t$ is weakly holomorphic on X and satisfies the integral equation $t^2 - z_1 = 0$. Here we have $\tilde{\mathcal{O}}_{X,0} = \mathbb{C}\{t\}$.

Proof. a) Let $f = g/h$ be an element in $\mathcal{M}_{X,x}$ satisfying an integral equation

$$f^m + a_1 f^{m-1} + \ldots + a_m = 0, \qquad a_k \in \mathcal{O}_{X,x}.$$

Set $A = h^{-1}(0)$. Then f is holomorphic on $X \smallsetminus A$ near x, and Lemma 4.10 shows that f is bounded on a neighborhood of x. By Remark 4.2, f can be extended as a holomorphic function on X_{reg} in a neighborhood of x, thus $f \in \tilde{\mathcal{O}}_{X,x}$.

b) Let $f \in \tilde{\mathcal{O}}_{X,x}$ and let $\pi : X \cap \Delta \longrightarrow \Delta'$ be a ramified covering in a neighborhood of x, with ramification locus S. As in the proof of Th. 6.1, f satisfies an equation

$$f^q - \sigma_1 f^{q-1} + \cdots + (-1)^q \sigma_q = 0, \qquad \sigma_k \in \mathcal{O}(\Delta') \ ;$$

indeed the elementary symmetric functions $\sigma_k(z')$ are holomorphic on $\Delta' \smallsetminus S$ and bounded, so they extend holomorphically to Δ'. Hence $\widetilde{\mathcal{O}}_{X,x}$ is integral over $\mathcal{O}_{X,x}$ and we already know that $\widetilde{\mathcal{O}}_{X,x} \subset \mathcal{M}_{X,x}$.

c) Finally, the cluster set $\bigcap_{V \ni x} \overline{f(X_{\mathrm{reg}} \cap V)}$ is connected, because there is a fundamental system of neighborhoods V of x such that $X_{\mathrm{reg}} \cap V$ is connected, and any intersection of a decreasing sequence of compact connected sets is connected. However the limit set is contained in the finite set of roots of equation b) at point $x' \in \Delta'$, so it must be reduced to one element. □

§7.2. Normal Spaces

Normal spaces are spaces for which all weakly holomorphic functions are actually holomorphic. These spaces will be seen later to have "simpler" singularities than general analytic spaces.

(7.4) Definition. *A complex space X is said to be normal at a point x if (X, x) is irreducible and $\widetilde{\mathcal{O}}_{X,x} = \mathcal{O}_{X,x}$, that is, $\mathcal{O}_{X,x}$ is integrally closed in its field of quotients. The set of normal* (*resp. non-normal*) *points will be denoted X_{norm}* (*resp. $X_{\text{n-n}}$*). *The space X itself is said to be normal if X is normal at every point.*

Observe that any regular point x is normal: in fact $\mathcal{O}_{X,x} \simeq \mathcal{O}_n$ is then factorial, hence integrally closed. Therefore $X_{\text{n-n}} \subset X_{\text{sing}}$.

(7.5) Theorem. *The non-normal set $X_{\text{n-n}}$ is an analytic subset of X. In particular, X_{norm} is open in X.*

Proof. We give here a beautifully simple proof due to (Grauert and Remmert 1984). Let h be a universal denominator on a neighborhood V of a given point and let $\mathcal{J} = \sqrt{h\mathcal{O}_X}$ be the sheaf of ideals of $h^{-1}(0)$ by Hilbert's Nullstellensatz. Finally, let $\mathcal{F} = \mathrm{hom}_{\mathcal{O}}(\mathcal{J}, \mathcal{J})$ be the sheaf of $\mathcal{O}_X$-endomorphisms of $\mathcal{J}$. Since $\mathcal{J}$ is coherent, so is $\mathcal{F}$ (cf. Exercise 10.?). Clearly, the homotheties of $\mathcal{J}$ give an injection $\mathcal{O}_X \subset \mathcal{F}$ over V. We claim that there is a natural injection $\mathcal{F} \subset \widetilde{\mathcal{O}}_X$. In fact, any endomorphism of $\mathcal{J}$ yields by restriction a homomorphism $h\mathcal{O}_X \longrightarrow \mathcal{O}_X$, and by $\mathcal{O}_X$-linearity such a homomorphism is obtained by multiplication by an element in $h^{-1}\mathcal{O}_X$. Thus $\mathcal{F} \subset h^{-1}\mathcal{O}_X \subset \mathcal{M}_X$. Since each stalk $\mathcal{J}_x$ is a finite $\mathcal{O}_{X,x}$-module containing non-zero divisors, it follows that that any meromorphic germ f such that $f\mathcal{J}_x \subset \mathcal{J}_x$ is integral over $\mathcal{O}_{X,x}$ (Lang 1965, Chapter IX, §1), hence $\mathcal{F}_x \subset \widetilde{\mathcal{O}}_{X,x}$. Thus we have inclusions $\mathcal{O}_X \subset \mathcal{F} \subset \widetilde{\mathcal{O}}_X$. Now, we assert that

$$X_{\text{n-n}} \cap V = \{x \in V\,;\, \mathcal{F}_x \neq \mathcal{O}_{X,x}\} = \mathcal{F}/\mathcal{O}_X.$$

This will imply the theorem by 5.10. To prove the equality, we first observe that $\mathcal{F}_x \neq \mathcal{O}_{X,x}$ implies $\widetilde{\mathcal{O}}_{X,x} \neq \mathcal{O}_{X,x}$, thus $x \in X_{\text{n-n}}$. Conversely, assume that

x is non normal, that is, $\tilde{\mathcal{O}}_{X,x} \neq \mathcal{O}_{X,x}$. Let k be the smallest integer such that $\mathcal{J}_x^k \tilde{\mathcal{O}}_{X,x} \subset \mathcal{O}_{X,x}$; such an integer exists since $\mathcal{J}_x^l \tilde{\mathcal{O}}_{X,x} \subset h\tilde{\mathcal{O}}_{X,x} \subset \mathcal{O}_{X,x}$ for l large. Then there is an element $w \in \mathcal{J}_x^{k-1}\tilde{\mathcal{O}}_{X,x}$ such that $w \notin \mathcal{O}_{X,x}$. We have $w\mathcal{J}_x \subset \mathcal{O}_{X,x}$; moreover, as w is locally bounded near X_{sing}, any germ wg in $w\mathcal{J}_x$ satisfies $\lim w(z)g(z) = 0$ when $z \in X_{\mathrm{reg}}$ tends to a point of the zero variety $h^{-1}(0)$ of $\mathcal{J}_x$. Hence $w\mathcal{J}_x \subset \mathcal{J}_x$, i.e. $w \in \mathcal{F}_x$, but $w \notin \mathcal{O}_{X,x}$, so $\mathcal{F}_x \neq \mathcal{O}_{X,x}$. □

(7.6) Theorem. *If $x \in X$ is a normal point, then (X_{sing}, x) has codimension at least 2 in (X, x).*

Proof. We suppose that $\Sigma = X_{\mathrm{sing}}$ has codimension 1 in a neighborhood of x and try to get a contradiction. By restriction to a smaller neighborhood, we may assume that X itself is normal and irreducible (since X_{norm} is open), $\dim X = n$, that Σ has pure dimension $n-1$ and that the ideal sheaf $\mathcal{J}_\Sigma$ has global generators $(g_1, \ldots, g_k)$. Then $\Sigma \subset \bigcup g_j^{-1}(0)$; both sets have pure dimension $n-1$ and thus singular sets of dimension $\le n-2$. Hence there is a point $a \in \Sigma$ that is regular on Σ and on $\bigcup g_j^{-1}(0)$, in particular there is a neighborhood V of a such that $g_1^{-1}(0) \cap V = \ldots = g_k^{-1}(0) \cap V = \Sigma \cap V$ is a smooth $(n-1)$-dimensional manifold. Since $\mathrm{codim}_X \Sigma = 1$ and a is a singular point of X, $\mathcal{J}_{\Sigma,a}$ cannot have less than 2 generators in $\mathcal{O}_{X,a}$ by Cor. 4.33. Take $(g_1, \ldots, g_l)$, $l \ge 2$, to be a minimal subset of generators. Then $f = g_2/g_1$ cannot belong to $\mathcal{O}_{X,a}$, but f is holomorphic on $V \smallsetminus \Sigma$. We may assume that there is a sequence $a_\nu \in V \smallsetminus \Sigma$ converging to a such that $f(a_\nu)$ remains bounded (otherwise reverse g_1 and g_2 and pass to a subsequence). Since $g_1^{-1}(0) \cap V = \Sigma \cap V$, Hilbert's Nullstellensatz gives an integer m such that $\mathcal{J}_{\Sigma,a}^m \subset g_1\mathcal{O}_{X,a}$, hence $f_a\mathcal{J}_{\Sigma,a}^m \subset \mathcal{O}_{X,a}$. We take m to be the smallest integer such that the latter inclusion holds. Then there is a product $g^\alpha = g_1^{\alpha_1} \ldots g_l^{\alpha_l}$ with $|\alpha| = m-1$ such that $fg^\alpha \notin \mathcal{O}_{X,a}$ but $fg^\alpha g_j \in \mathcal{O}_{X,a}$ for each j. Since the sequence $f(a_\nu)$ is bounded we conclude that $fg^\alpha g_j$ vanishes at a. The zero set of this function has dimension $n-1$ and is contained in $\bigcup g_k^{-1}(0) \cap V = \Sigma \cap V$ so it must contain the germ (Σ, a). Hence $fg^\alpha g_j \in \mathcal{J}_{\Sigma,a}$ and $fg^\alpha \mathcal{J}_{\Sigma,a} \subset \mathcal{J}_{\Sigma,a}$. As $\mathcal{J}_{\Sigma,a}$ is a finitely generated $\mathcal{O}_{X,a}$-module, this implies $fg^\alpha \in \tilde{\mathcal{O}}_{X,a} = \mathcal{O}_{X,a}$, a contradiction. □

(7.7) Corollary. *A complex curve is normal if and only if it is regular.*

(7.8) Corollary. *Let X be a normal complex space and Y an analytic subset of X such that* $\dim(Y, x) \le \dim(X, x) - 2$ *for any $x \in X$. Then any holomorphic function on $X \smallsetminus Y$ can be extended to a holomorphic function on X.*

Proof. By Cor. 1.4.5, every holomorphic function f on $X_{\mathrm{reg}} \smallsetminus Y$ extends to X_{reg}. Since $\mathrm{codim}\, X_{\mathrm{sing}} \ge 2$, Th. 6.1 shows that f is locally bounded near X_{sing}. Therefore f extends to X by definition of a normal space. □

§7.3. The Oka Normalization Theorem

The important normalization theorem of (Oka 1950) shows that $\widetilde{\mathcal{O}}_X$ can be used to define the structure sheaf of a new analytic space $\tilde{X}$ which is normal and is obtained by "simplifying" the singular set of X. More precisely:

(7.9) Definition. *Let X be a complex space. A normalization (Y, π) of X is a normal complex space Y together with a holomorphic map $\pi : Y \longrightarrow X$ such that the following conditions are satisfied.*

a) *$\pi : Y \longrightarrow X$ is proper and has finite fibers;*

b) *if Σ is the set of singular points of X and $A = \pi^{-1}(\Sigma)$, then $Y \smallsetminus A$ is dense in Y and $\pi : Y \smallsetminus A \longrightarrow X \smallsetminus \Sigma = X_{\rm reg}$ is an analytic isomorphism.*

It follows from b) that $Y \smallsetminus A \subset Y_{\rm reg}$. Thus Y is obtained from X by a suitable "modification" of its singular points. Observe that $Y_{\rm reg}$ may be larger than $Y \smallsetminus A$, as is the case in the following two examples.

(7.10) Examples.

a) Let $X = \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$ be the complex curve $z_1 z_2 = 0$ in $\mathbb{C}^2$. Then the normalization of X is the disjoint union $Y = \mathbb{C} \times \{1, 2\}$ of two copies of $\mathbb{C}$, with the map $\pi(t_1) = (t_1, 0)$, $\pi(t_2) = (0, t_2)$. The set $A = \pi^{-1}(0, 0)$ consists of exactly two points.

b) The cubic curve $X : z_1^3 = z_2^2$ is normalized by the map $\pi : \mathbb{C} \longrightarrow X$, $t \longmapsto (t^2, t^3)$. Here π is a homeomorphism but π^{-1} is not analytic at $(0, 0)$. □

We first show that the normalization is essentially unique up to isomorphism and postpone the proof of its existence for a while.

(7.11) Lemma. *If (Y_1, π_1) and (Y_2, π_2) are normalizations of X, there is a unique analytic isomorphism $\varphi : Y_1 \longrightarrow Y_2$ such that $\pi_1 = \pi_2 \circ \varphi$.*

Proof. Let Σ be the set of singular points of X and $A_j = \pi_j^{-1}(\Sigma)$, $j = 1, 2$. Let $\varphi' : Y_1 \smallsetminus A_1 \longrightarrow Y_2 \smallsetminus A_2$ be the analytic isomorphism $\pi_2^{-1} \circ \pi_1$. We assert that φ' can be extended into a map $\varphi : Y_1 \longrightarrow Y_2$. In fact, let $a \in A_1$ and $s = \pi_1(a) \in \Sigma$. Then $\pi_2^{-1}(s)$ consists of a finite set of points $y_j \in Y_2$. Take disjoint neighborhoods U_j of y_j such that U_j is an analytic subset in an open set $\Omega_j \subset\subset \mathbb{C}^N$. Since π_2 is proper, there is a neighborhood V of s in X such that $\pi_2^{-1}(V) \subset \bigcup U_j$ and by continuity of π_1 a neighborhood W of a such that $\pi_1(W) \subset V$. Then $\varphi' = \pi_2^{-1} \circ \pi_1$ maps $W \smallsetminus A_1$ into $\bigcup U_j$ and can be seen as a bounded holomorphic map into $\mathbb{C}^N$ through the embeddings $U_j \subset \Omega_j \subset\subset \mathbb{C}^N$. Since Y_1 is normal, φ' extends to W, and the extension takes values in $\bigcup \overline{U}_j$ which is contained in Y_2 (shrink U_j if necessary). Thus

φ' extends into a map $\varphi : Y_1 \longrightarrow Y_2$ and similarly φ'^{-1} extends into a map $\psi : Y_2 \longrightarrow Y_1$. By density of $Y_j \smallsetminus A_j$, we have $\psi \circ \varphi = \mathrm{Id}_{Y_1}$, $\varphi \circ \psi = \mathrm{Id}_{Y_2}$. $\square$

(7.12) Oka normalization theorem. *Let X be any complex space. Then X has a normalization (Y, π).*

Proof. Because of the previous lemma, it suffices to prove that any point $x \in X$ has a neighborhood U such that U admits a normalization; all these local normalizations will then glue together. Hence we may suppose that X is an analytic set in an open set of $\mathbb{C}^n$. Moreover, if (X, x) splits into irreducible components (X_j, x) and if (Y_j, π_j) is a normalization of $X_j \cap U$, then the disjoint union $Y = \coprod Y_j$ with $\pi = \coprod \pi_j$ is easily seen to be a normalization of $X \cap U$. We may therefore assume that (X, x) is irreducible. Let h be a universal denominator in a neighborhood of x. Then $\widetilde{\mathcal{O}}_{X,x}$ is isomorphic to its image $h\widetilde{\mathcal{O}}_{X,x} \subset \mathcal{O}_{X,x}$, so it is a finitely generated $\mathcal{O}_{X,x}$-module. Let $(f_1, \ldots, f_m)$ be a finite set of generators of $\mathcal{O}_{X,x}$. After shrinking X again, we may assume the following two points:

- X is an analytic set in an open set $\Omega \subset \mathbb{C}^n$, (X, x) is irreducible and X_{reg} is connected;
- f_j is holomorphic in X_{reg}, can be written $f_j = g_j/h$ on X with g_j, h in $\mathcal{O}_n(\Omega)$ and satisfies an integral equation $P_j(z\,;\,f_j(z)) = 0$ where $P_j(z\,;\,T)$ is a unitary polynomial with holomorphic coefficients on X.

Set $X' = X \smallsetminus h^{-1}(0)$. Consider the holomorphic map

$$F : X_{\mathrm{reg}} \longrightarrow \Omega \times \mathbb{C}^m, \qquad z \longmapsto \big(z, f_1(z), \ldots, f_m(z)\big)$$

and the image $Y' = F(X')$. We claim that the closure Y of Y' in $\Omega \times \mathbb{C}^m$ is an analytic set. In fact, the set

$$Z = \big\{(z, w) \in \Omega \times \mathbb{C}^m\,;\, z \in X\,,\ h(z)w_j = g_j(z)\big\}$$

is analytic and $Y' = Z \smallsetminus \{h(z) = 0\}$, so we may apply Cor. 5.4. Observe that Y' is contained in the set defined by $P_j(z\,;w_j) = 0$, thus so is its closure Y. The first projection $\Omega \times \mathbb{C}^m \longrightarrow \Omega$ gives a holomorphic map $\pi : Y \longrightarrow X$ such that $\pi \circ F = \mathrm{Id}$ on X', hence also on X_{reg}. If $\Sigma = X_{\mathrm{sing}}$ and $A = \pi^{-1}(\Sigma)$, the restriction $\pi : Y \smallsetminus A \longrightarrow X \smallsetminus \Sigma = X_{\mathrm{reg}}$ is thus an analytic isomorphism and F is its inverse. Since (X, x) is irreducible, each f_j has a limit ℓ_j at x by Th. 7.3 and the fiber $\pi^{-1}(x)$ is reduced to the single point $y = (x, \ell)$. The other fibers $\pi^{-1}(z)$ are finite because they are contained in the finite set of roots of the equations $P_j(z\,;\,w_j) = 0$. The same argument easily shows that π is proper (use Lemma 4.10).

Next, we show that Y is normal at the point $y = \pi^{-1}(x)$. In fact, for any bounded holomorphic function u on (Y_{reg}, y) the function $u \circ F$ is bounded and holomorphic on (X_{reg}, x). Hence $u \circ F \in \widetilde{\mathcal{O}}_{X,x} = \mathcal{O}_{X,x}[f_1, \ldots, f_m]$ and we can write $u \circ F(z) = Q(z\,;\,f_1(z), \ldots, f_m(z)) = Q \circ F(z)$ where $Q(z\,;\,w) =$

$\sum a_\alpha(z)w^\alpha$ is a polynomial in w with coefficients in $\mathcal{O}_{X,x}$. Thus u coincides with Q on (Y_{reg}, y), and as Q is holomorphic on $(X,x)\times\mathbb{C}^m \supset (Y,y)$, we conclude that $u\in\mathcal{O}_{Y,y}$. Therefore $\tilde{\mathcal{O}}_{Y,y}=\mathcal{O}_{Y,y}$.

Finally, by Th. 7.5, there is a neighborhood $V\subset Y$ of y such that every point of V is normal. As π is proper, we can find a neighborhood U of x with $\pi^{-1}(U)\subset V$. Then $\pi:\pi^{-1}(U)\longrightarrow U$ is the required normalization in a neighborhood of x. □

The proof of Th. 7.12 shows that the fiber $\pi^{-1}(x)$ has exactly one point y_j for each irreducible component (X_j,x) of (X,x). As a one-to-one proper map is a homeomorphism, we get in particular:

(7.13) Corollary. *If X is a locally irreducible complex space, the normalization $\pi : Y\longrightarrow X$ is a homeomorphism.* □

(7.14) Remark. In general, for any open set $U\subset X$, we have an isomorphism

$$(7.15)\qquad \pi^\star : \tilde{\mathcal{O}}_X(U) \xrightarrow{\simeq} \mathcal{O}_Y\big(\pi^{-1}(U)\big),$$

whose inverse is given by the comorphism of $\pi^{-1}: X_{\mathrm{reg}}\longrightarrow Y$; note that $\tilde{\mathcal{O}}_Y(U)=\mathcal{O}_Y(U)$ since Y is normal. Taking the direct limit over all neighborhoods U of a given point $x\in X$, we get an isomorphism

$$(7.15')\qquad \pi^\star : \tilde{\mathcal{O}}_{X,x} \longrightarrow \bigoplus_{y_j\in\pi^{-1}(x)} \mathcal{O}_{Y,y_j}.$$

In other words, $\tilde{\mathcal{O}}_X$ is isomorphic to the direct image sheaf $\pi_\star\mathcal{O}_Y$, see (1.12). We will prove later on the deep fact that the direct image of a coherent sheaf by a proper holomorphic map is always coherent (Grauert 1960, see 9.?.1). Hence $\tilde{\mathcal{O}}_X=\pi_\star\mathcal{O}_Y$ is a coherent sheaf over $\mathcal{O}_X$.

§8. Holomorphic Mappings and Extension Theorems

§8.1. Rank of a Holomorphic Mapping

Our goal here is to introduce the general concept of the rank of a holomorphic map and to relate the rank to the dimension of the fibers. As in the smooth case, the rank is shown to satisfy semi-continuity properties.

(8.1) Lemma. *Let $F : X\longrightarrow Y$ be a holomorphic map from a complex space X to a complex space Y.*

a) *If F is finite, i.e. proper with finite fibers, then* $\dim X\le\dim Y$.

b) *If F is finite and surjective, then* $\dim X = \dim Y$.

Proof. a) Let $x \in X$, (X_j, x) an irreducible component and $m = \dim(X_j, x)$. If (Y_k, y) are the irreducible components of Y at $y = F(x)$, then (X_j, x) is contained in $\bigcup F^{-1}(Y_k)$, hence (X_j, x) is contained in one of the sets $F^{-1}(Y_k)$. If $p = \dim(Y_k, y)$, there is a ramified covering π from some neighborhood of y in Y_k onto a polydisk in $\Delta' \subset \mathbb{C}^p$. Replacing X by some neighborhood of x in X_j and F by the finite map $\pi \circ F_{\restriction X_j} : X_j \longrightarrow \Delta'$, we may suppose that $Y = \Delta'$ and that X is irreducible, $\dim X = m$. Let $r = \operatorname{rank} dF_{x_0}$ be the maximum of the rank of the differential of F on X_{reg}. Then $r \le \min\{m, p\}$ and the rank of dF is constant equal to r on a neighborhood U of x_0. The constant rank theorem implies that the fibers $F^{-1}(y) \cap U$ are $(m-r)$-dimensional submanifolds, hence $m - r = 0$ and $m = r \le p$.

b) We only have to show that $\dim X \ge \dim Y$. Fix a regular point $y \in Y$ of maximal dimension. By taking the restriction $F : F^{-1}(U) \longrightarrow U$ to a small neighborhood U of y, we may assume that Y is an open subset of $\mathbb{C}^p$. If $\dim X < \dim Y$, then X is a union of analytic manifolds of dimension $< \dim Y$ and Sard's theorem implies that $F(X)$ has zero Lebesgue measure in Y, a contradiction. □

(8.2) Proposition. *For any holomorphic map $F : X \longrightarrow Y$, the fiber dimension* $\dim\big(F^{-1}(F(x)), x\big)$ *is an upper semi-continuous function of x.*

Proof. Without loss of generality, we may suppose that X is an analytic set in $\Omega \subset \mathbb{C}^n$, that $F(X)$ is contained in a small neighborhood of $F(x)$ in Y which is embedded in $\mathbb{C}^N$, and that $x = 0$, $F(x) = 0$. Set $A = F^{-1}(0)$ and $s = \dim(A, 0)$. We can find a linear form ξ_1 on $\mathbb{C}^n$ such that $\dim(A \cap \xi_1^{-1}(0), 0) = s - 1$; in fact we need only select a point $x_j \ne 0$ on each irreducible component $(A_j, 0)$ of $(A, 0)$ and take $\xi_1(x_j) \ne 0$. By induction, we can find linearly independent forms $\xi_1, \ldots, \xi_s$ on $\mathbb{C}^n$ such that

$$\dim\big(A \cap \xi_1^{-1}(0) \cap \ldots \cap \xi_j^{-1}(0), 0\big) = s - j$$

for all $j = 1, \ldots, s$; in particular 0 is an isolated point in the intersection when $j = s$. After a change of coordinates, we may suppose that $\xi_j(z) = z_j$. Fix r'' so small that the ball $\overline{B}'' \subset \mathbb{C}^{n-s}$ of center 0 and radius r'' satisfies $A \cap (\{0\} \times \overline{B}'') = \{0\}$. Then A is disjoint from the compact set $\{0\} \times \partial B''$, so there exists a small ball $B' \subset \mathbb{C}^s$ of center 0 such that $A \cap (\overline{B}' \times \partial B'') = \emptyset$, i.e. F does not vanish on the compact set $K = X \cap (\overline{B}' \times \partial B'')$. Set $\varepsilon = \min_K |F|$. Then for $|y| < \varepsilon$ the fiber $F^{-1}(y)$ does not intersect $\overline{B}' \times \partial B''$. This implies that the projection map $\pi : F^{-1}(y) \cap (B' \times B'') \longrightarrow B'$ is proper. The fibers of π are then compact analytic subsets of B'', so they are finite by 5.9. Lemma 8.1 a) implies

$$\dim F^{-1}(y) \cap (B' \times B'') \le \dim B' = s = \dim(A, 0) = \dim(F^{-1}(0), 0). \quad \square$$

Let X be a pure dimensional complex space and $F : X \longrightarrow Y$ a holomorphic map. For any point $x \in X$, we define the *rank of F at x* by

$$\text{(8.3)}\quad \rho_F(x) = \dim(X, x) - \dim\big(F^{-1}(F(x)), x\big).$$

By the above proposition, ρ_F is a lower semi-continuous function on X. In particular, if ρ_F is maximum at some point x_0, it must be constant in a neighborhood of x_0. The maximum $\overline{\rho}(F) = \max_X \rho_F$ is thus attained on $X_{\rm reg}$ or on any dense open subset $X' \subset X_{\rm reg}$. If X is not pure dimensional, we define $\overline{\rho}(F) = \max_\alpha \overline{\rho}(F_{\restriction X_\alpha})$ where (X_α) are the irreducible components of X. For a map $F : X \longrightarrow \mathbb{C}^N$, the constant rank theorem implies that $\overline{\rho}(F)$ is equal to the maximum of the rank of the jacobian matrix dF at points of $X_{\rm reg}$ (or of X').

(8.4) Proposition. *If $F : X \longrightarrow Y$ is a holomorphic map and Z an analytic subset of X, then $\overline{\rho}(F_{\restriction Z}) \le \overline{\rho}(F)$.*

Proof. Since each irreducible component of Z is contained in an irreducible component of X, we may assume X irreducible. Let $\pi : \tilde{X} \longrightarrow X$ be the normalization of X and $\tilde{Z} = \pi^{-1}(Z)$. Since π is finite and surjective, the fiber of $F \circ \pi$ at point x has the same dimension than the fiber of F at $\pi(x)$ by Lemma 8.1 b). Therefore $\overline{\rho}(F \circ \pi) = \overline{\rho}(F)$ and $\overline{\rho}(F \circ \pi_{\restriction \tilde{Z}}) = \overline{\rho}(F_{\restriction Z})$, so we may assume X normal. By induction on $\dim X$, we may also suppose that Z has pure codimension 1 in X (every point of Z has a neighborhood $V \subset X$ such that $Z \cap V$ is contained in a pure one codimensional analytic subset of V). But then $Z_{\rm reg} \cap X_{\rm reg}$ is dense in $Z_{\rm reg}$ because $\operatorname{codim} X_{\rm sing} \ge 2$. Thus we are reduced to the case when X is a manifold and Z a submanifold, and this case is clear if we consider the rank of the jacobian matrix. □

(8.5) Theorem. *Let $F : X \longrightarrow Y$ be a holomorphic map. If Y is pure dimensional and $\overline{\rho}(F) < \dim Y$, then $F(X)$ has empty interior in Y.*

Proof. Taking the restriction of F to $F^{-1}(Y_{\rm reg})$, we may assume that Y is a manifold. Since X is a countable union of compact sets, so is $F(X)$, and Baire's theorem shows that the result is local for X. By Prop. 8.4 and an induction on $\dim X$, $F(X_{\rm sing})$ has empty interior in Y. The set $Z \subset X_{\rm reg}$ of points where the jacobian matrix of F has rank $< \overline{\rho}(F)$ is an analytic subset hence, by induction again, $F(Z)$ has empty interior. The constant rank theorem finally shows that every point $x \in X_{\rm reg} \smallsetminus Z$ has a neighborhood V such that $F(V)$ is a submanifold of dimension $\overline{\rho}(F)$ in Y, thus $F(V)$ has empty interior and Baire's theorem completes the proof. □

(8.6) Corollary. *Let $F : X \longrightarrow Y$ be a surjective holomorphic map. Then $\dim Y = \overline{\rho}(F)$.*

Proof. By the remark before Prop. 8.4, there is a regular point $x_0 \in X$ such that the jacobian matrix of F has rank $\overline{\rho}(F)$. Hence, by the constant rank theorem $\dim Y \geq \overline{\rho}(F)$. Conversely, let Y_α be an irreducible component of Y of dimension equal to $\dim Y$, and $Z = F^{-1}(Y_\alpha) \subset X$. Then $F(Z) = Y_\alpha$ and Th. 8.5 implies $\overline{\rho}(F) \geq \overline{\rho}(F_{\restriction Z}) \geq \dim Y_\alpha$. □

§8.2. Remmert and Remmert-Stein Theorems

We are now ready to prove two important results: the extension theorem for analytic subsets due to (Remmert and Stein 1953) and the theorem of (Remmert 1956,1957) which asserts that the image of a complex space under a proper holomorphic map is an analytic set. These will be obtained by a simultaneous induction on the dimension.

(8.7) Remmert-Stein theorem. *Let X be a complex space, A an analytic subset of X and Z an analytic subset of $X \smallsetminus A$. Suppose that there is an integer $p \geq 0$ such that* $\dim A \leq p$, *while* $\dim(Z, x) > p$ *for all $x \in Z$. Then the closure $\overline{Z}$ of Z in X is an analytic subset.*

(8.8) Remmert's proper mapping theorem. *Let $F : X \longrightarrow Y$ be a proper holomorphic map. Then $F(X)$ is an analytic subset of Y.*

Proof. We let (8.7_m) denote statement (8.7) for $\dim Z \leq m$ and (8.8_m) denote statement (8.8) for $\dim X \leq m$. We proceed by induction on m in two steps:

Step 1. (8.7_m) and (8.8_{m-1}) imply (8.8_m).
Step 2. (8.8_{m-1}) implies (8.7_m).

As (8.8_m) is obvious for $m = 0$, our statements will then be valid for all m, i.e. for all complex spaces of bounded dimension. However, Th. 8.7 is local on X and Th. 8.8 is local on Y, so the general case is immediately reduced to the finite dimensional case.

Proof of step 1. The analyticity of $F(X)$ is a local question in Y. Since $F : F^{-1}(U) \longrightarrow U$ is proper for any open set $U \subset Y$ and $F^{-1}(U) \subset\subset X$ if $U \subset\subset Y$, we may suppose that Y is embedded in an open set $\Omega \subset \mathbb{C}^n$ and that X only has finitely many irreducible components X_α. Then we have $F(X) = \bigcup F(X_\alpha)$ and we are reduced to the case when X is irreducible, $\dim X = m$ and $Y = \Omega$.

First assume that X is a manifold and that the rank of dF is constant. The constant rank theorem implies that every point in X has a neighborhood V such that $F(V)$ is a closed submanifold in a neighborhood W of $F(x)$ in Y. For any point $y \in Y$, the fiber $F^{-1}(y)$ can be covered by finitely many neighborhoods V_j of points $x_j \in F^{-1}(y)$ such that $F(V_j)$ is a closed submanifold in a neighborhood W_j of y. Then there is a neighborhood of y

$W \subset \bigcap W_j$ such that $F^{-1}(W) \subset \bigcup V_j$, so $F(X) \cap W = \bigcup F(V_j) \cap W$ is a finite union of closed submanifolds in W and $F(X)$ is analytic in Y.

Now suppose that X is a manifold, set $r = \overline{\rho}(F)$ and let $Z \subset X$ be the analytic subset of points x where the rank of dF_x is $< r$. Since $\dim Z < m = \dim X$, the hypothesis (8.8_{m-1}) shows that $F(Z)$ is analytic. We have $\dim F(Z) = \overline{\rho}(F_{\restriction Z}) < r$. If $F(Z) = F(X)$, then $F(X)$ is analytic. Otherwise $A = F^{-1}\big(F(Z)\big)$ is a proper analytic subset of X, dF has constant rank on $X \smallsetminus A \subset X \smallsetminus Z$ and the morphism $F : X \smallsetminus A \longrightarrow Y \smallsetminus F(Z)$ is proper. Hence the image $F(X \smallsetminus A)$ is analytic in $Y \smallsetminus F(Z)$. Since $\dim F(X \smallsetminus A) = r \le m$ and $\dim F(Z) < r$, hypothesis (8.7_m) implies that $F(X) = \overline{F(X \smallsetminus A)}$ is analytic in Y. When X is not a manifold, we apply the same reasoning with $Z = X_{\mathrm{sing}}$ in order to be reduced to the case of $F : X \smallsetminus A \longrightarrow Y \smallsetminus F(Z)$ where $X \smallsetminus A$ is a manifold. □

Proof of step 2. Since Th. 8.7 is local on X, we may suppose that X is an open set $\Omega \subset \mathbb{C}^n$. Then we use induction on p to reduce the situation to the case when A is a p-dimensional submanifold (if this case is taken for granted, the closure of Z in $\Omega \smallsetminus A_{\mathrm{sing}}$ is analytic and we conclude by the induction hypothesis). By a local analytic change of coordinates, we may assume that $0 \in A$ and that $A = \Omega \cap L$ where L is a vector subspace of $\mathbb{C}^n$ of dimension p. By writing $Z = \bigcup_{p<s\le m} Z_s$ where Z_s is an analytic subset of $\Omega \smallsetminus Y$ of pure dimension s, we may suppose that Z has pure dimension s, $p < s \le m$. We are going to show that $\overline{Z}$ is analytic in a neighborhood of 0.

Let ξ_1 be a linear form on $\mathbb{C}^n$ which is not identically zero on L nor on any irreducible component of Z (just pick a point x_ν on each component and take $\xi_1(x_\nu) \ne 0$ for all ν). Then $\dim L \cap \xi_1^{-1}(0) = p - 1$ and the analytic set $Z \cap \xi_1^{-1}(0)$ has pure dimension $s - 1$. By induction, there exist linearly independent forms $\xi_1, \ldots, \xi_s$ such that

$$\begin{aligned} &\dim L \cap \xi_1^{-1}(0) \cap \ldots \cap \xi_j^{-1}(0) = p - j, \qquad 1 \le j \le p,\\ (8.9)\qquad &\dim Z \cap \xi_1^{-1}(0) \cap \ldots \cap \xi_j^{-1}(0) = s - j, \qquad 1 \le j \le s. \end{aligned}$$

By adding a suitable linear combination of $\xi_1, \ldots, \xi_p$ to each ξ_j, $p < j \le s$, we may take $\xi_{j\restriction L} = 0$ for $p < j \le s$. After a linear change of coordinates, we may suppose that $\xi_j(z) = z_j$, $L = \mathbb{C}^p \times \{0\}$ and $A = \Omega \cap (\mathbb{C}^p \times \{0\})$. Let $\xi = (\xi_1, \ldots, \xi_s) : \mathbb{C}^n \longrightarrow \mathbb{C}^s$ be the projection onto the first s variables. As Z is closed in $\Omega \smallsetminus A$, $Z \cup A$ is closed in Ω. Moreover, our construction gives $(Z \cup A) \cap \xi^{-1}(0) = \big(Z \cap \xi^{-1}(0)\big) \cup \{0\}$ and the case $j = s$ of (8.9) shows that $Z \cap \xi^{-1}(0)$ is a locally finite sequence in $\Omega \cap (\{0\} \times \mathbb{C}^{n-s}) \smallsetminus \{0\}$. Therefore, we can find a small ball $\overline{B}''$ of center 0 in $\mathbb{C}^{n-s}$ such that $Z \cap (\{0\} \times \partial B'') = \emptyset$. As $\{0\} \times \partial B''$ is compact and disjoint from the closed set $Z \cup A$, there is a small ball B' of center 0 in $\mathbb{C}^s$ such that $(Z \cup A) \cap (\overline{B}' \times \partial B'') = \emptyset$. This implies that the projection $\xi : (Z \cup A) \cap (B' \times B'') \longrightarrow B'$ is proper. Set $A' = B' \cap (\mathbb{C}^p \times \{0\})$. Then the restriction

Fig. 3 Projection $\pi : Z \cap ((B' \smallsetminus A') \times B'') \longrightarrow B' \smallsetminus A'$.

$$\pi = \xi : Z \cap (B' \times B'') \smallsetminus (A' \times B'') \longrightarrow B' \smallsetminus A'$$

is proper, and $Z \cap (B' \times B'')$ is analytic in $(B' \times B'') \smallsetminus A$, so π has finite fibers by Th. 5.9. By definition of the rank we have $\overline{\rho}(\pi) = s$. Let $S_1 = Z_{\rm sing} \cap \pi^{-1}(B' \smallsetminus A')$ and $S_1' = \pi(S_1)$; further, let S_2 be the set of points $x \in Z \cap \pi^{-1}\big(B' \smallsetminus (A' \cup S_1')\big) \subset Z_{\rm reg}$ such that $d\pi_x$ has rank $< s$ and $S_2' = \pi(S_2)$. We have $\dim S_j \le s - 1 \le m - 1$. Hypothesis $(8.8)_{m-1}$ implies that S_1' is analytic in $B' \smallsetminus A'$ and that S_2' is analytic in $B' \smallsetminus (A' \cup S_1')$. By Remark 4.2, $B' \smallsetminus (A' \cup S_1' \cup S_2')$ is connected and every bounded holomorphic function on this set extends to B'. As π is a (non ramified) covering over $B' \smallsetminus (A' \cup S_1' \cup S_2')$, the sheet number is a constant q.

Let $\lambda(z) = \sum_{j>s} \lambda_j z_j$ be a linear form on $\mathbb{C}^n$ in the coordinates of index $j > s$. For $z' \in B' \smallsetminus (A' \cup S_1' \cup S_2')$, we let $\sigma_j(z')$ be the elementary symmetric functions in the q complex numbers $\lambda(z)$ corresponding to $z \in \pi^{-1}(z')$. Then these functions can be extended as bounded holomorphic functions on B' and we get a polynomial $P_\lambda(z' ; T)$ such that $P_\lambda\big(z' ; \lambda(z'')\big)$ vanishes identically on $Z \smallsetminus \pi^{-1}(A' \cup S_1' \cup S_2')$. Since π is finite, $Z \cap \pi^{-1}(A' \cup S_1' \cup S_2')$ is a union of three (non necessarily closed) analytic subsets of dimension $\le s - 1$, thus has empty interior in Z. It follows that the closure $\overline{Z} \cap (B' \times B'')$ is contained in the analytic set $W \subset B' \times B''$ equal to the common zero set of all functions

$P_\lambda\big(z'\,;\,\lambda(z'')\big)$. Moreover, by construction,

$$Z \smallsetminus \pi^{-1}(A' \cup S_1' \cup S_2') = W \smallsetminus \pi^{-1}(A' \cup S_1' \cup S_2').$$

As in the proof of Cor. 5.4, we easily conclude that $\overline{Z} \cap (B' \times B'')$ is equal to the union of all irreducible components of W that are not contained in $\pi^{-1}(A' \cup S_1' \cup S_2')$. Hence $\overline{Z}$ is analytic. □

Finally, we give two interesting applications of the Remmert-Stein theorem. We assume here that the reader knows what is the complex projective space $\mathbb{P}^n$. For more details, see Sect. 5.15.

(8.10) Chow's theorem (Chow 1949). *Let A be an analytic subset of the complex projective space $\mathbb{P}^n$. Then A is algebraic, i.e. A is the common zero set of finitely many homogeneous polynomials $P_j(z_0, \ldots, z_n)$, $1 \le j \le N$.*

Proof. Let $\pi : \mathbb{C}^{n+1} \smallsetminus \{0\} \longrightarrow \mathbb{P}^n$ be the natural projection and $Z = \pi^{-1}(A)$. Then Z is an analytic subset of $\mathbb{C}^{n+1} \smallsetminus \{0\}$ which is invariant by homotheties and $\dim Z = \dim A + 1 \ge 1$. The Remmert-Stein theorem implies that $\overline{Z} = Z \cup \{0\}$ is an analytic subset of $\mathbb{C}^{n+1}$. Let $f_1, \ldots, f_N$ be holomorphic functions on a small polydisk $\Delta \subset \mathbb{C}^{n+1}$ of center 0 such that $\overline{Z} \cap \Delta = \bigcap f_j^{-1}(0)$. The Taylor series at 0 gives an expansion $f_j = \sum_{k=0}^{+\infty} P_{j,k}$ where $P_{j,k}$ is a homogeneous polynomial of degree k. We claim that $\overline{Z}$ coincides with the common zero W set of the polynomials $P_{j,k}$. In fact, we clearly have $W \cap \Delta \subset \bigcap f_j^{-1}(0) = \overline{Z} \cap \Delta$. Conversely, for $z \in \overline{Z} \cap \Delta$, the invariance of Z by homotheties shows that $f_j(tz) = \sum P_{j,k}(z) t^k$ vanishes for every complex number t of modulus < 1, so all coefficients $P_{j,k}(z)$ vanish and $z \in W \cap \Delta$. By homogeneity $\overline{Z} = W$; since $\mathbb{C}[z_0, \ldots, z_n]$ is Noetherian, W can be defined by finitely many polynomial equations. □

(8.11) E.E. Levi's continuation theorem. *Let X be a normal complex space and A an analytic subset such that* $\dim(A, x) \le \dim(X, x) - 2$ *for all $x \in A$. Then every meromorphic function on $X \smallsetminus A$ has a meromorphic extension to X.*

Proof. We may suppose X irreducible, $\dim X = n$. Let f be a meromorphic function on $X \smallsetminus A$. By Th. 6.13, the pole set P_f has pure dimension $(n-1)$, so the Remmert-Stein theorem implies that $\overline{P}_f$ is analytic in X. Fix a point $x \in A$. There is a connected neighborhood V of x and a non zero holomorphic function $h \in \mathcal{O}_X(V)$ such that $\overline{P}_f \cap V$ has finitely many irreducible components $\overline{P}_{f,j}$ and $\overline{P}_f \cap V \subset h^{-1}(0)$. Select a point x_j in $\overline{P}_{f,j} \smallsetminus (X_{\mathrm{sing}} \cup (\overline{P}_f)_{\mathrm{sing}} \cup A)$. As x_j is a regular point on X and on $\overline{P}_f$, there is a local coordinate $z_{1,j}$ at x_j defining an equation of $\overline{P}_{f,j}$, such that $z_{1,j}^{m_j} f \in \mathcal{O}_{X,x_j}$ for some integer m_j. Since h vanishes along P_f, we have $h^{m_j} f \in \mathcal{O}_{X,x}$. Thus, for $m = \max\{m_j\}$, the pole set P_g of $g = h^m f$ in $V \smallsetminus A$ does not contain x_j. As P_g is $(n-1)$-dimensional and contained in

$P_f \cap V$, it is a union of irreducible components $\overline{P}_{f,j} \smallsetminus A$. Hence P_g must be empty and g is holomorphic on $V \smallsetminus A$. By Cor. 7.8, g has an extension to a holomorphic function $\tilde{g}$ on V. Then $\tilde{g}/h^m$ is the required meromorphic extension of f on V. □

§9. Complex Analytic Schemes

Our goal is to introduce a generalization of the notion of complex space given in Def. 5.2. A complex space is a space locally isomorphic to an analytic set A in an open subset $\Omega \subset \mathbb{C}^n$, together with the sheaf of rings $\mathcal{O}_A = (\mathcal{O}_\Omega/\mathcal{I}_A)_{\restriction A}$. Our desire is to enrich the structure sheaf $\mathcal{O}_A$ by replacing $\mathcal{I}_A$ with a possibly smaller ideal $\mathcal{J}$ defining the same zero variety $V(\mathcal{J}) = A$. In this way holomorphic functions are described not merely by their values on A, but also possibly by some "transversal derivatives" along A.

§9.1. Ringed Spaces

We start by an abstract notion of ringed space on an arbitrary topological space.

(9.1) Definition. *A ringed space is a pair $(X, \mathcal{R}_X)$ consisting of a topological space X and of a sheaf of rings $\mathcal{R}_X$ on X, called the structure sheaf. A morphism*

$$F : (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$$

of ringed spaces is a pair $(f, F^\star)$ where $f : X \longrightarrow Y$ is a continuous map and

$$F^\star \ :\ f^{-1}\mathcal{R}_Y \longrightarrow \mathcal{R}_X, \qquad F^\star_x \ :\ (\mathcal{R}_Y)_{f(x)} \longrightarrow (\mathcal{R}_X)_x$$

a homomorphism of sheaves of rings on X, called the comorphism of F.

If $F : (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$ and $G : (Y, \mathcal{R}_Y) \longrightarrow (Z, \mathcal{R}_Z)$ are morphisms of ringed spaces, the composite $G \circ F$ is the pair consisting of the map $g \circ f : X \longrightarrow Z$ and of the comorphism $(G \circ F)^\star = F^\star \circ f^{-1}G^\star$:

$$\text{(9.2)} \quad \begin{aligned} F^\star \circ f^{-1}G^\star \ &:\ f^{-1}g^{-1}\mathcal{R}_Z \xrightarrow{f^{-1}G^\star} f^{-1}\mathcal{R}_Y \xrightarrow{F^\star} \mathcal{R}_X, \\ F^\star_x \circ G^\star_{f(x)} \ &:\ (\mathcal{R}_Z)_{g\circ f(x)} \longrightarrow (\mathcal{R}_Y)_{f(x)} \longrightarrow (\mathcal{R}_X)_x. \end{aligned}$$

§9.2. Definition of Complex Analytic Schemes

We begin by a description of what will be the local model of an analytic scheme. Let $\Omega \subset \mathbb{C}^n$ be an open subset, $\mathcal{J} \subset \mathcal{O}_\Omega$ a coherent sheaf of ideals and $A = V(\mathcal{J})$ the analytic set in Ω defined locally as the zero set of a system of generators of $\mathcal{J}$. By Hilbert's Nullstellensatz 4.22 we have $\mathcal{I}_A = \sqrt{\mathcal{J}}$, but $\mathcal{I}_A$ differs in general from $\mathcal{J}$. The sheaf of rings $\mathcal{O}_\Omega/\mathcal{J}$ is supported on A, i.e. $(\mathcal{O}_\Omega/\mathcal{J})_x = 0$ if $x \notin A$. Ringed spaces of the type $(A, \mathcal{O}_\Omega/\mathcal{J})$ will be used as the local models of analytic schemes.

(9.3) Definition. *A morphism*

$$F = (f, F^\star) : (A, \mathcal{O}_\Omega/\mathcal{J}_{\restriction A}) \longrightarrow (A', \mathcal{O}_{\Omega'}/\mathcal{J}'_{\restriction A'})$$

is said to be analytic if for every point $x \in A$ there exists a neighborhood W_x of x in Ω and a holomorphic function $\Phi : W_x \longrightarrow \Omega'$ such that $f_{\restriction A\cap W_x} = \Phi_{\restriction A\cap W_x}$ and such that the comorphism

$$F^\star_x : (\mathcal{O}_{\Omega'}/\mathcal{J}')_{f(x)} \longrightarrow (\mathcal{O}_\Omega/\mathcal{J})_x$$

is induced by $\Phi^\star : \mathcal{O}_{\Omega',f(x)} \ni u \longmapsto u \circ \Phi \in \mathcal{O}_{\Omega,x}$ with $\Phi^\star\mathcal{J}' \subset \mathcal{J}$.

(9.4) Example. Take $\Omega = \mathbb{C}^n$ and $\mathcal{J} = (z_n^2)$. Then A is the hyperplane $\mathbb{C}^{n-1} \times \{0\}$, and the sheaf $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}$ can be identified with the sheaf of rings of functions $u + z_n u'$, $u, u' \in \mathcal{O}_{\mathbb{C}^{n-1}}$, with the relation $z_n^2 = 0$. In particular, z_n is a nilpotent element of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{J}$. A morphism F of $(A, \mathcal{O}_{\mathbb{C}^n}/\mathcal{J})$ into itself is induced (at least locally) by a holomorphic map $\Phi = (\widetilde{\Phi}, \Phi_n)$ defined on a neighborhood of A in $\mathbb{C}^n$ with values in $\mathbb{C}^n$, such that $\Phi(A) \subset A$, i.e. $\Phi_{n\restriction A} = 0$. We see that F is completely determined by the data

$$\begin{aligned} f(z_1, \ldots, z_{n-1}) &= \widetilde{\Phi}(z_1, \ldots, z_{n-1}, 0), \qquad & f &: \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1}, \\ f'(z_1, \ldots, z_{n-1}) &= \frac{\partial \Phi}{\partial z_n}(z_1, \ldots, z_{n-1}, 0), & f' &: \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^n, \end{aligned}$$

which can be chosen arbitrarily.

(9.5) Definition. *A complex analytic scheme is a ringed space $(X, \mathcal{O}_X)$ over a separable Hausdorff topological space X, satisfying the following property: there exist an open covering (U_λ) of X and isomorphisms of ringed spaces*

$$G_\lambda : (U_\lambda, \mathcal{O}_{X\restriction U_\lambda}) \longrightarrow (A_\lambda, \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda})$$

where A_λ is the zero set of a coherent sheaf of ideals $\mathcal{J}_\lambda$ on an open subset $\Omega_\lambda \subset \mathbb{C}^{N_\lambda}$, such that every transition morphism $G_\lambda \circ G_\mu^{-1}$ is a holomorphic isomorphism from $g_\mu(U_\lambda \cap U_\mu) \subset A_\mu$ onto $g_\lambda(U_\lambda \cap U_\mu) \subset A_\lambda$, equipped with the respective structure sheaves $\mathcal{O}_{\Omega_\mu}/\mathcal{J}_{\mu\restriction A_\mu}$, $\mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda}$.

We shall often consider the maps G_λ as identifications and write simply $U_\lambda = A_\lambda$. A morphism $F : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of analytic schemes obtained by gluing patches $(A_\lambda, \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_{\lambda\restriction A_\lambda})$ and $(A'_\mu, \mathcal{O}_{\Omega'_\mu}/\mathcal{J}'_{\mu\, A'_\mu})$, respectively, is a morphism F of ringed spaces such that for each pair (λ, μ), the restriction of F from $A_\lambda \cap f^{-1}(A'_\mu) \subset X$ to $A'_\mu \subset Y$ is holomorphic in the sense of Def. 9.3.

§9.3. Nilpotent Elements and Reduced Schemes

Let $(X, \mathcal{O}_X)$ be an analytic scheme. The set of *nilpotent elements* is the sheaf of ideals of $\mathcal{O}_X$ defined by

$$(9.6)\quad \mathcal{N}_X = \{u \in \mathcal{O}_X\,;\, u^k = 0 \ \text{ for some } \ k \in \mathbb{N}\}.$$

Locally, we have $\mathcal{O}_{X\restriction A_\lambda} = (\mathcal{O}_{\Omega_\lambda}/\mathcal{J}_\lambda)_{\restriction A_\lambda}$, thus

$$(9.7)\qquad \mathcal{N}_{X\restriction A_\lambda} = (\sqrt{\mathcal{J}_\lambda}/\mathcal{J}_\lambda)_{\restriction A_\lambda},$$

$$(9.8)\qquad (\mathcal{O}_X/\mathcal{N}_X)_{\restriction A_\lambda} \simeq (\mathcal{O}_{\Omega_\lambda}/\sqrt{\mathcal{J}_\lambda})_{\restriction A_\lambda} = (\mathcal{O}_{\Omega_\lambda}/\mathcal{I}_{A_\lambda})_{\restriction A_\lambda} = \mathcal{O}_{A_\lambda}.$$

The scheme $(X, \mathcal{O}_X)$ is said to be *reduced* if $\mathcal{N}_X = 0$. The associated ringed space $(X, \mathcal{O}_X/\mathcal{N}_X)$ is reduced by construction; it is called the *reduced scheme* of $(X, \mathcal{O}_X)$. We shall often denote the original scheme by the letter X merely, the associated reduced scheme by X_{red}, and let $\mathcal{O}_{X,\mathrm{red}} = \mathcal{O}_X/\mathcal{N}_X$. There is a canonical morphism $X_{\mathrm{red}} \to X$ whose comorphism is the reduction morphism

$$(9.9)\quad \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\mathrm{red}}(U) = (\mathcal{O}_X/\mathcal{N}_X)(U), \qquad \forall U \text{ open set in } X.$$

By (9.8), the notion of reduced scheme is equivalent to the notion of complex space introduced in Def. 5.2. It is easy to see that a morphism F of reduced schemes X, Y is completely determined by the set-theoretic map $f : X \longrightarrow Y$.

§9.4. Coherent Sheaves on Analytic Schemes

If $(X, \mathcal{O}_X)$ is an analytic scheme, a sheaf $\mathcal{S}$ of $\mathcal{O}_X$-modules is said to be *coherent* if it satisfies the same properties as those already stated when X is a manifold:

(9.10) $\mathcal{S}$ is locally finitely generated over $\mathcal{O}_X$;

(9.10′) for any open set $U \subset X$ and any sections $G_1, \ldots, G_q \in \mathcal{S}(U)$, the relation sheaf $\mathcal{R}(G_1, \ldots, G_q) \subset \mathcal{O}^{\oplus q}_{X\restriction U}$ is locally finitely generated.

Locally, we have $\mathcal{O}_{X\restriction A_\lambda} = \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_\lambda$, so if $i_\lambda : A_\lambda \to \Omega_\lambda$ is the injection, the direct image $\mathcal{S}_\lambda = (i_\lambda)_\star(\mathcal{S}_{\restriction A_\lambda})$ is a module over $\mathcal{O}_{\Omega_\lambda}$ such that $\mathcal{J}_\lambda.\mathcal{S}_\lambda = 0$. It is clear that $\mathcal{S}_{\restriction\Omega_\lambda}$ is coherent if and only if $\mathcal{S}_\lambda$ is coherent as a module over $\mathcal{O}_{\Omega_\lambda}$. It follows immediately that the Oka theorem and its consequences 3.16–20 are still valid over analytic schemes.

§9.5. Subschemes

Let X be an analytic scheme and $\mathcal{G}$ a coherent sheaf of ideals in $\mathcal{O}_X$. The image of $\mathcal{G}$ in $\mathcal{O}_{X,\mathrm{red}}$ is a coherent sheaf of ideals, and its zero set Y is clearly an analytic subset of X_{red}. We can make Y into a scheme by introducing the structure sheaf

$$(9.11) \quad \mathcal{O}_Y = (\mathcal{O}_X/\mathcal{G})_{\restriction Y},$$

and we have a scheme morphism $F : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ such that f is the inclusion and $F^\star : f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ the obvious map of $\mathcal{O}_{X\restriction Y}$ onto its quotient $\mathcal{O}_Y$. The scheme $(Y, \mathcal{O}_Y)$ will be denoted $V(\mathcal{G})$. When the analytic set Y is given, the structure sheaf of $V(\mathcal{G})$ depends of course on the choice of the equations of Y in the ideal $\mathcal{G}$; in general $\mathcal{O}_Y$ has nilpotent elements.

§9.6. Inverse Images of Coherent Sheaves

Let $F : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of analytic schemes and $\mathcal{S}$ a coherent sheaf over Y. The sheaf theoretic inverse image $f^{-1}\mathcal{S}$, whose stalks are $(f^{-1}\mathcal{S})_x = \mathcal{S}_{f(x)}$, is a sheaf of modules over $f^{-1}\mathcal{O}_Y$. We define the *analytic inverse image* $F^\star\mathcal{S}$ by

$$(9.12) \quad F^\star\mathcal{S} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{S}, \qquad (F^\star\mathcal{S})_x = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{S}_{f(x)}.$$

Here the tensor product is taken with respect to the comorphism $F^\star : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, which yields a ring morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. If $\mathcal{S}$ is given over $U \subset Y$ by a local presentation

$$\mathcal{O}^{\oplus p}_{Y\restriction U} \xrightarrow{A} \mathcal{O}^{\oplus q}_{Y\restriction U} \longrightarrow \mathcal{S}_{\restriction U} \longrightarrow 0$$

where A is a $(q \times p)$-matrix with coefficients in $\mathcal{O}_Y(U)$, our definition shows that $F^\star\mathcal{S}$ is a coherent sheaf over $\mathcal{O}_X$, given over $f^{-1}(U)$ by the local presentation

$$(9.13) \quad \mathcal{O}^{\oplus p}_{X\restriction f^{-1}(U)} \xrightarrow{F^\star A} \mathcal{O}^{\oplus q}_{X\restriction f^{-1}(U)} \longrightarrow F^\star\mathcal{S}_{\restriction f^{-1}(U)} \longrightarrow 0.$$

§9.7. Products of Analytic Schemes

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be analytic schemes, and let $(A_\lambda, \mathcal{O}_{\Omega_\lambda}/\mathcal{J}_\lambda)$, $(B_\mu, \mathcal{O}_{\Omega'_\mu}/\mathcal{J}'_\mu)$ be local models of X, Y, respectively. The *product scheme* $(X \times Y, \mathcal{O}_{X\times Y})$ is obtained by gluing the open patches

$$(9.14) \quad \Big(A_\lambda \times B_\mu \ , \ \mathcal{O}_{\Omega_\lambda\times\Omega'_\mu}/\big(\mathrm{pr}_1^{-1}\mathcal{J}_\lambda + \mathrm{pr}_2^{-1}\mathcal{J}'_\mu\big)\mathcal{O}_{\Omega_\lambda\times\Omega'_\mu}\Big).$$

In other words, if A_λ, B_μ are the subschemes of Ω_λ, Ω'_μ defined by the equations $g_{\lambda,j}(x) = 0$, $g'_{\mu,k}(y) = 0$, where $(g_{\lambda,j})$ and $(g'_{\mu,k})$ are generators of

$\mathcal{J}_\lambda$ and $\mathcal{J}'_\mu$ respectively, then $A_\lambda \times B_\mu$ is equipped with the structure sheaf $\mathcal{O}_{\Omega_\lambda \times \Omega'_\mu} / \big(g_{\lambda,j}(x), g'_{\mu,k}(y)\big)$.

Now, let $\mathcal{S}$ be a coherent sheaf over $\mathcal{O}_X$ and let $\mathcal{S}'$ be a coherent sheaf over $\mathcal{O}_Y$. The (analytic) *external tensor product* $\mathcal{S} \boxtimes \mathcal{S}'$ is defined to be

$$(9.15) \quad \mathcal{S} \boxtimes \mathcal{S}' = \mathrm{pr}_1^\star \mathcal{S} \otimes_{\mathcal{O}_{X\times Y}} \mathrm{pr}_2^\star \mathcal{S}'.$$

If we go back to the definition of the inverse image, we see that the stalks of $\mathcal{S} \boxtimes \mathcal{S}'$ are given by

$$(9.15') \quad (\mathcal{S} \boxtimes \mathcal{S}')_{(x,y)} = \mathcal{O}_{X\times Y,(x,y)} \otimes_{\mathcal{O}_{X,x} \otimes \mathcal{O}_{Y,y}} (\mathcal{S}_x \otimes_{\mathbb{C}} \mathcal{S}'_y)\ ,$$

in particular $(\mathcal{S} \boxtimes \mathcal{S}')_{(x,y)}$ *does not coincide* with the sheaf theoretic tensor product $\mathcal{S}_x \otimes \mathcal{S}'_y$ which is merely a module over $\mathcal{O}_{X,x} \otimes \mathcal{O}_{Y,y}$. If $\mathcal{S}$ and $\mathcal{S}'$ are given by local presentations

$$\mathcal{O}_{X\restriction U}^{\oplus p} \xrightarrow{A} \mathcal{O}_{X\restriction U}^{\oplus q} \longrightarrow \mathcal{S}_{\restriction U} \longrightarrow 0, \qquad \mathcal{O}_{Y\restriction U'}^{p'} \xrightarrow{B} \mathcal{O}_{Y\restriction U'}^{q'} \longrightarrow \mathcal{S}'_{\restriction U'} \longrightarrow 0,$$

then $\mathcal{S} \boxtimes \mathcal{S}'$ is the coherent sheaf given by

$$\mathcal{O}_{X\times Y\restriction U\times U'}^{pq'\oplus qp'} \xrightarrow{(A(x)\otimes \mathrm{Id}, \mathrm{Id}\otimes B(y))} \mathcal{O}_{X\times Y\restriction U\times U'}^{qq'} \longrightarrow (\mathcal{S} \boxtimes \mathcal{S}')_{\restriction U\times U'} \longrightarrow 0.$$

§9.8. Zariski Embedding Dimension

If x is a point of an analytic scheme $(X, \mathcal{O}_X)$, the *Zariski embedding dimension* of the germ (X,x) is the smallest integer N such that (X,x) can be embedded in $\mathbb{C}^N$, i.e. such that there exists a patch of X near x isomorphic to $(A, \mathcal{O}_\Omega/\mathcal{J})$ where Ω is an open subset of $\mathbb{C}^N$. This dimension is denoted

$$(9.16) \quad \mathrm{embdim}(X,x) = \text{ smallest such } N.$$

Consider the maximal ideal $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$ of functions which vanish at point x. If (X,x) is embedded in $(\Omega, x) = (\mathbb{C}^N, 0)$, then $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is generated by $z_1, \ldots, z_N$, so $d = \dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \le N$. Let $s_1, \ldots, s_d$ be germs in $\mathfrak{m}_{\Omega,x}$ which yield a basis of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \simeq \mathfrak{m}_{\Omega,x}/(\mathfrak{m}_{\Omega,x}^2 + \mathcal{J}_x)$. We can write

$$z_j = \sum_{1\le k\le d} c_{jk} s_k + u_j + f_j, \quad c_{jk} \in \mathbb{C}, \quad u_j \in \mathfrak{m}_{\Omega,x}^2, \quad f_j \in \mathcal{J}_x, \quad 1 \le j \le n.$$

Then we find $dz_j = \sum c_{jk}\, ds_k(x) + df_j(x)$, so that the rank of the system of differentials $\big(df_j(x)\big)$ is at least $N-d$. Assume for example that $df_1(x), \ldots, df_{N-d}(x)$ are linearly independant . By the implicit function theorem, the equations $f_1 = \ldots = f_{N-d} = 0$ define a germ of smooth subvariety $(Z,x) \subset (\Omega, x)$ of dimension d which contains (X,x). We have $\mathcal{O}_Z = \mathcal{O}_\Omega/(f_1, \ldots, f_{N-d})$ in a neighborhood of x, thus

$$\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{J} \simeq \mathcal{O}_Z/\mathcal{J}' \quad \text{where } \mathcal{J}' = \mathcal{J}/(f_1, \ldots, f_{N-d}).$$

This shows that (X,x) can be imbedded in $\mathbb{C}^d$, and we get

(9.17) $\operatorname{embdim}(X,x) = \dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$.

(9.18) Remark. For a given dimension $n = \dim(X,x)$, the embedding dimension d can be arbitrarily large. Consider for example the curve $\Gamma \subset \mathbb{C}^N$ parametrized by $\mathbb{C} \ni t \longmapsto (t^N, t^{N+1}, \ldots, t^{2N-1})$. Then $\mathcal{O}_{\Gamma,0}$ is the ring of convergent series in $\mathbb{C}\{t\}$ which have no terms $t, t^2, \ldots, t^{N-1}$, and $\mathfrak{m}_{\Gamma,0}/\mathfrak{m}_{\Gamma,0}^2$ admits precisely $z_1 = t^N, \ldots, z_N = t^{2N-1}$ as a basis. Therefore $n = 1$ but $d = N$ can be as large as we want.

§10. Bimeromorphic maps, Modifications and Blow-ups

It is a very frequent situation in analytic or algebraic geometry that two complex spaces have isomorphic dense open subsets but are nevertheless different along some analytic subset. These ideas are made precise by the notions of modification and bimeromorphic map. This will also lead us to generalize the notion of meromorphic function to maps between analytic schemes. If $(X, \mathcal{O}_X)$ is an analytic scheme, $\mathcal{M}_X$ denotes the sheaf of meromorphic functions on X, defined at the beginning of § 6.2.

(10.1) Definition. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be analytic schemes. An analytic morphism $F : X \to Y$ is said to be a modification if F is proper and if there exists a nowhere dense closed analytic subset $B \subset Y$ such that the restriction $F : X \smallsetminus F^{-1}(B) \to Y \smallsetminus B$ is an isomorphism.*

(10.2) Definition. *If $F : X \to Y$ is a modification, then the comorphism $F^\star : f^\star\mathcal{O}_Y \to \mathcal{O}_X$ induces an isomorphism $F^\star : f^\star\mathcal{M}_Y \to \mathcal{M}_X$ for the sheaves of meromorphic functions on X and Y.*

Proof. Let $v = g/h$ be a section of $\mathcal{M}_Y$ on a small open set Ω where u is actually given as a quotient of functions $g, h \in \mathcal{O}_Y(\Omega)$. Then $F^\star u = (g \circ F)/(h \circ F)$ is a section of $\mathcal{M}_X$ on $F^{-1}(\Omega)$, for $h \circ F$ cannot vanish identically on any open subset W of $F^{-1}(\Omega)$ (otherwise h would vanish on the open subset $F(W \smallsetminus F^{-1}(B))$ of $\Omega \smallsetminus B$). Thus the extension of the comorphism to sheaves of meromorphic functions is well defined. Our claim is that this is an isomorphism. The injectivity of $F^\star$ is clear: $F^\star u = 0$ implies $g \circ F = 0$, which implies $g = 0$ on $\Omega \smallsetminus B$ and thus $g = 0$ on Ω because B is nowhere dense. In order to prove surjectivity, we need only show that every section $u \in \mathcal{O}_X(F^{-1}(\Omega))$ is in the image of $\mathcal{M}_Y(\Omega)$ by $F^\star$. For this, we may shrink Ω into a relatively compact subset $\Omega' \subset\subset \Omega$ and thus assume that u is bounded (here we use the properness of F through the fact that $F^{-1}(\Omega')$ is relatively compact in $F^{-1}(\Omega)$). Then $v = u \circ F^{-1}$ defines a bounded holomorphic function on $\Omega \setminus B$. By Th. 7.2, it follows that v is weakly holomorphic for the reduced structure of Y. Our claim now follows from the following Lemma. $\square$

(10.3) Lemma. *If $(X, \mathcal{O}_X)$ is an analytic scheme, then every holomorphic function v in the complement of a nowhere dense analytic subset $B \subset Y$ which is weakly holomorphic on X_{red} is meromorphic on X.*

Proof. It is enough to argue with the germ of v at any point $x \in Y$, and thus we may suppose that $(Y, \mathcal{O}_Y) = (A, \mathcal{O}_\Omega/\mathcal{J})$ is embedded in $\mathbb{C}^N$. Because v is weakly holomorphic, we can write $v = g/h$ in Y_{red}, for some germs of holomorphic functions g, h. Let $\widetilde{g}$ and $\widetilde{h}$ be extensions of g, h to $\mathcal{O}_{\Omega,x}$. Then there is a neighborhood U of x such that $\widetilde{g} - v\widetilde{h}$ is a nilpotent section of $cO_\Omega(U \setminus B)$ which is in $\mathcal{J}$ on

(10.4) Definition. *A meromorphic map $F : X \dashrightarrow Y$ is a scheme morphism $F : X \setminus A \to Y$ defined in the complement of a nowhere dense analytic subset $A \subset X$, such that the closure of the graph of F in $X \times Y$ is an analytic subset (for the reduced complex space structure of $X \times Y$).*

§11. Exercises

11.1. Let $\mathcal{A}$ be a sheaf on a topological space X. If the sheaf space $\widetilde{\mathcal{A}}$ is Hausdorff, show that $\mathcal{A}$ satisfies the following *unique continuation principle*: any two sections $s, s' \in \mathcal{A}(U)$ on a connected open set U which coincide on some non empty open subset $V \subset U$ must coincide identically on U. Show that the converse holds if X is Hausdorff and locally connected.

11.2. Let $\mathcal{A}$ be a sheaf of abelian groups on X and let $s \in \mathcal{A}(X)$. The support of s, denoted Supp s, is defined to be $\{x \in X\,;\, s(x) \neq 0\}$. Show that Supp s is a closed subset of X. The support of $\mathcal{A}$ is defined to be Supp $\mathcal{A} = \{x \in X\,;\, \mathcal{A}_x \neq 0\}$. Show that Supp $\mathcal{A}$ is not necessarily closed: if Ω is an open set in X, consider the sheaf $\mathcal{A}$ such that $\mathcal{A}(U)$ is the set of continuous functions $f \in \mathcal{C}(U)$ which vanish on a neighborhood of $U \cap (X \setminus \Omega)$.

11.3. Let $\mathcal{A}$ be a sheaf of rings on a topological space X and let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{A}$-modules. We define a presheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ such that $\mathcal{H}(U)$ is the module of all sheaf-homomorphisms $\mathcal{F}_{\restriction U} \to \mathcal{G}_{\restriction U}$ which are $\mathcal{A}$-linear.

a) Show that $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ is a sheaf and that there is a canonical homomorphism $\varphi_x : \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})_x \longrightarrow \hom_{\mathcal{A}_x}(\mathcal{F}_x, \mathcal{G}_x)$ for every $x \in X$.

b) If $\mathcal{F}$ is locally finitely generated, then φ_x is injective, and if $\mathcal{F}$ has local finite presentations as in (3.12), then φ_x is bijective.

c) Suppose that $\mathcal{A}$ is a coherent sheaf of rings and that $\mathcal{F}$, $\mathcal{G}$ are coherent modules over $\mathcal{A}$. Then $\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{A}$-module.

Hint: observe that the result is true if $\mathcal{F} = \mathcal{A}^{\oplus p}$ and use a local presentation of $\mathcal{F}$ to get the conclusion.

11.4. Let $f : X \to Y$ be a continuous map of topological spaces. Given sheaves of abelian groups $\mathcal{A}$ on X and $\mathcal{B}$ on Y, show that there is a natural isomorphism

$$\mathrm{hom}_X(f^{-1}\mathcal{B}, \mathcal{A}) = \mathrm{hom}_Y(\mathcal{B}, f_\star\mathcal{A}).$$

Hint: use the natural morphisms (2.17).

11.5. Show that the sheaf of polynomials over $\mathbb{C}^n$ is a coherent sheaf of rings (with either the ordinary topology or the Zariski topology on $\mathbb{C}^n$). Extend this result to the case of regular algebraic functions on an algebraic variety.
Hint: check that the proof of the Oka theorem still applies.

11.6. Let P be a non zero polynomial on $\mathbb{C}^n$. If P is irreducible in $\mathbb{C}[z_1, \ldots, z_n]$, show that the hypersurface $H = P^{-1}(0)$ is globally irreducible as an analytic set. In general, show that the irreducible components of H are in a one-to-one correspondence with the irreducible factors of P.
Hint: for the first part, take coordinates such that $P(0, \ldots, 0, z_n)$ has degree equal to P; if H splits in two components H_1, H_2, then P can be written as a product P_1P_2 where the roots of $P_j(z', z_n)$ correspond to points in H_j.

11.7. Prove the following facts:

a) For every algebraic variety A of pure dimension p in $\mathbb{C}^n$, there are coordinates $z' = (z_1, \ldots, z_p)$, $z'' = (z_{p+1}, \ldots, z_n)$ such that $\pi : A \to \mathbb{C}^p$, $z \mapsto z''$ is proper with finite fibers, and such that A is entirely contained in a cone

$$|z''| \le C(|z'| + 1).$$

Hint: imitate the proof of Cor. 4.11.

b) Conversely if an analytic set A of pure dimension p in $\mathbb{C}^n$ is contained in a cone $|z''| \le C(|z'| + 1)$, then A is algebraic.
Hint: first apply (5.9) to conclude that the projection $\pi : A \to \mathbb{C}^p$ is finite. Then repeat the arguments used in the final part of the proof of Th. 4.19.

c) Deduce from a), b) that an algebraic set in $\mathbb{C}^n$ is irreducible if and only if it is irreducible as an analytic set.

11.8. Let $\Gamma : f(x, y) = 0$ be a germ of analytic curve in $\mathbb{C}^2$ through $(0, 0)$ and let $(\Gamma_j, 0)$ be the irreducible components of $(\Gamma, 0)$. Suppose that $f(0, y) \not\equiv 0$. Show that the roots y of $f(x, y) = 0$ corresponding to points of Γ near 0 are given by *Puiseux expansions* of the form $y = g_j(x^{1/q_j})$, where $g_j \in \mathcal{O}_{\mathbb{C},0}$ and where q_j is the sheet number of the projection $\Gamma_j \to \mathbb{C}$, $(x, y) \mapsto x$.
Hint: special case of the parametrization obtained in (4.27).

11.9. The goal of this exercise is to prove the existence and the analyticity of the *tangent cone* to an arbitrary analytic germ $(A, 0)$ in $\mathbb{C}^n$. Suppose that A is defined by holomorphic equations $f_1 = \ldots = f_N = 0$ in a ball $\Omega = B(0, r)$. Then the (set theoretic) tangent cone to A at 0 is the set $C(A, 0)$ of all limits of sequences $t_\nu^{-1}z_\nu$ with $z_\nu \in A$ and $\mathbb{C}^\star \ni t_\nu$ converging to 0.

a) Let E be the set of points $(z, t) \in \Omega \times \mathbb{C}^\star$ such that $z \in t^{-1}A$. Show that the closure $\overline{E}$ in $\Omega \times \mathbb{C}$ is analytic.

Hint: observe that $E = A \smallsetminus (\Omega \times \{0\})$ where $A = \{f_j(tz) = 0\}$ and apply Cor. 5.4.

b) Show that $C(A, 0)$ is a conic set and that $\overline{E} \cap (\Omega \times \{0\}) = C(A, 0) \times \{0\}$ and conclude. Infer from this that $C(A, 0)$ is an algebraic subset of $\mathbb{C}^n$.

11.10. Give a new proof of Theorem 5.5 based on the coherence of ideal sheaves and on the strong noetherian property.

11.11. Let X be an analytic space and let A, B be analytic subsets of pure dimensions. Show that $\operatorname{codim}_X(A \cap B) \le \operatorname{codim}_X A + \operatorname{codim}_X B$ if A or B is a local complete intersection, but that the equality does not necessarily hold in general. *Hint*: see Remark (6.5).

11.12. Let Γ be the curve in $\mathbb{C}^3$ parametrized by $\mathbb{C} \ni t \longmapsto (x, y, z) = (t^3, t^4, t^5)$. Show that the ideal sheaf $\mathcal{J}_\Gamma$ is generated by the polynomials $xz - y^2$, $x^3 - yz$ and $x^2y - z^2$, and that the germ $(\Gamma, 0)$ is not a (sheaf theoretic) complete intersection. *Hint*: Γ is smooth except at the origin. Let $f(x, y, z) = \sum a_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma$ be a convergent power series near 0. Show that $f \in \mathcal{J}_{\Gamma,0}$ if and only if all weighted homogeneous components $f_k = \sum_{3\alpha+4\beta+5\gamma=k} a_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma$ are in $\mathcal{J}_{\Gamma,0}$. By means of suitable substitutions, reduce the proof to the case when $f = f_k$ is homogeneous with all non zero monomials satisfying $\alpha \le 2$, $\beta \le 1$, $\gamma \le 1$; then check that there is at most one such monomial in each weighted degree ≤ 15 the product of a power of x by a homogeneous polynomial of weighted degree ≤ 8.

Chapter III
Positive Currents and Lelong Numbers

In 1957, P. Lelong introduced natural positivity concepts for currents of pure bidimension (p,p) on complex manifolds. With every analytic subset is associated a current of integration over its set of regular points and all such currents are positive and closed. The important closedness property is proved here via the Skoda-El Mir extension theorem. Positive currents have become an important tool for the study of global geometric problems as well as for questions related to local algebra and intersection theory. We develope here a differential geometric approach to intersection theory through a detailed study of wedge products of closed positive currents (Monge-Ampère operators). The Lelong-Poincaré equation and the Jensen-Lelong formula are basic in this context, providing a useful tool for studying the location and multiplicities of zeroes of entire functions on $\mathbb{C}^n$ or on a manifold, in relation with the growth at infinity. Lelong numbers of closed positive currents are then introduced; these numbers can be seen as a generalization to currents of the notion of multiplicity of a germ of analytic set at a singular point. We prove various properties of Lelong numbers (e.g. comparison theorems, semi-continuity theorem of Siu, transformation under holomorphic maps). As an application to Number Theory, we prove a general Schwarz lemma in $\mathbb{C}^n$ and derive from it Bombieri's theorem on algebraic values of meromorphic maps and the famous theorems of Gelfond-Schneider and Baker on the transcendence of exponentials and logarithms of algebraic numbers.

1. Basic Concepts of Positivity

1.A. Positive and Strongly Positive Forms

Let V be a complex vector space of dimension n and $(z_1,\ldots,z_n)$ coordinates on V. We denote by $(\partial/\partial z_1,\ldots,\partial/\partial z_n)$ the corresponding basis of V, by $(dz_1,\ldots,dz_n)$ its dual basis in $V^\star$ and consider the exterior algebra

$$\Lambda V_{\mathbb{C}}^\star = \bigoplus \Lambda^{p,q}V^\star, \qquad \Lambda^{p,q}V^\star = \Lambda^p V^\star \otimes \Lambda^q \overline{V^\star}.$$

We are of course especially interested in the case where $V = T_xX$ is the tangent space to a complex manifold X, but we want to emphasize here that our considerations only involve linear algebra. Let us first observe that V has a canonical orientation, given by the (n,n)-form

$$\tau(z) = \mathrm{i}dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}dz_n \wedge d\overline{z}_n = 2^n\, dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$$

where $z_j = x_j + iy_j$. In fact, if $(w_1, \ldots, w_n)$ are other coordinates, we find

$$dw_1 \wedge \ldots \wedge dw_n = \det(\partial w_j/\partial z_k)\, dz_1 \wedge \ldots \wedge dz_n,$$
$$\tau(w) = \big|\det(\partial w_j/\partial z_k)\big|^2\, \tau(z).$$

In particular, a complex manifold always has a canonical orientation. More generally, natural positivity concepts for (p,p)-forms can be defined.

(1.1) Definition. *A (p,p)-form $u \in \Lambda^{p,p}V^\star$ is said to be positive if for all $\alpha_j \in V^\star$, $1 \le j \le q = n-p$, then*

$$u \wedge \mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_q \wedge \overline{\alpha}_q$$

is a positive (n,n)-form. A (q,q)-form $v \in \Lambda^{q,q}V^\star$ is said to be strongly positive if v is a convex combination

$$v = \sum \gamma_s\ \mathrm{i}\alpha_{s,1} \wedge \overline{\alpha}_{s,1} \wedge \ldots \wedge \mathrm{i}\alpha_{s,q} \wedge \overline{\alpha}_{s,q}$$

where $\alpha_{s,j} \in V^\star$ and $\gamma_s \ge 0$.

(1.2) Example. Since $\mathrm{i}^p(-1)^{p(p-1)/2} = \mathrm{i}^{p^2}$, we have the commutation rules

$$\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p = i^{p^2}\alpha \wedge \overline{\alpha}, \qquad \forall \alpha = \alpha_1 \wedge \ldots \wedge \alpha_p \in \Lambda^{p,0}V^\star,$$
$$i^{p^2}\beta \wedge \overline{\beta} \wedge i^{m^2}\gamma \wedge \overline{\gamma} = i^{(p+m)^2}\beta \wedge \gamma \wedge \overline{\beta \wedge \gamma}, \qquad \forall \beta \in \Lambda^{p,0}V^\star,\ \forall \gamma \in \Lambda^{m,0}V^\star.$$

Take $m = q$ to be the complementary degree of p. Then $\beta \wedge \gamma = \lambda dz_1 \wedge \ldots \wedge dz_n$ for some $\lambda \in \mathbb{C}$ and $i^{n^2}\beta \wedge \gamma \wedge \overline{\beta \wedge \gamma} = |\lambda|^2 \tau(z)$. If we set $\gamma = \alpha_1 \wedge \ldots \wedge \alpha_q$, we find that $i^{p^2}\beta \wedge \overline{\beta}$ is a positive (p,p)-form for every $\beta \in \Lambda^{p,0}V^\star$; in particular, strongly positive forms are positive. □

The sets of positive and strongly positive forms are closed convex cones, i.e. closed and stable under convex combinations. By definition, the positive cone is dual to the strongly positive cone via the pairing

$$(1.3)\qquad \begin{aligned} \Lambda^{p,p}V^\star \times \Lambda^{q,q}V^\star &\longrightarrow \mathbb{C} \\ (u,v) &\longmapsto u \wedge v/\tau, \end{aligned}$$

that is, $u \in \Lambda^{p,p}V^\star$ is positive if and only if $u \wedge v \ge 0$ for all strongly positive forms $v \in \Lambda^{q,q}V^\star$. Since the bidual of an arbitrary convex cone Γ is equal to its closure $\overline{\Gamma}$, we also obtain that v is strongly positive if and only if $v \wedge u = u \wedge v$ is ≥ 0 for all positive forms u. Later on, we will need the following elementary lemma.

(1.4) Lemma. *Let $(z_1, \ldots, z_n)$ be arbitrary coordinates on V. Then $\Lambda^{p,p}V^\star$ admits a basis consisting of strongly positive forms*

$$\beta_s = \mathrm{i}\beta_{s,1} \wedge \overline{\beta}_{s,1} \wedge \ldots \wedge \mathrm{i}\beta_{s,p} \wedge \overline{\beta}_{s,p}, \qquad 1 \le s \le \binom{n}{p}^2$$

where each $\beta_{s,l}$ is of the type $dz_j \pm dz_k$ or $dz_j \pm \mathrm{i}dz_k$, $1 \le j,k \le n$.

Proof. Since one can always extract a basis from a set of generators, it is sufficient to see that the family of forms of the above type generates $\Lambda^{p,p}V^\star$. This follows from the identities

$$\begin{aligned} 4dz_j \wedge d\overline{z}_k = \ & (dz_j + dz_k) \wedge (\overline{dz_j + dz_k}) - (dz_j - dz_k) \wedge (\overline{dz_j - dz_k}) \\ & + \mathrm{i}(dz_j + \mathrm{i}dz_k) \wedge (\overline{dz_j + \mathrm{i}dz_k}) - \mathrm{i}(dz_j - \mathrm{i}dz_k) \wedge (\overline{dz_j - \mathrm{i}dz_k}), \end{aligned}$$

$$dz_{j_1} \wedge \ldots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \ldots \wedge d\overline{z}_{k_p} = \pm \bigwedge_{1 \le s \le p} dz_{j_s} \wedge d\overline{z}_{k_s}. \qquad \square$$

(1.5) Corollary. *All positive forms u are real, i.e. satisfy $\overline{u} = u$. In terms of coordinates, if $u = i^{p^2} \sum_{|I|=|J|=p} u_{I,J}\, dz_I \wedge d\overline{z}_J$, then the coefficients satisfy the hermitian symmetry relation $\overline{u_{I,J}} = u_{J,I}$.*

Proof. Clearly, every strongly positive (q,q)-form is real. By Lemma 1.4, these forms generate over $\mathbb{R}$ the real elements of $\Lambda^{q,q}V^\star$, so we conclude by duality that positive (p,p)-forms are also real. $\square$

(1.6) Criterion. *A form $u \in \Lambda^{p,p}V^\star$ is positive if and only if its restriction $u_{\restriction S}$ to every p-dimensional subspace $S \subset V$ is a positive volume form on S.*

Proof. If S is an arbitrary p-dimensional subspace of V we can find coordinates $(z_1, \ldots, z_n)$ on V such that $S = \{z_{p+1} = \ldots = z_n = 0\}$. Then

$$u_{\restriction S} = \lambda_S\, \mathrm{i}dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}dz_p \wedge d\overline{z}_p$$

where λ_S is given by

$$u \wedge \mathrm{i}dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots \wedge \mathrm{i}dz_n \wedge d\overline{z}_n = \lambda_S\, \tau(z).$$

If u is positive then $\lambda_S \ge 0$ so $u_{\restriction S}$ is positive for every S. The converse is true because the $(n-p, n-p)$-forms $\bigwedge_{j>p} \mathrm{i}dz_j \wedge d\overline{z}_j$ generate all strongly positive forms when S runs over all p-dimensional subspaces. $\square$

(1.7) Corollary. *A form $u = \mathrm{i}\sum_{j,k} u_{jk}\, dz_j \wedge d\overline{z}_k$ of bidegree $(1,1)$ is positive if and only if $\xi \mapsto \sum u_{jk}\xi_j\overline{\xi}_k$ is a semi-positive hermitian form on $\mathbb{C}^n$.*

Proof. If S is the complex line generated by ξ and $t \mapsto t\xi$ the parametrization of S, then $u_{\restriction S} = \big(\sum u_{jk}\xi_j\overline{\xi}_k\big)\, \mathrm{i}dt \wedge d\overline{t}$. $\square$

Observe that there is a canonical one-to-one correspondence between hermitian forms and real $(1,1)$-forms on V. The correspondence is given by

$$(1.8)\qquad h=\sum_{1\le j,k\le n} h_{jk}(z)\,dz_j\otimes d\overline{z}_k \longmapsto u=\mathrm{i}\sum_{1\le j,k\le n} h_{jk}(z)\,dz_j\wedge d\overline{z}_k$$

and does not depend on the choice of coordinates: indeed, as $\overline{h}_{jk}=h_{kj}$, one finds immediately

$$u(\xi,\eta)=\mathrm{i}\sum h_{jk}(z)(\xi_j\overline{\eta}_k-\eta_j\overline{\xi}_k)=-2\,\mathrm{Im}\,h(\xi,\eta),\qquad \forall\xi,\eta\in TX.$$

Moreover, h is ≥ 0 as a hermitian form if and only if $u\ge 0$ as a $(1,1)$-form. A diagonalization of h shows that every positive $(1,1)$-form $u\in\Lambda^{1,1}V^\star$ can be written

$$u=\sum_{1\le j\le r}\mathrm{i}\gamma_j\wedge\overline{\gamma}_j\ ,\qquad \gamma\in V^\star,\quad r=\text{rank of}\quad u,$$

in particular, every positive $(1,1)$-form is strongly positive. By duality, this is also true for $(n-1,n-1)$-forms.

(1.9) Corollary. *The notions of positive and strongly positive* (p,p)*-forms coincide for* $p=0,1,n-1,n$. □

(1.10) Remark. It is not difficult to see, however, that positivity and strong positivity differ in all bidegrees (p,p) such that $2\le p\le n-2$. Indeed, a positive form $i^{p^2}\beta\wedge\overline{\beta}$ with $\beta\in\Lambda^{p,0}V^\star$ is strongly positive if and only if β is decomposable as a product $\beta_1\wedge\ldots\wedge\beta_p$. To see this, suppose that

$$i^{p^2}\beta\wedge\overline{\beta}=\sum_{1\le j\le N} i^{p^2}\gamma_j\wedge\overline{\gamma}_j$$

where all $\gamma_j\in\Lambda^{p,0}V^\star$ are decomposable. Take N minimal. The equality can be also written as an equality of hermitian forms $|\beta|^2=\sum|\gamma_j|^2$ if β,γ_j are seen as linear forms on Λ^pV. The hermitian form $|\beta|^2$ has rank one, so we must have $N=1$ and $\beta=\lambda\gamma_j$, as desired. Note that there are many non decomposable p-forms in all degrees p such that $2\le p\le n-2$, e.g. $(dz_1\wedge dz_2+dz_3\wedge dz_4)\wedge\ldots\wedge dz_{p+2}$: if a p-form is decomposable, the vector space of its contractions by elements of $\bigwedge^{p-1}V$ is a p-dimensional subspace of $V^\star$; in the above example the dimension is $p+2$.

(1.11) Proposition. *If* $u_1,\ldots,u_s$ *are positive forms, all of them strongly positive* (*resp. all except perhaps one*), *then* $u_1\wedge\ldots\wedge u_s$ *is strongly positive* (*resp. positive*).

Proof. Immediate consequence of Def. 1.1. Observe however that the wedge product of two positive forms is not positive in general (otherwise we would infer that positivity coincides with strong positivity). □

(1.12) Proposition. *If $\Phi : W \longrightarrow V$ is a complex linear map and $u \in \Lambda^{p,p}V^\star$ is (strongly) positive, then $\Phi^\star u \in \Lambda^{p,p}W^\star$ is (strongly) positive.*

Proof. This is clear for strong positivity, since

$$\Phi^\star(\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p) = \mathrm{i}\beta_1 \wedge \overline{\beta}_1 \wedge \ldots \wedge \mathrm{i}\beta_p \wedge \overline{\beta}_p$$

with $\beta_j = \Phi^\star\alpha_j \in W^\star$, for all $\alpha_j \in V^\star$. For u positive, we may apply Criterion 1.6: if S is a p-dimensional subspace of W, then $u_{\restriction\Phi(S)}$ and $(\Phi^\star u)_{\restriction S} = (\Phi_{\restriction}S)^\star u_{\restriction\Phi(S)}$ are positive when $\Phi_{\restriction S} : S \longrightarrow \Phi(S)$ is an isomorphism; otherwise we get $(\Phi^\star u)_{\restriction S} = 0$. □

1.B. Positive Currents

The duality between the positive and strongly positive cones of forms can be used to define corresponding positivity notions for currents.

(1.13) Definition. *A current $T \in \mathcal{D}'_{p,p}(X)$ is said to be positive (resp. strongly positive) if $\langle T, u\rangle \geq 0$ for all test forms $u \in \mathcal{D}_{p,p}(X)$ that are strongly positive (resp. positive) at each point. The set of positive (resp. strongly positive) currents of bidimension (p,p) will be denoted*

$$\mathcal{D}'^{+}_{p,p}(X), \qquad \text{resp.} \quad \mathcal{D}'^{\oplus}_{p,p}(X).$$

It is clear that (strong) positivity is a local property and that the sets $\mathcal{D}'^{\oplus}_{p,p}(X) \subset \mathcal{D}'^{+}_{p,p}(X)$ are closed convex cones with respect to the weak topology. Another way of stating Def. 1.13 is:

T is positive (strongly positive) if and only if $T \wedge u \in \mathcal{D}'_{0,0}(X)$ is a positive measure for all strongly positive (positive) forms $u \in C^\infty_{p,p}(X)$.

This is so because a distribution $S \in \mathcal{D}'(X)$ such that $S(f) \geq 0$ for every non-negative function $f \in \mathcal{D}(X)$ is a positive measure.

(1.14) Proposition. *Every positive current $T = i^{(n-p)^2} \sum T_{I,J}\, dz_I \wedge d\overline{z}_J$ in $\mathcal{D}'^{+}_{p,p}(X)$ is real and of order 0, i.e. its coefficients $T_{I,J}$ are complex measures and satisfy $\overline{T_{I,J}} = T_{J,I}$ for all multi-indices $|I| = |J| = n-p$. Moreover $T_{I,I} \geq 0$, and the absolute values $|T_{I,J}|$ of the measures $T_{I,J}$ satisfy the inequality*

$$\lambda_I\lambda_J\,|T_{I,J}| \leq 2^p \sum_M \lambda_M^2\, T_{M,M}, \qquad I \cap J \subset M \subset I \cup J$$

where $\lambda_k \geq 0$ are arbitrary coefficients and $\lambda_I = \prod_{k\in I}\lambda_k$.

Proof. Since positive forms are real, positive currents have to be real by duality. Let us denote by $K = \complement I$ and $L = \complement J$ the ordered complementary multi-indices of I, J in $\{1, 2, \ldots, n\}$. The distribution $T_{I,I}$ is a positive measure because

$$T_{I,I}\ \tau = T \wedge i^{p^2} dz_K \wedge d\overline{z}_K \geq 0.$$

On the other hand, the proof of Lemma 1.4 yields

$$T_{I,J}\ \tau = \pm T \wedge i^{p^2} dz_K \wedge d\overline{z}_L = \sum_{a \in (\mathbb{Z}/4\mathbb{Z})^p} \varepsilon_a\, T \wedge \gamma_a \qquad \text{where}$$

$$\gamma_a = \bigwedge_{1 \leq s \leq p} \frac{\mathrm{i}}{4} (dz_{k_s} + i^{a_s} dz_{l_s}) \wedge (\overline{dz_{k_s} + i^{a_s} dz_{l_s}}), \qquad \varepsilon_a = \pm 1, \pm i.$$

Now, each $T \wedge \gamma_a$ is a positive measure, hence $T_{I,J}$ is a complex measure and

$$\begin{aligned}
|T_{I,J}|\,\tau &\leq \sum_a T \wedge \gamma_a = T \wedge \sum_a \gamma_a \\
&= T \wedge \bigwedge_{1 \leq s \leq p} \Big(\sum_{a_s \in \mathbb{Z}/4\mathbb{Z}} \frac{\mathrm{i}}{4} (dz_{k_s} + i^{a_s} dz_{l_s}) \wedge (\overline{dz_{k_s} + i^{a_s} dz_{l_s}}) \Big) \\
&= T \wedge \bigwedge_{1 \leq s \leq p} \big(\mathrm{i} dz_{k_s} \wedge d\overline{z}_{k_s} + \mathrm{i} dz_{l_s} \wedge d\overline{z}_{l_s} \big).
\end{aligned}$$

The last wedge product is a sum of at most 2^p terms, each of which is of the type $i^{p^2} dz_M \wedge d\overline{z}_M$ with $|M| = p$ and $M \subset K \cup L$. Since $T \wedge i^{p^2} dz_M \wedge d\overline{z}_M = T_{\complement M, \complement M}\ \tau$ and $\complement M \supset \complement K \cap \complement L = I \cap J$, we find

$$|T_{I,J}| \leq 2^p \sum_{M \supset I \cap J} T_{M,M}.$$

Now, consider a change of coordinates $(z_1, \ldots, z_n) = \Lambda w = (\lambda_1 w_1, \ldots, \lambda_n w_n)$ with $\lambda_1, \ldots, \lambda_n > 0$. In the new coordinates, the current T becomes $\Lambda^\star T$ and its coefficients become $\lambda_I \lambda_J\, T_{I,J}(\Lambda w)$. Hence, the above inequality implies

$$\lambda_I \lambda_J\, |T_{I,J}| \leq 2^p \sum_{M \supset I \cap J} \lambda_M^2\, T_{M,M}.$$

This inequality is still true for $\lambda_k \geq 0$ by passing to the limit. The inequality of Prop. 1.14 follows when all coefficients λ_k, $k \notin I \cup J$, are replaced by 0, so that $\lambda_M = 0$ for $M \not\subset I \cup J$. □

(1.15) Remark. If T is of order 0, we define the *mass measure* of T by $\|T\| = \sum |T_{I,J}|$ (of course $\|T\|$ depends on the choice of coordinates). By the Radon-Nikodym theorem, we can write $T_{I,J} = f_{I,J} \|T\|$ with a Borel function $f_{I,J}$ such that $\sum |f_{I,J}| = 1$. Hence

$$T = \|T\|\, f, \qquad \text{where} \ \ f = i^{(n-p)^2} \sum f_{I,J}\, dz_I \wedge d\overline{z}_J.$$

Then T is (strongly) positive if and only if the form $f(x) \in \Lambda^{n-p,n-p}T^\star_x X$ is (strongly) positive at $\|T\|$-almost all points $x \in X$. Indeed, this condition is clearly sufficient. On the other hand, if T is (strongly) positive and u_j is a sequence of forms with constant coefficients in $\Lambda^{p,p}T^\star X$ which is dense in the set of strongly positive (positive) forms, then $T \wedge u_j = \|T\|\, f \wedge u_j$, so $f(x) \wedge u_j$ has to be a positive (n,n)-form except perhaps for x in a set $N(u_j)$ of $\|T\|$-measure 0. By a simple density argument, we see that $f(x)$ is (strongly) positive outside the $\|T\|$-negligible set $N = \bigcup N(u_j)$.

As a consequence of this proof, T is positive (strongly positive) if and only if $T \wedge u$ is a positive measure for all strongly positive (positive) forms u of bidegree (p,p) with *constant coefficients* in the given coordinates $(z_1, \ldots, z_n)$. It follows that if T is (strongly) positive in a coordinate patch Ω, then the convolution $T \star \rho_\varepsilon$ is (strongly) positive in $\Omega_\varepsilon = \{x \in \Omega\,;\, d(x, \partial\Omega) > \varepsilon\}$. □

(1.16) Corollary. *If $T \in \mathcal{D}'_{p,p}(X)$ and $v \in C^0_{s,s}(X)$ are positive, one of them (resp. both of them) strongly positive, then the wedge product $T \wedge v$ is a positive (resp. strongly positive) current.*

This follows immediately from Remark 1.15 and Prop. 1.11 for forms. Similarly, Prop. 1.12 on pull-backs of positive forms easily shows that positivity properties of currents are preserved under direct or inverse images by holomorphic maps.

(1.17) Proposition. *Let $\Phi : X \longrightarrow Y$ be a holomorphic map between complex analytic manifolds.*

a) *If $T \in \mathcal{D}'^{+}_{p,p}(X)$ and $\Phi_{\restriction \mathrm{Supp}\,T}$ is proper, then $\Phi_\star T \in \mathcal{D}'^{+}_{p,p}(Y)$.*

b) *If $T \in \mathcal{D}'^{+}_{p,p}(Y)$ and if Φ is a submersion with m-dimensional fibers, then $\Phi^\star T \in \mathcal{D}'^{+}_{p+m,p+m}(X)$.*

Similar properties hold for strongly positive currents. □

1.C. Basic Examples of Positive Currents

We present here two fundamental examples which will be of interest in many circumstances.

(1.18) Current Associated to a Plurisubharmonic Function Let X be a complex manifold and $u \in \mathrm{Psh}(X) \cap L^1_{\mathrm{loc}}(X)$ a plurisubharmonic function. Then

$$T = \mathrm{i}d'd''u = \mathrm{i} \sum_{1 \le j,k \le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\, dz_j \wedge d\overline{z}_k$$

is a positive current of bidegree $(1,1)$. Moreover T is closed (we always mean here d-closed, that is, $dT = 0$). Assume conversely that Θ is a closed real

$(1,1)$-current on X. Poincaré's lemma implies that every point $x_0 \in X$ has a neighborhood Ω_0 such that $\Theta = dS$ with $S \in \mathcal{D}'_1(\Omega_0, \mathbb{R})$. Write $S = S^{1,0} + S^{0,1}$, where $S^{0,1} = \overline{S^{1,0}}$. Then $d''S = \Theta^{0,2} = 0$, and the Dolbeault-Grothendieck lemma shows that $S^{0,1} = d''v$ on some neighborhood $\Omega \subset \Omega_0$, with $v \in \mathcal{D}'(\Omega, \mathbb{C})$. Thus

$$S = \overline{d''v} + d''v = d'\overline{v} + d''v,$$
$$\Theta = dS = d'd''(v - \overline{v}) = \mathrm{i}d'd''u,$$

where $u = 2\,\mathrm{Re}\,v \in \mathcal{D}'(\Omega, \mathbb{R})$. If $\Theta \in C^\infty_{1,1}(X)$, the hypoellipticity of d'' in bidegree $(p,0)$ shows that $d'u$ is of class C^∞, so $u \in C^\infty(\Omega)$. When Θ is positive, the distribution u is a plurisubharmonic function (Th. I.3.31). We have thus proved:

(1.19) Proposition. *If $\Theta \in \mathcal{D}'^{+}_{n-1,n-1}(X)$ is a closed positive current of bidegree $(1,1)$, then for every point $x_0 \in X$ there exists a neighborhood Ω of x_0 and $u \in \mathrm{Psh}(\Omega)$ such that $\Theta = \mathrm{i}d'd''u$.* □

(1.20) Current of Integration on a Complex Submanifold Let $Z \subset X$ be a closed p-dimensional complex submanifold with its canonical orientation and $T = [Z]$. Then $T \in \mathcal{D}'^{\oplus}_{p,p}(X)$. Indeed, every (r,s)-form of total degree $r + s = 2p$ has zero restriction to Z unless $(r,s) = (p,p)$, therefore we have $[Z] \in \mathcal{D}'_{p,p}(X)$. Now, if $u \in \mathcal{D}_{p,p}(X)$ is a positive test form, then $u_{\restriction Z}$ is a positive volume form on Z by Criterion 1.6, therefore

$$\langle [Z], u \rangle = \int_Z u_{\restriction Z} \geq 0.$$

In this example the current $[Z]$ is also closed, because $d[Z] = \pm[\partial Z] = 0$ by Stokes' theorem. □

1.D. Trace Measure and Wirtinger's Inequality

We discuss now some questions related to the concept of area on complex submanifolds. Assume that X is equipped with a hermitian metric h, i.e. a positive definite hermitian form $h = \sum h_{jk} dz_j \otimes d\overline{z}_k$ of class C^∞ ; we denote by $\omega = \mathrm{i} \sum h_{jk} dz_j \wedge d\overline{z}_k \in C^\infty_{1,1}(X)$ the associated positive $(1,1)$-form.

(1.21) Definition. *For every $T \in \mathcal{D}'^{+}_{p,p}(X)$, the trace measure of T with respect to ω is the positive measure*

$$\sigma_T = \frac{1}{2^p p!} T \wedge \omega^p.$$

If $(\zeta_1, \ldots, \zeta_n)$ is an orthonormal frame of $T^\star X$ with respect to h on an open subset $U \subset X$, we may write

$$\omega = \mathrm{i}\sum_{1\le j\le n} \zeta_j\wedge\overline{\zeta}_j, \qquad \omega^p = i^{p^2}p!\sum_{|K|=p}\zeta_K\wedge\overline{\zeta}_K,$$
$$T = i^{(n-p)^2}\sum_{|I|=|J|=n-p} T_{I,J}\,\zeta_I\wedge\overline{\zeta}_J, \qquad T_{I,J}\in\mathcal{D}'(U),$$

where $\zeta_I = \zeta_{i_1}\wedge\ldots\wedge\zeta_{i_{n-p}}$. An easy computation yields

$$(1.22)\qquad \sigma_T = 2^{-p}\Big(\sum_{|I|=n-p} T_{I,I}\Big)\,\mathrm{i}\zeta_1\wedge\overline{\zeta}_1\wedge\ldots\wedge\mathrm{i}\zeta_n\wedge\overline{\zeta}_n.$$

For $X=\mathbb{C}^n$ with the standard hermitian metric $h=\sum dz_j\otimes d\overline{z}_j$, we get in particular

$$(1.22')\qquad \sigma_T = 2^{-p}\Big(\sum_{|I|=n-p} T_{I,I}\Big)\,\mathrm{i}dz_1\wedge d\overline{z}_1\wedge\ldots\wedge\mathrm{i}dz_n\wedge d\overline{z}_n.$$

Proposition 1.14 shows that the mass measure $||T||=\sum|T_{I,J}|$ of a positive current T is always dominated by $C\sigma_T$ where $C>0$ is a constant. It follows easily that the weak topology of $\mathcal{D}'_p(X)$ and of $\mathcal{D}^0_p{}'(X)$ coincide on $\mathcal{D}'^+_p(X)$, which is moreover a metrizable subspace: its weak topology is in fact defined by the collection of semi-norms $T\longmapsto|\langle T,f_\nu\rangle|$ where (f_ν) is an arbitrary dense sequence in $\mathcal{D}_p(X)$. By the Banach-Alaoglu theorem, the unit ball in the dual of a Banach space is weakly compact, thus:

(1.23) Proposition. *Let δ be a positive continuous function on X. Then the set of currents $T\in\mathcal{D}'^+_p(X)$ such that $\int_X\delta\,T\wedge\omega^p\le 1$ is weakly compact.*

Proof. Note that our set is weakly closed, since a weak limit of positive currents is positive and $\int_X\delta\,T\wedge\omega^p=\sup\langle T,\theta\delta\omega^p\rangle$ when θ runs over all elements of $\mathcal{D}(X)$ such that $0\le\theta\le 1$. □

Now, let Z be a p-dimensional complex analytic submanifold of X. We claim that

$$(1.24)\qquad \sigma_{[Z]} = \frac{1}{2^p p!}[Z]\wedge\omega^p = \text{Riemannian volume measure on } Z.$$

This result is in fact a special case of the following important inequality.

(1.25) Wirtinger's inequality. *Let Y be an oriented real submanifold of class C^1 and dimension $2p$ in X, and let dV_Y be the Riemannian volume form on Y associated with the metric $h_{\restriction Y}$. Set*

$$\frac{1}{2^p p!}\omega^p_{\restriction Y} = \alpha\,dV_Y, \qquad \alpha\in C^0(Y).$$

Then $|\alpha| \le 1$ and the equality holds if and only if Y is a complex analytic submanifold of X. In that case $\alpha = 1$ if the orientation of Y is the canonical one, $\alpha = -1$ otherwise.

Proof. The restriction $\omega_{\restriction Y}$ is a real antisymmetric 2-form on TY. At any point $z \in Y$, we can thus find an oriented orthonormal $\mathbb{R}$-basis $(e_1, e_2, \ldots, e_{2p})$ of T_zY such that

$$\frac{1}{2}\omega = \sum_{1\le k\le p} \alpha_k\, e^\star_{2k-1} \wedge e^\star_{2k} \quad \text{on} \;\; T_zY, \qquad \text{where}$$
$$\alpha_k = \frac{1}{2}\omega(e_{2k-1}, e_{2k}) = -\operatorname{Im} h(e_{2k-1}, e_{2k}).$$

We have $dV_Y = e^\star_1 \wedge \ldots \wedge e^\star_{2p}$ by definition of the Riemannian volume form. By taking the p-th power of ω, we get

$$\frac{1}{2^p p!}\omega^p_{\restriction T_zY} = \alpha_1 \ldots \alpha_p\, e^\star_1 \wedge \ldots \wedge e^\star_{2p} = \alpha_1 \ldots \alpha_p\, dV_Y.$$

Since (e_k) is an orthonormal $\mathbb{R}$-basis, we have $\operatorname{Re} h(e_{2k-1}, e_{2k}) = 0$, thus $h(e_{2k-1}, e_{2k}) = -\mathrm{i}\alpha_k$. As $|e_{2k-1}| = |e_{2k}| = 1$, we get

$$0 \le |e_{2k} \pm Je_{2k-1}|^2 = 2\big(1 \pm \operatorname{Re} h(Je_{2k-1}, e_{2k})\big) = 2(1 \pm \alpha_k).$$

Therefore

$$|\alpha_k| \le 1, \qquad |\alpha| = |\alpha_1 \ldots \alpha_p| \le 1,$$

with equality if and only if $\alpha_k = \pm 1$ for all k, i.e. $e_{2k} = \pm Je_{2k-1}$. In this case $T_zY \subset T_zX$ is a complex vector subspace at every point $z \in Y$, thus Y is complex analytic by Lemma I.4.23. Conversely, assume that Y is a $\mathbb{C}$-analytic submanifold and that $(e_1, e_3, \ldots, e_{2p-1})$ is an orthonormal complex basis of T_zY. If $e_{2k} := Je_{2k-1}$, then $(e_1, \ldots, e_{2p})$ is an orthonormal $\mathbb{R}$-basis corresponding to the canonical orientation and

$$\frac{1}{2}\omega_{\restriction Y} = \sum_{1\le k\le p} e^\star_{2k-1} \wedge e^\star_{2k}, \qquad \frac{1}{2^p p!}\omega^p_{\restriction Y} = e^\star_1 \wedge \ldots \wedge e^\star_{2p} = dV_Y. \qquad \square$$

Note that in the case of the standard hermitian metric ω on $X = \mathbb{C}^n$, the form $\omega = \mathrm{i}\sum dz_j \wedge d\overline{z}_j = d\big(\mathrm{i}\sum z_j\, d\overline{z}_j\big)$ is globally exact. Under this hypothesis, we are going to see that $\mathbb{C}$-analytic submanifolds are always *minimal surfaces* for the Plateau problem, which consists in finding a compact subvariety Y of minimal area with prescribed boundary ∂Y.

(1.26) Theorem. *Assume that the $(1,1)$-form ω is exact, say $\omega = d\gamma$ with $\gamma \in C^\infty_1(X, \mathbb{R})$, and let $Y, Z \subset X$ be $(2p)$-dimensional oriented compact real* *submanifolds of class C^1 with boundary. If $\partial Y = \partial Z$ and Z is complex analytic, then*

$$\mathrm{Vol}(Y) \geq \mathrm{Vol(Z)}.$$

Proof. Write $\omega = d\gamma$. Wirtinger's inequality and Stokes' theorem imply

$$\mathrm{Vol}(Y) \geq \frac{1}{2^p p!}\Big|\int_Y \omega^p\Big| = \frac{1}{2^p p!}\Big|\int_Y d(\omega^{p-1}\wedge\gamma)\Big| = \frac{1}{2^p p!}\Big|\int_{\partial Y}\omega^{p-1}\wedge\gamma\Big|,$$
$$\mathrm{Vol}(Z) = \frac{1}{2^p p!}\int_Z \omega^p = \frac{1}{2^p p!}\int_{\partial Z}\omega^{p-1}\wedge\gamma = \pm\frac{1}{2^p p!}\int_{\partial Y}\omega^{p-1}\wedge\gamma. \qquad \square$$

2. Closed Positive Currents

2.A. The Skoda-El Mir Extension Theorem

We first prove the Skoda-El Mir extension theorem (Skoda 1982, El Mir 1984), which shows in particular that a closed positive current defined in the complement of an analytic set E can be extended through E if (and only if) the mass of the current is locally finite near E. El Mir simplified Skoda's argument and showed that it is enough to assume E complete pluripolar. We follow here the exposition of Sibony's survey article (Sibony 1985).

(2.1) Definition. *A subset $E \subset X$ is said to be complete pluripolar in X if for every point $x_0 \in X$ there exist a neighborhood Ω of x_0 and a function $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ such that $E \cap \Omega = \{z \in \Omega\ ;\ u(z) = -\infty\}$.*

Note that any closed analytic subset $A \subset X$ is complete pluripolar: if $g_1 = \ldots = g_N = 0$ are holomorphic equations of A on an open set $\Omega \subset X$, we can take $u = \log(|g_1|^2 + \ldots + |g_N|^2)$.

(2.2) Lemma. *Let $E \subset X$ be a closed complete pluripolar set. If $x_0 \in X$ and Ω is a sufficiently small neighborhood of x_0, there exists:*

a) *a function $v \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega \smallsetminus E)$ such that $v = -\infty$ on $E \cap \Omega$;*

b) *an increasing sequence $v_k \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega)$, $0 \leq v_k \leq 1$, converging uniformly to 1 on every compact subset of $\Omega \smallsetminus E$, such that $v_k = 0$ on a neighborhood of $E \cap \Omega$.*

Proof. Assume that $\Omega_0 \subset\subset X$ is a coordinate patch of X containing x_0 and that $E \cap \Omega_0 = \{z \in \Omega_0\ ;\ u(z) = -\infty\}$, $u \in \mathrm{Psh}(\Omega_0)$. In addition, we can achieve $u \leq 0$ by shrinking Ω_0 and subtracting a constant to u. Select a convex increasing function $\chi \in C^\infty([0,1],\mathbb{R})$ such that $\chi(t) = 0$ on $[0,1/2]$ and $\chi(1) = 1$. We set

$$u_k = \chi\big(\exp(u/k)\big).$$

Then $0 \le u_k \le 1$, u_k is plurisubharmonic on Ω_0, $u_k = 0$ in a neighborhood ω_k of $E \cap \Omega_0$ and $\lim u_k = 1$ on $\Omega_0 \smallsetminus E$. Let $\Omega \subset\subset \Omega_0$ be a neighborhood of x_0, let $\delta_0 = d(\Omega, \complement\Omega_0)$ and $\varepsilon_k \in]0,\delta_0[$ be such that $\varepsilon_k < d(E \cap \overline{\Omega}, \overline{\Omega} \smallsetminus \omega_k)$. Then

$$w_k := \max_{j \le k}\{u_j \star \rho_{\varepsilon_j}\} \in \mathrm{Psh}(\Omega) \cap C^0(\Omega),$$

$0 \le w_k \le 1$, $w_k = 0$ on a neighborhood of $E \cap \Omega$ and w_k is an increasing sequence converging to 1 on $\Omega \smallsetminus E$ (note that $w_k \ge u_k$). Hence, the convergence is uniform on every compact subset of $\Omega \smallsetminus E$ by Dini's lemma. We may therefore choose a subsequence w_{k_s} such that $w_{k_s}(z) \ge 1 - 2^{-s}$ on an increasing sequence of open sets Ω'_s with $\bigcup \Omega'_s = \Omega \smallsetminus E$. Then

$$w(z) := |z|^2 + \sum_{s=0}^{+\infty}(w_{k_s}(z) - 1)$$

is a strictly plurisubharmonic function on Ω that is continuous on $\Omega \smallsetminus E$, and $w = -\infty$ on $E \cap \Omega$. Richberg's theorem I.3.40 applied on $\Omega \smallsetminus E$ produces $v \in \mathrm{Psh}(\Omega \smallsetminus E) \cap C^\infty(\Omega \smallsetminus E)$ such that $w \le v \le w+1$. If we set $v = -\infty$ on $E \cap \Omega$, then v is plurisubharmonic on Ω and has the properties required in a). After subtraction of a constant, we may assume $v \le 0$ on Ω. Then the sequence (v_k) of statement b) is obtained by letting $v_k = \chi\big(\exp(v/k)\big)$. □

(2.3) Theorem (El Mir)**.** *Let $E \subset X$ be a closed complete pluripolar set and $T \in \mathcal{D}'^{+}_{p,p}(X \smallsetminus E)$ a closed positive current. Assume that T has finite mass in a neighborhood of every point of E. Then the trivial extension $\tilde{T} \in \mathcal{D}'^{+}_{p,p}(X)$ obtained by extending the measures $T_{I,J}$ by 0 on E is closed on X.*

Proof. The statement is local on X, so we may work on a small open set Ω such that there exists a sequence $v_k \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega)$ as in 2.2 b). Let $\theta \in C^\infty([0,1])$ be a function such that $\theta = 0$ on $[0,1/3]$, $\theta = 1$ on $[2/3,1]$ and $0 \le \theta \le 1$. Then $\theta \circ v_k = 0$ near $E \cap \Omega$ and $\theta \circ v_k = 1$ for k large on every fixed compact subset of $\Omega \smallsetminus E$. Therefore $\tilde{T} = \lim_{k\to+\infty}(\theta \circ v_k)T$ and

$$d'\tilde{T} = \lim_{k\to+\infty} T \wedge d'(\theta \circ v_k)$$

in the weak topology of currents. It is therefore sufficient to check that $T \wedge d'(\theta \circ v_k)$ converges weakly to 0 in $\mathcal{D}'_{p-1,p}(\Omega)$ (note that $d''\tilde{T}$ is conjugate to $d'\tilde{T}$, thus $d''\tilde{T}$ will also vanish).

Assume first that $p = 1$. Then $T \wedge d'(\theta \circ v_k) \in \mathcal{D}'_{0,1}(\Omega)$, and we have to show that

$$\langle T \wedge d'(\theta \circ v_k), \overline{\alpha} \rangle = \langle T, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle \longrightarrow 0, \qquad \forall \alpha \in \mathcal{D}_{1,0}(\Omega).$$

As $\gamma \longmapsto \langle T, \mathrm{i}\gamma \wedge \overline{\gamma} \rangle$ is a non-negative hermitian form on $\mathcal{D}_{1,0}(\Omega)$, the Cauchy-Schwarz inequality yields

$$\big|\langle T, \mathrm{i}\beta \wedge \overline{\gamma} \rangle\big|^2 \leq \langle T, \mathrm{i}\beta \wedge \overline{\beta} \rangle \; \langle T, \mathrm{i}\gamma \wedge \overline{\gamma} \rangle, \qquad \forall \beta, \gamma \in \mathcal{D}_{1,0}(\Omega).$$

Let $\psi \in \mathcal{D}(\Omega)$, $0 \leq \psi \leq 1$, be equal to 1 in a neighborhood of $\operatorname{Supp} \alpha$. We find

$$\big|\langle T, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle\big|^2 \leq \langle T, \psi \mathrm{i} d'v_k \wedge d''v_k \rangle \; \langle T, \theta'(v_k)^2 \mathrm{i}\alpha \wedge \overline{\alpha} \rangle.$$

By hypothesis $\int_{\Omega \smallsetminus E} T \wedge \mathrm{i}\alpha \wedge \overline{\alpha} < +\infty$ and $\theta'(v_k)$ converges everywhere to 0 on Ω, thus $\langle T, \theta'(v_k)^2 \mathrm{i}\alpha \wedge \overline{\alpha} \rangle$ converges to 0 by Lebesgue's dominated convergence theorem. On the other hand

$$\mathrm{i}d'd''v_k^2 = 2v_k \, \mathrm{i}d'd''v_k + 2\mathrm{i}d'v_k \wedge d''v_k \geq 2\mathrm{i}d'v_k \wedge d''v_k,$$
$$2\langle T, \psi \mathrm{i}d'v_k \wedge d''v_k \rangle \leq \langle T, \psi \mathrm{i}d'd''v_k^2 \rangle.$$

As $\psi \in \mathcal{D}(\Omega)$, $v_k = 0$ near E and $d'T = d''T = 0$ on $\Omega \smallsetminus E$, an integration by parts yields

$$\langle T, \psi \mathrm{i}d'd''v_k^2 \rangle = \langle T, v_k^2 \mathrm{i}d'd''\psi \rangle \leq C \int_{\Omega \smallsetminus E} \|T\| < +\infty$$

where C is a bound for the coefficients of ψ. Thus $\langle T, \psi \mathrm{i}d'v_k \wedge d''v_k \rangle$ is bounded, and the proof is complete when $p = 1$.

In the general case, let $\beta_s = \mathrm{i}\beta_{s,1} \wedge \overline{\beta}_{s,1} \wedge \ldots \wedge \mathrm{i}\beta_{s,p-1} \wedge \overline{\beta}_{s,p-1}$ be a basis of forms of bidegree $(p-1, p-1)$ with constant coefficients (Lemma 1.4). Then $T \wedge \beta_s \in \mathcal{D}'^{+}_{1,1}(\Omega \smallsetminus E)$ has finite mass near E and is closed on $\Omega \smallsetminus E$. Therefore $d(\tilde{T} \wedge \beta_s) = (d\tilde{T}) \wedge \beta_s = 0$ on Ω for all s, and we conclude that $d\tilde{T} = 0$. □

(2.4) Corollary. *If $T \in \mathcal{D}'^{+}_{p,p}(X)$ is closed, if $E \subset X$ is a closed complete pluripolar set and $\mathbb{1}_E$ is its characteristic function, then $\mathbb{1}_E T$ and $\mathbb{1}_{X \smallsetminus E} T$ are closed (and, of course, positive).*

Proof. If we set $\Theta = T_{\restriction X \smallsetminus E}$, then Θ has finite mass near E and we have $\mathbb{1}_{X \smallsetminus E} T = \tilde{\Theta}$ and $\mathbb{1}_E T = T - \tilde{\Theta}$. □

2.B. Current of Integration over an Analytic Set

Let A be a pure p-dimensional analytic subset of a complex manifold X. We would like to generalize Example 1.20 and to define a current of integration $[A]$ by letting

$$(2.5) \quad \langle [A], v \rangle = \int_{A_{\mathrm{reg}}} v, \qquad v \in \mathcal{D}_{p,p}(X).$$

One difficulty is of course to verify that the integral converges near $A_{\rm sing}$. This follows from the following lemma, due to (Lelong 1957).

(2.6) Lemma. *The current* $[A_{\rm reg}] \in \mathcal{D}'^{+}_{p,p}(X \smallsetminus A_{\rm sing})$ *has finite mass in a neighborhood of every point* $z_0 \in A_{\rm sing}$.

Proof. Set $T = [A_{\rm reg}]$ and let $\Omega \ni z_0$ be a coordinate open set. If we write the monomials $dz_K \wedge d\overline{z}_L$ in terms of an arbitrary basis of $\Lambda^{p,p}T^\star\Omega$ consisting of decomposable forms $\beta_s = \mathrm{i}\beta_{s,1}\wedge\overline{\beta}_{s,1}\wedge\ldots\wedge\beta_{s,p}\wedge\overline{\beta}_{s,p}$ (cf. Lemma 1.4), we see that the measures $T_{I,J}\,.\,\tau$ are linear combinations of the positive measures $T\wedge\beta_s$. It is thus sufficient to prove that all $T\wedge\beta_s$ have finite mass near $A_{\rm sing}$. Without loss of generality, we may assume that (A, z_0) is irreducible. Take new coordinates $w = (w_1, \ldots, w_n)$ such that $w_j = \beta_{s,j}(z - z_0)$, $1 \le j \le p$. After a slight perturbation of the $\beta_{s,j}$, we may assume that each projection

$$\pi_s : A \cap (\Delta' \times \Delta''), \qquad w \longmapsto w' = (w_1, \ldots, w_p)$$

defines a ramified covering of A (cf. Prop. II.3.8 and Th. II.3.19), and that (β_s) remains a basis of $\Lambda^{p,p}T^\star\Omega$. Let S be the ramification locus of π_s and $A_S = A \cap \big((\Delta' \smallsetminus S) \times \Delta''\big) \subset A_{\rm reg}$. The restriction of π_s: $A_S \longrightarrow \Delta' \smallsetminus S$ is then a covering with finite sheet number q_s and we find

$$\begin{aligned}\int_{\Delta'\times\Delta''}[A_{\rm reg}]\wedge\beta_s &= \int_{A_{\rm reg}\cap(\Delta'\times\Delta'')} \mathrm{i}dw_1\wedge d\overline{w}_1\wedge\ldots\wedge \mathrm{i}dw_p\wedge d\overline{w}_p\\ &= \int_{A_S}\mathrm{i}dw_1\wedge d\overline{w}_1\ldots\wedge d\overline{w}_p = q_s\int_{\Delta'\smallsetminus S}\mathrm{i}dw_1\wedge d\overline{w}_1\ldots\wedge d\overline{w}_p < +\infty.\end{aligned}$$

The second equality holds because A_S is the complement in $A_{\rm reg}\cap(\Delta'\times\Delta'')$ of an analytic subset (such a set is of zero Lebesgue measure in $A_{\rm reg}$). □

(2.7) Theorem (Lelong, 1957)**.** *For every pure p-dimensional analytic subset $A \subset X$, the current of integration* $[A] \in \mathcal{D}'^{+}_{p,p}(X)$ *is a closed positive current on X.*

Proof. Indeed, $[A_{\rm reg}]$ has finite mass near $A_{\rm sing}$ and $[A]$ is the trivial extension of $[A_{\rm reg}]$ to X through the complete pluripolar set $E = A_{\rm sing}$. Theorem 2.7 is then a consequence of El Mir's theorem. □

2.C. Support Theorems and Lelong-Poincaré Equation

Let $M \subset X$ be a closed C^1 real submanifold of X. The *holomorphic tangent space* at a point $x \in M$ is

$$(2.8)\qquad {}^hT_xM = T_xM \cap JT_xM,$$

that is, the largest complex subspace of T_xX contained in T_xM. We define the *Cauchy-Riemann dimension* of M at x by $\mathrm{CRdim}_x M = \dim_{\mathbb{C}} {}^hT_xM$ and say that M is a *CR submanifold* of X if $\mathrm{CRdim}_x M$ is a constant. In general, we set

$$\text{(2.9)} \quad \mathrm{CRdim}\, M = \max_{x\in M} \mathrm{CRdim}_x M = \max_{x\in M} \dim_{\mathbb{C}} {}^hT_xM.$$

A current Θ is said to be *normal* if Θ and $d\Theta$ are currents of order 0. For instance, every closed positive current is normal. We are going to prove two important theorems describing the structure of normal currents with support in CR submanifolds.

(2.10) First theorem of support. *Let $\Theta \in \mathcal{D}'_{p,p}(X)$ be a normal current. If* $\mathrm{Supp}\,\Theta$ *is contained in a real submanifold M of CR dimension $< p$, then $\Theta = 0$.*

Proof. Let $x_0 \in M$ and let $g_1, \ldots, g_m$ be real C^1 functions in a neighborhood Ω of x_0 such that $M = \{z \in \Omega\ ;\ g_1(z) = \ldots = g_m(z) = 0\}$ and $dg_1 \wedge \ldots \wedge dg_m \neq 0$ on Ω. Then

$${}^hTM = TM \cap JTM = \bigcap_{1\le k\le m} \ker dg_k \cap \ker(dg_k \circ J) = \bigcap_{1\le k\le m} \ker d'g_k$$

because $d'g_k = \frac{1}{2}\big(dg_k - i(dg_k)\circ J\big)$. As $\dim_{\mathbb{C}} {}^hTM < p$, the rank of the system of $(1,0)$-forms $(d'g_k)$ must be $> n-p$ at every point of $M\cap\Omega$. After a change of the ordering, we may assume for example that $\zeta_1 = d'g_1$, $\zeta_2 = d'g_2$, …, $\zeta_{n-p+1} = d'g_{n-p+1}$ are linearly independent on Ω (shrink Ω if necessary). Complete $(\zeta_1, \ldots, \zeta_{n-p+1})$ into a continuous frame $(\zeta_1, \ldots, \zeta_n)$ of $T^\star X_{\restriction\Omega}$ and set

$$\Theta = \sum_{|I|=|J|=n-p} \Theta_{I,J}\,\zeta_I \wedge \overline{\zeta}_J \quad \text{on} \quad \Omega.$$

As Θ and $d'\Theta$ have measure coefficients supported on M and $g_k = 0$ on M, we get $g_k\Theta = g_k d'\Theta = 0$, thus

$$d'g_k \wedge \Theta = d'(g_k\Theta) - g_k d'\Theta = 0, \qquad 1 \le k \le m,$$

in particular $\zeta_k \wedge \Theta = 0$ for all $1 \le k \le n-p+1$. When $|I| = n-p$, the multi-index $\complement I$ contains at least one of the elements $1, \ldots, n-p+1$, hence $\Theta \wedge \zeta_{\complement I} \wedge \overline{\zeta}_{\complement J} = 0$ and $\Theta_{I,J} = 0$. □

(2.11) Corollary. *Let $\Theta \in \mathcal{D}'_{p,p}(X)$ be a normal current. If* $\mathrm{Supp}\,\Theta$ *is contained in an analytic subset A of dimension $< p$, then $\Theta = 0$.*

Proof. As A_{reg} is a submanifold of CRdim $< p$ in $X \smallsetminus A_{\mathrm{sing}}$, Theorem 2.9 shows that $\mathrm{Supp}\,\Theta \subset A_{\mathrm{sing}}$ and we conclude by induction on $\dim A$. □

Now, assume that $M \subset X$ is a CR submanifold of class C^1 with $\mathrm{CRdim}\, M = p$ and that ${}^hTM$ is an integrable subbundle of TM; this means that the Lie bracket of two vector fields in ${}^hTM$ is in ${}^hTM$. The Frobenius integrability theorem then shows that M is locally fibered by complex analytic p-dimensional submanifolds. More precisely, in a neighborhood of every point of M, there is a submersion $\sigma : M \longrightarrow Y$ onto a real C^1 manifold Y such that the tangent space to each fiber $F_t = \sigma^{-1}(t)$, $t \in Y$, is the holomorphic tangent space ${}^hTM$; moreover, the fibers F_t are necessarily complex analytic in view of Lemma 1.7.18. Under these assumptions, with any complex measure μ on Y we associate a current Θ with support in M by

$$(2.12)\qquad \Theta = \int_{t\in Y} [F_t]\, d\mu(t), \qquad \text{i.e. } \langle \Theta, u\rangle = \int_{t\in Y} \Big(\int_{F_t} u\Big)\, d\mu(t)$$

for all $u \in \mathcal{D}'_{p,p}(X)$. Then clearly $\Theta \in \mathcal{D}'_{p,p}(X)$ is a closed current of order 0, for all fibers $[F_t]$ have the same properties. When the fibers F_t are connected, the following converse statement holds:

(2.13) Second theorem of support. *Let $M \subset X$ be a CR submanifold of CR dimension p such that there is a submersion $\sigma : M \longrightarrow Y$ of class C^1 whose fibers $F_t = \sigma^{-1}(t)$ are connected and are the integral manifolds of the holomorphic tangent space ${}^hTM$. Then any closed current $\Theta \in \mathcal{D}'_{p,p}(X)$ of order 0 with support in M can be written $\Theta = \int_Y [F_t]\, d\mu(t)$ with a unique complex measure μ on Y. Moreover Θ is (strongly) positive if and only if the measure μ is positive.*

Proof. Fix a compact set $K \subset Y$ and a C^1 retraction ρ from a neighborhood V of M onto M. By means of a partition of unity, it is easy to construct a positive form $\alpha \in \mathcal{D}^0_{p,p}(V)$ such that $\int_{F_t} \alpha = 1$ for each fiber F_t with $t \in K$. Then the uniqueness and positivity statements for μ follow from the obvious formula

$$\int_Y f(t)\, d\mu(t) = \langle \Theta, (f \circ \rho)\, \alpha\rangle, \qquad \forall f \in C^0(Y), \quad \mathrm{Supp}\, f \subset K.$$

Now, let us prove the existence of μ. Let $x_0 \in M$. There is a small neighborhood Ω of x_0 and real coordinates $(x_1, y_1, \ldots, x_p, y_p, t_1, \ldots, t_q, g_1, \ldots, g_m)$ such that

• $z_j = x_j + \mathrm{i}y_j$, $1 \le j \le p$, are holomorphic functions on Ω that define complex coordinates on all fibers $F_t \cap \Omega$.

• $t_1, \ldots, t_q$ restricted to $M \cap \Omega$ are pull-backs by $\sigma : M \to Y$ of local coordinates on an open set $U \subset Y$ such that $\sigma_{\restriction \Omega} : M \cap \Omega \longrightarrow U$ is proper and surjective.

• $g_1 = \ldots = g_m = 0$ are equations of M in Ω.

Then $TF_t = \{dt_j = dg_k = 0\}$ equals ${}^hTM = \{d'g_k = 0\}$ and the rank of $(d'g_1, \ldots, d'g_m)$ is equal to $n - p$ at every point of $M \cap \Omega$. After a change of the ordering we may suppose that $\zeta_1 = d'g_1, \ldots, \zeta_{n-p} = d'g_{n-p}$ are linearly independent on Ω. As in Prop. 2.10, we get $\zeta_k \wedge \Theta = \overline{\zeta}_k \wedge \Theta = 0$ for $1 \le k \le n-p$ and infer that $\Theta \wedge \zeta_{\complement I} \wedge \overline{\zeta}_{\complement J} = 0$ unless $I = J = L$ where $L = \{1, 2, \ldots, n-p\}$. Hence

$$\Theta = \Theta_{L,L}\ \zeta_1 \wedge \ldots \wedge \zeta_{n-p} \wedge \overline{\zeta}_1 \wedge \ldots \wedge \overline{\zeta}_{n-p} \quad \text{ on } \ \Omega.$$

Now $\zeta_1 \wedge \ldots \wedge \overline{\zeta}_{n-p}$ is proportional to $dt_1 \wedge \ldots dt_q \wedge dg_1 \wedge \ldots \wedge dg_m$ because both induce a volume form on the quotient space $TX_{\restriction M}/{}^hTM$. Therefore, there is a complex measure ν supported on $M \cap \Omega$ such that

$$\Theta = \nu\ dt_1 \wedge \ldots dt_q \wedge dg_1 \wedge \ldots \wedge dg_m \quad \text{ on } \ \Omega.$$

As Θ is supposed to be closed, we have $\partial\nu/\partial x_j = \partial\nu/\partial y_j = 0$. Hence ν is a measure depending only on (t, g), with support in $g = 0$. We may write $\nu = d\mu_U(t) \otimes \delta_0(g)$ where μ_U is a measure on $U = \sigma(M \cap \Omega)$ and δ_0 is the Dirac measure at 0. If $j : M \longrightarrow X$ is the injection, this means precisely that $\Theta = j_\star \sigma^\star \mu_U$ on Ω, i.e.

$$\Theta = \int_{t \in U} [F_t]\, d\mu_U(t) \quad \text{ on } \ \Omega.$$

The uniqueness statement shows that for two open sets Ω_1, Ω_2 as above, the associated measures μ_{U_1} and μ_{U_2} coincide on $\sigma(M \cap \Omega_1 \cap \Omega_2)$. Since the fibers F_t are connected, there is a unique measure μ which coincides with all measures μ_U. □

(2.14) Corollary. *Let A be an analytic subset of X with global irreducible components A_j of pure dimension p. Then any closed current $\Theta \in \mathcal{D}'_{p,p}(X)$ of order 0 with support in A is of the form $\Theta = \sum \lambda_j [A_j]$ where $\lambda_j \in \mathbb{C}$. Moreover, Θ is (strongly) positive if and only if all coefficients λ_j are ≥ 0.*

Proof. The regular part $M = A_{\rm reg}$ is a complex submanifold of $X \smallsetminus A_{\rm sing}$ and its connected components are $A_j \cap A_{\rm reg}$. Thus, we may apply Th. 2.13 in the case where Y is discrete to see that $\Theta = \sum \lambda_j [A_j]$ on $X \smallsetminus A_{\rm sing}$. Now $\dim A_{\rm sing} < p$ and the difference $\Theta - \sum \lambda_j [A_j] \in \mathcal{D}'_{p,p}(X)$ is a closed current of order 0 with support in $A_{\rm sing}$, so this current must vanish by Cor. 2.11. □

(2.15) Lelong-Poincaré equation. *Let $f \in \mathcal{M}(X)$ be a meromorphic function which does not vanish identically on any connected component of X and let $\sum m_j Z_j$ be the divisor of f. Then the function $\log|f|$ is locally integrable on X and satisfies the equation*

$$\frac{\mathrm{i}}{\pi} d'd'' \log|f| = \sum\ m_j\, [Z_j]$$

in the space $\mathcal{D}'_{n-1,n-1}(X)$ of currents of bidimension $(n-1, n-1)$.

We refer to Sect. 2.6 for the definition of divisors, and especially to (2.6.14). Observe that if f is holomorphic, then $\log|f| \in \mathrm{Psh}(X)$, the coefficients m_j are positive integers and the right hand side is a positive current in $\mathcal{D}'^{+}_{n-1,n-1}(X)$.

Proof. Let $Z = \bigcup Z_j$ be the support of $\mathrm{div}(f)$. Observe that the sum in the right hand side is locally finite and that $d'd''\log|f|$ is supported on Z, since

$$d'\log|f|^2 = d'\log(f\overline{f}) = \frac{\overline{f}\,df}{f\overline{f}} = \frac{df}{f} \quad \text{on} \quad X \smallsetminus Z.$$

In a neighborhood Ω of a point $a \in Z_j \cap Z_{\mathrm{reg}}$, we can find local coordinates $(w_1, \ldots, w_n)$ such that $Z_j \cap \Omega$ is given by the equation $w_1 = 0$. Then Th. 2.6.6 shows that f can be written $f(w) = u(w) w_1^{m_j}$ with an invertible holomorphic function u on a smaller neighborhood $\Omega' \subset \Omega$. Then we have

$$\mathrm{i}d'd''\log|f| = \mathrm{i}d'd''\big(\log|u| + m_j \log|w_1|\big) = m_j\,\mathrm{i}d'd''\log|w_1|.$$

For $z \in \mathbb{C}$, Cor. I.3.4 implies

$$\mathrm{i}d'd''\log|z|^2 = -\mathrm{i}d''\Big(\frac{dz}{z}\Big) = -\mathrm{i}\pi\delta_0\, d\overline{z} \wedge dz = 2\pi\,[0].$$

If $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}$ is the projection $z \longmapsto z_1$ and $H \subset \mathbb{C}^n$ the hyperplane $\{z_1 = 0\}$, formula (1.2.19) shows that

$$\mathrm{i}d'd''\log|z_1| = \mathrm{i}d'd''\log|\Phi(z)| = \Phi^\star(\mathrm{i}d'd''\log|z|) = \pi\Phi^\star([0]) = \pi\,[H],$$

because Φ is a submersion. We get therefore $\frac{\mathrm{i}}{\pi}d'd''\log|f| = m_j[Z_j]$ in Ω'. This implies that the Lelong-Poincaré equation is valid at least on $X \smallsetminus Z_{\mathrm{sing}}$. As $\dim Z_{\mathrm{sing}} < n-1$, Cor. 2.11 shows that the equation holds everywhere on X. $\square$

3. Definition of Monge-Ampère Operators

Let X be a n-dimensional complex manifold. We denote by $d = d' + d''$ the usual decomposition of the exterior derivative in terms of its $(1,0)$ and $(0,1)$ parts, and we set

$$d^c = \frac{1}{2\mathrm{i}\pi}(d' - d'').$$

It follows in particular that d^c is a real operator, i.e. $\overline{d^c u} = d^c\overline{u}$, and that $dd^c = \frac{\mathrm{i}}{\pi}d'd''$. Although not quite standard, the $1/2\mathrm{i}\pi$ normalization is very convenient for many purposes, since we may then forget the factor 2π almost everywhere (e.g. in the Lelong-Poincaré equation (2.15)). In this context, we have the following integration by part formula.

(3.1) Formula. *Let $\Omega \subset\subset X$ be a smoothly bounded open set in X and let f, g be forms of class C^2 on $\overline{\Omega}$ of pure bidegrees (p,p) and (q,q) with $p+q=n-1$. Then*

$$\int_{\Omega} f \wedge dd^c g - dd^c f \wedge g = \int_{\partial\Omega} f \wedge d^c g - d^c f \wedge g.$$

Proof. By Stokes' theorem the right hand side is the integral over Ω of

$$d(f \wedge d^c g - d^c f \wedge g) = f \wedge dd^c g - dd^c f \wedge g + (df \wedge d^c g + d^c f \wedge dg).$$

As all forms of total degree $2n$ and bidegree $\neq (n,n)$ are zero, we get

$$df \wedge d^c g = \frac{1}{2\mathrm{i}\pi}(d''f \wedge d'g - d'f \wedge d''g) = -d^c f \wedge dg. \qquad \square$$

Let u be a plurisubharmonic function on X and let T be a closed positive current of bidimension (p,p), i.e. of bidegree $(n-p, n-p)$. Our desire is to define the wedge product $dd^c u \wedge T$ even when neither u nor T are smooth. A priori, this product does not make sense because $dd^c u$ and T have measure coefficients and measures cannot be multiplied; see (Kiselman 1983) for interesting counterexamples. Assume however that u is a *locally bounded* plurisubharmonic function. Then the current uT is well defined since u is a locally bounded Borel function and T has measure coefficients. According to (Bedford-Taylor 1982) we define

$$dd^c u \wedge T = dd^c(uT)$$

where $dd^c(\ \ )$ is taken in the sense of distribution (or current) theory.

(3.2) Proposition. *The wedge product $dd^c u \wedge T$ is again a closed positive current.*

Proof. The result is local. In an open set $\Omega \subset \mathbb{C}^n$, we can use convolution with a family of regularizing kernels to find a decreasing sequence of smooth plurisubharmonic functions $u_k = u \star \rho_{1/k}$ converging pointwise to u. Then $u \le u_k \le u_1$ and Lebesgue's dominated convergence theorem shows that $u_k T$ converges weakly to uT; thus $dd^c(u_k T)$ converges weakly to $dd^c(uT)$ by the weak continuity of differentiations. However, since u_k is smooth, $dd^c(u_k T)$ coincides with the product $dd^c u_k \wedge T$ in its usual sense. As $T \ge 0$ and as $dd^c u_k$ is a positive $(1,1)$-form, we have $dd^c u_k \wedge T \ge 0$, hence the weak limit $dd^c u \wedge T$ is ≥ 0 (and obviously closed). $\square$

Given locally bounded plurisubharmonic functions $u_1, \ldots, u_q$, we define inductively

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T = dd^c(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T).$$

By (3.2) the product is a closed positive current. In particular, when u is a locally bounded plurisubharmonic function, the bidegree (n,n) current $(dd^c u)^n$ is well defined and is a positive measure. If u is of class C^2, a computation in local coordinates gives

$$(dd^c u)^n = \det\Big(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\Big) \cdot \frac{n!}{\pi^n}\, \mathrm{i}dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}dz_n \wedge d\overline{z}_n.$$

The expression "Monge-Ampère operator" classically refers to the non-linear partial differential operator $u \longmapsto \det(\partial^2 u/\partial z_j \partial \overline{z}_k)$. By extension, all operators $(dd^c)^q$ defined above are also called Monge-Ampère operators.

Now, let Θ be a current of order 0. When $K \subset\subset X$ is an arbitrary compact subset, we define a *mass* semi-norm

$$||\Theta||_K = \sum_j \int_{K_j} \sum_{I,J} |\Theta_{I,J}|$$

by taking a partition $K = \bigcup K_j$ where each $\overline{K_j}$ is contained in a coordinate patch and where $\Theta_{I,J}$ are the corresponding measure coefficients. Up to constants, the semi-norm $||\Theta||_K$ does not depend on the choice of the coordinate systems involved. When K itself is contained in a coordinate patch, we set $\beta = dd^c|z|^2$ over K; then, if $\Theta \geq 0$, there are constants $C_1, C_2 > 0$ such that

$$C_1 ||\Theta||_K \leq \int_K \Theta \wedge \beta^p \leq C_2 ||\Theta||_K.$$

We denote by $L^1(K)$, resp. by $L^\infty(K)$, the space of integrable (resp. bounded measurable) functions on K with respect to any smooth positive density on X.

(3.3) Chern-Levine-Nirenberg inequalities (1969)**.** *For all compact subsets K, L of X with $L \subset K^\circ$, there exists a constant $C_{K,L} \geq 0$ such that*

$$||dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T||_L \leq C_{K,L}\ ||u_1||_{L^\infty(K)} \ldots ||u_q||_{L^\infty(K)}\, ||T||_K.$$

Proof. By induction, it is sufficient to prove the result for $q = 1$ and $u_1 = u$. There is a covering of L by a family of balls $B'_j \subset\subset B_j \subset K$ contained in coordinate patches of X. Let $\chi \in \mathcal{D}(B_j)$ be equal to 1 on $\overline{B}'_j$. Then

$$||dd^c u \wedge T||_{L\cap \overline{B}'_j} \leq C \int_{\overline{B}'_j} dd^c u \wedge T \wedge \beta^{p-1} \leq C \int_{B_j} \chi\, dd^c u \wedge T \wedge \beta^{p-1}.$$

As T and β are closed, an integration by parts yields

$$||dd^c u \wedge T||_{L\cap \overline{B}'_j} \leq C \int_{B_j} u\, T \wedge dd^c \chi \wedge \beta^{p-1} \leq C' ||u||_{L^\infty(K)} ||T||_K$$

where C' is equal to C multiplied by a bound for the coefficients of the smooth form $dd^c\chi \wedge \beta^{p-1}$. □

(3.4) Remark. With the same notations as above, any plurisubharmonic function V on X satisfies inequalities of the type

a) $||dd^cV||_L \le C_{K,L}\,||V||_{L^1(K)}$.

b) $\sup_L V_+ \le C_{K,L}\,||V||_{L^1(K)}$.

In fact the inequality

$$\int_{L\cap\overline{B}'_j} dd^cV \wedge \beta^{n-1} \le \int_{B_j} \chi dd^cV \wedge \beta^{n-1} = \int_{B_j} V dd^c\chi \wedge \beta^{n-1}$$

implies a), and b) follows from the mean value inequality.

(3.5) Remark. Products of the form $\Theta = \gamma_1 \wedge \ldots \wedge \gamma_q \wedge T$ with mixed $(1,1)$-forms $\gamma_j = dd^cu_j$ or $\gamma_j = dv_j \wedge d^cw_j + dw_j \wedge d^cv_j$ are also well defined whenever u_j, v_j, w_j are locally bounded plurisubharmonic functions. Moreover, for $L \subset K^\circ$, we have

$$||\Theta||_L \le C_{K,L}||T||_K \prod ||u_j||_{L^\infty(K)} \prod ||v_j||_{L^\infty(K)} \prod ||w_j||_{L^\infty(K)}.$$

To check this, we may suppose $v_j, w_j \ge 0$ and $||v_j|| = ||w_j|| = 1$ in $L^\infty(K)$. Then the inequality follows from (3.3) by the polarization identity

$$2(dv_j \wedge d^cw_j + dw_j \wedge d^cv_j) = dd^c(v_j + w_j)^2 - dd^cv_j^2 - dd^cw_j^2 - v_jdd^cw_j - w_jdd^cv_j$$

in which all dd^c operators act on plurisubharmonic functions.

(3.6) Corollary. *Let $u_1, \ldots, u_q$ be continuous (finite) plurisubharmonic functions and let $u_1^k, \ldots, u_q^k$ be sequences of plurisubharmonic functions converging locally uniformly to $u_1, \ldots, u_q$. If T_k is a sequence of closed positive currents converging weakly to T, then*

a) $u_1^k dd^cu_2^k \wedge \ldots \wedge dd^cu_q^k \wedge T_k \longrightarrow u_1dd^cu_2 \wedge \ldots \wedge dd^cu_q \wedge T$ *weakly.*

b) $dd^cu_1^k \wedge \ldots \wedge dd^cu_q^k \wedge T_k \longrightarrow dd^cu_1 \wedge \ldots \wedge dd^cu_q \wedge T$ *weakly.*

Proof. We observe that b) is an immediate consequence of a) by the weak continuity of dd^c. By using induction on q, it is enough to prove result a) when $q = 1$. If (u^k) converges locally uniformly to a finite continuous plurisubharmonic function u, we introduce local regularizations $u_\varepsilon = u \star \rho_\varepsilon$ and write

$$u^kT_k - uT = (u^k - u)T_k + (u - u_\varepsilon)T_k + u_\varepsilon(T_k - T).$$

As the sequence T_k is weakly convergent, it is locally uniformly bounded in mass, thus $||(u^k - u)T_k||_K \le ||u^k - u||_{L^\infty(K)}||T_k||_K$ converges to 0 on every compact set K. The same argument shows that $||(u-u_\varepsilon)T_k||_K$ can be made arbitrarily small by choosing ε small enough. Finally u_ε is smooth, so $u_\varepsilon(T_k-T)$ converges weakly to 0. □

Now, we prove a deeper monotone continuity theorem due to (Bedford-Taylor 1982) according to which the continuity and uniform convergence assumptions can be dropped if each sequence (u_j^k) is decreasing and T_k is a constant sequence.

(3.7) Theorem. *Let $u_1,\ldots,u_q$ be locally bounded plurisubharmonic functions and let $u_1^k,\ldots,u_q^k$ be decreasing sequences of plurisubharmonic functions converging pointwise to $u_1,\ldots,u_q$. Then*

a) $u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

b) $dd^c u_1^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ *weakly.*

Proof. Again by induction, observing that a) $\Longrightarrow$ b) and that a) is obvious for $q=1$ thanks to Lebesgue's bounded convergence theorem. To proceed with the induction step, we first have to make some slight modifications of our functions u_j and u_j^k.

As the sequence (u_j^k) is decreasing and as u_j is locally bounded, the family $(u_j^k)_{k\in\mathbb{N}}$ is locally uniformly bounded. The results are local, so we can work on a Stein open set $\Omega \subset\subset X$ with strongly pseudoconvex boundary. We use the following notations:

(3.8) let ψ be a strongly plurisubharmonic function of class C^∞ near $\overline{\Omega}$ with $\psi<0$ on Ω and $\psi=0$, $d\psi\neq 0$ on $\partial\Omega$;

(3.8′) we set $\Omega_\delta=\{z\in\Omega\,;\,\psi(z)<-\delta\}$ for all $\delta>0$.

After addition of a constant we can assume that $-M\le u_j^k\le -1$ near $\overline{\Omega}$. Let us denote by $(u_j^{k,\varepsilon})$, $\varepsilon\in\,]0,\varepsilon_0]$, an increasing family of regularizations converging to u_j^k as $\varepsilon\to 0$ and such that $-M\le u_j^{k,\varepsilon}\le -1$ on $\overline{\Omega}$. Set $A=M/\delta$ with $\delta>0$ small and replace u_j^k by $v_j^k=\max\{A\psi,u_j^k\}$ and $u_j^{k,\varepsilon}$ by $v_j^{k,\varepsilon}=\max_\varepsilon\{A\psi,u_j^{k,\varepsilon}\}$ where $\max_\varepsilon=\max\star\rho_\varepsilon$ is a regularized max function. Then v_j^k coincides with u_j^k on Ω_δ since $A\psi<-A\delta=-M$ on Ω_δ, and v_j^k is equal to $A\psi$ on the corona $\Omega\setminus\Omega_{\delta/M}$. Without loss of generality, we can therefore assume that all u_j^k (and similarly all $u_j^{k,\varepsilon}$) coincide with $A\psi$ on a fixed neighborhood of $\partial\Omega$. We need a lemma.

(3.9) Lemma. *Let f_k be a decreasing sequence of upper semi-continuous functions converging to f on some separable locally compact space X and μ_k a sequence of positive measures converging weakly to μ on X. Then every weak limit ν of $f_k\mu_k$ satisfies $\nu\le f\mu$.*

Indeed if (g_p) is a decreasing sequence of continuous functions converging to f_{k_0} for some k_0, then $f_k\mu_k\le f_{k_0}\mu_k\le g_p\mu_k$ for $k\ge k_0$, thus $\nu\le g_p\mu$

Fig. 1 Construction of v_j^k

as $k \to +\infty$. The monotone convergence theorem then gives $\nu \le f_{k_0}\mu$ as $p \to +\infty$ and $\nu \le f\mu$ as $k_0 \to +\infty$. □

Proof of Theorem 3.7 (end). Assume that a) has been proved for $q-1$. Then

$$S^k = dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow S = dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T.$$

By 3.3 the sequence $(u_1^k S^k)$ has locally bounded mass, hence is relatively compact for the weak topology. In order to prove a), we only have to show that every weak limit Θ of $u_1^k S^k$ is equal to $u_1 S$. Let (m,m) be the bidimension of S and let γ be an arbitrary smooth and strongly positive form of bidegree (m,m). Then the positive measures $S^k \wedge \gamma$ converge weakly to $S \wedge \gamma$ and Lemma 3.9 shows that $\Theta \wedge \gamma \le u_1 S \wedge \gamma$, hence $\Theta \le u_1 S$. To get the equality, we set $\beta = dd^c\psi > 0$ and show that $\int_\Omega u_1 S \wedge \beta^m \le \int_\Omega \Theta \wedge \beta^m$, i.e.

$$\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \le \liminf_{k \to +\infty} \int_\Omega u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \beta^m.$$

As $u_1 \le u_1^k \le u_1^{k,\varepsilon_1}$ for every $\varepsilon_1 > 0$, we get

$$\begin{aligned}
&\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\
&\qquad \le \int_\Omega u_1^{k,\varepsilon_1} dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\
&\qquad = \int_\Omega dd^c u_1^{k,\varepsilon_1} \wedge u_2 dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m
\end{aligned}$$

after an integration by parts (there is no boundary term because u_1^{k,ε_1} and u_2 both vanish on $\partial\Omega$). Repeating this argument with $u_2, \ldots, u_q$, we obtain

$$\begin{aligned}\int_\Omega & u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\ &\le \int_\Omega dd^c u_1^{k,\varepsilon_1} \wedge \ldots \wedge dd^c u_{q-1}^{k,\varepsilon_{q-1}} \wedge u_q T \wedge \beta^m \\ &\le \int_\Omega u_1^{k,\varepsilon_1} dd^c u_2^{k,\varepsilon_2} \wedge \ldots \wedge dd^c u_q^{k,\varepsilon_q} \wedge T \wedge \beta^m.\end{aligned}$$

Now let $\varepsilon_q \to 0, \ldots, \varepsilon_1 \to 0$ in this order. We have weak convergence at each step and $u_1^{k,\varepsilon_1} = 0$ on the boundary; therefore the integral in the last line converges and we get the desired inequality

$$\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \le \int_\Omega u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \beta^m. \square$$

(3.10) Corollary. *The product $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ is symmetric with respect to $u_1, \ldots, u_q$.*

Proof. Observe that the definition was unsymmetric. The result is true when $u_1, \ldots, u_q$ are smooth and follows in general from Th. 3.7 applied to the sequences $u_j^k = u_j \star \rho_{1/k}$, $1 \le j \le q$. $\square$

(3.11) Proposition. *Let K, L be compact subsets of X such that $L \subset K^\circ$. For any plurisubharmonic functions $V, u_1, \ldots, u_q$ on X such that $u_1, \ldots, u_q$ are locally bounded, there is an inequality*

$$||V dd^c u_1 \wedge \ldots \wedge dd^c u_q||_L \le C_{K,L}\, ||V||_{L^1(K)} ||u_1||_{L^\infty(K)} \ldots ||u_q||_{L^\infty(K)}.$$

Proof. We may assume that L is contained in a strongly pseudoconvex open set $\Omega = \{\psi < 0\} \subset K$ (otherwise we cover L by small balls contained in K). A suitable normalization gives $-2 \le u_j \le -1$ on K; then we can modify u_j on $\Omega \setminus L$ so that $u_j = A\psi$ on $\Omega \setminus \Omega_\delta$ with a fixed constant A and $\delta > 0$ such that $L \subset \Omega_\delta$. Let $\chi \ge 0$ be a smooth function equal to $-\psi$ on Ω_δ with compact support in Ω. If we take $||V||_{L^1(K)} = 1$, we see that V_+ is uniformly bounded on Ω_δ by 3.4 b); after subtraction of a fixed constant we can assume $V \le 0$ on Ω_δ. First suppose $q \le n-1$. As $u_j = A\psi$ on $\Omega \setminus \Omega_\delta$, we find

$$\begin{aligned}\int_{\Omega_\delta} & -V\, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q} \\ &= \int_\Omega V\, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q-1} \wedge dd^c\chi - A^q \int_{\Omega\setminus\Omega_\delta} V\, \beta^{n-1} \wedge dd^c\chi \\ &= \int_\Omega \chi\, dd^c V \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q-1} - A^q \int_{\Omega\setminus\Omega_\delta} V\, \beta^{n-1} \wedge dd^c\chi.\end{aligned}$$

The first integral of the last line is uniformly bounded thanks to 3.3 and 3.4 a), and the second one is bounded by $||V||_{L^1(\Omega)} \le$ constant. Inequality

3.11 follows for $q \le n-1$. If $q = n$, we can work instead on $X \times \mathbb{C}$ and consider $V, u_1, \ldots, u_q$ as functions on $X \times \mathbb{C}$ independent of the extra variable

4. Case of Unbounded Plurisubharmonic Functions

We would like to define $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ also in some cases when $u_1, \ldots, u_q$ are not bounded below everywhere, especially when the u_j have logarithmic poles. Consider first the case $q = 1$ and let u be a plurisubharmonic function on X. The *pole set* of u is by definition $P(u) = u^{-1}(-\infty)$. We define the *unbounded locus* $L(u)$ to be the set of points $x \in X$ such that u is unbounded in every neighborhood of x. Clearly $L(u)$ is closed and we have $L(u) \supset \overline{P(u)}$ but in general these sets are different: in fact, $u(z) = \sum k^{-2} \log(|z - 1/k| + e^{-k^3})$ is everywhere finite in $\mathbb{C}$ but $L(u) = \{0\}$.

(4.1) Proposition. *We make two additional assumptions:*

a) *T has non zero bidimension (p,p) (i.e. degree of $T < 2n$).*

b) *X is covered by a family of Stein open sets $\Omega \subset\subset X$ whose boundaries $\partial\Omega$ do not intersect $L(u) \cap \operatorname{Supp} T$.*

Then the current uT has locally finite mass in X.

For any current T, hypothesis 4.1 b) is clearly satisfied when u has a discrete unbounded locus $L(u)$; an interesting example is $u = \log|F|$ where $F = (F_1, \ldots, F_N)$ are holomorphic functions having a discrete set of common zeros. Observe that the current uT need not have locally finite mass when T has degree $2n$ (i.e. T is a measure); example: $T = \delta_0$ and $u(z) = \log|z|$ in $\mathbb{C}^n$. The result also fails when the sets Ω are not assumed to be Stein; example: $X =$ blow-up of $\mathbb{C}^n$ at 0, $T = [E] =$ current of integration on the exceptional divisor and $u(z) = \log|z|$ (see § 7.12 for the definition of blow-ups).

Proof. By shrinking Ω slightly, we may assume that Ω has a smooth strongly pseudoconvex boundary. Let ψ be a defining function of Ω as in (3.8). By subtracting a constant to u, we may assume $u \le -\varepsilon$ on $\overline{\Omega}$. We fix δ so small that $\overline{\Omega} \smallsetminus \Omega_\delta$ does not intersect $L(u) \cap \operatorname{Supp} T$ and we select a neighborhood ω of $(\overline{\Omega} \smallsetminus \Omega_\delta) \cap \operatorname{Supp} T$ such that $\overline{\omega} \cap L(u) = \emptyset$. Then we define

$$u_s(z) = \begin{cases} \max\{u(z), A\psi(z)\} & \text{on } \omega, \\ \max\{u(z), s\} & \text{on } \Omega_\delta = \{\psi < -\delta\}. \end{cases}$$

By construction $u \ge -M$ on ω for some constant $M > 0$. We fix $A \ge M/\delta$ and take $s \le -M$, so

$$\max\{u(z), A\psi(z)\} = \max\{u(z), s\} = u(z) \quad \text{on } \omega \cap \Omega_\delta$$

and our definition of u_s is coherent. Observe that u_s is defined on $\omega \cup \Omega_\delta$, which is a neighborhood of $\overline{\Omega} \cap \operatorname{Supp} T$. Now, $u_s = A\psi$ on $\omega \cap (\Omega \setminus \Omega_{\varepsilon/A})$, hence Stokes' theorem implies

$$\begin{aligned}\int_\Omega dd^c u_s \wedge T \wedge (dd^c\psi)^{p-1} &- \int_\Omega A dd^c\psi \wedge T \wedge (dd^c\psi)^{p-1} \\ &= \int_\Omega dd^c\big[(u_s - A\psi)T \wedge (dd^c\psi)^{p-1}\big] = 0\end{aligned}$$

because the current $[\ldots]$ has a compact support contained in $\overline{\Omega}_{\varepsilon/A}$. Since u_s and ψ both vanish on $\partial\Omega$, an integration by parts gives

$$\begin{aligned}\int_\Omega u_s T \wedge (dd^c\psi)^p &= \int_\Omega \psi dd^c u_s \wedge T \wedge (dd^c\psi)^{p-1} \\ &\geq -||\psi||_{L^\infty(\Omega)} \int_\Omega T \wedge dd^c u_s \wedge (dd^c\psi)^{p-1} \\ &= -||\psi||_{L^\infty(\Omega)} A \int_\Omega T \wedge (dd^c\psi)^p.\end{aligned}$$

Finally, take $A = M/\delta$, let s tend to $-\infty$ and use the inequality $u \geq -M$ on ω. We obtain

$$\begin{aligned}\int_\Omega u T \wedge (dd^c\psi)^p &\geq -M \int_\omega T \wedge (dd^c\psi)^p + \lim_{s\to-\infty} \int_{\Omega_\delta} u_s T \wedge (dd^c\psi)^p \\ &\geq -\big(M + ||\psi||_{L^\infty(\Omega)} M/\delta\big) \int_\Omega T \wedge (dd^c\psi)^p.\end{aligned}$$

The last integral is finite. This concludes the proof. □

(4.2) Remark. If Ω is smooth and strongly pseudoconvex, the above proof shows in fact that

$$||uT||_{\overline{\Omega}} \leq \frac{C}{\delta} ||u||_{L^\infty((\overline{\Omega} \setminus \Omega_\delta) \cap \operatorname{Supp} T)} ||T||_{\overline{\Omega}}$$

when $L(u) \cap \operatorname{Supp} T \subset \Omega_\delta$. In fact, if u is continuous and if ω is chosen sufficiently small, the constant M can be taken arbitrarily close to $||u||_{L^\infty((\overline{\Omega} \setminus \Omega_\delta) \cap \operatorname{Supp} T)}$. Moreover, the maximum principle implies

$$||u_+||_{L^\infty(\overline{\Omega} \cap \operatorname{Supp} T)} = ||u_+||_{L^\infty(\partial\Omega \cap \operatorname{Supp} T)},$$

so we can achieve $u < -\varepsilon$ on a neighborhood of $\overline{\Omega} \cap \operatorname{Supp} T$ by subtracting $||u||_{L^\infty((\overline{\Omega} \setminus \Omega_\delta) \cap \operatorname{Supp} T)} + 2\varepsilon$ [Proof of maximum principle: if $u(z_0) > 0$ at $z_0 \in \Omega \cap \operatorname{Supp} T$ and $u \leq 0$ near $\partial\Omega \cap \operatorname{Supp} T$, then

$$\int_\Omega u_+ T \wedge (dd^c\psi)^p = \int_\Omega \psi dd^c u_+ \wedge T \wedge (dd^c\psi)^{p-1} \leq 0,$$

a contradiction]. □

(4.3) Corollary. *Let $u_1, \ldots, u_q$ be plurisubharmonic functions on X such that X is covered by Stein open sets Ω with $\partial\Omega \cap L(u_j) \cap \operatorname{Supp} T = \emptyset$. We use again induction to define*

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T = dd^c(u_1 dd^c u_2 \ldots \wedge dd^c u_q \wedge T).$$

Then, if $u_1^k, \ldots, u_q^k$ are decreasing sequences of plurisubharmonic functions converging pointwise to $u_1, \ldots, u_q$, $q \le p$, properties (3.7 a, b) *hold.*

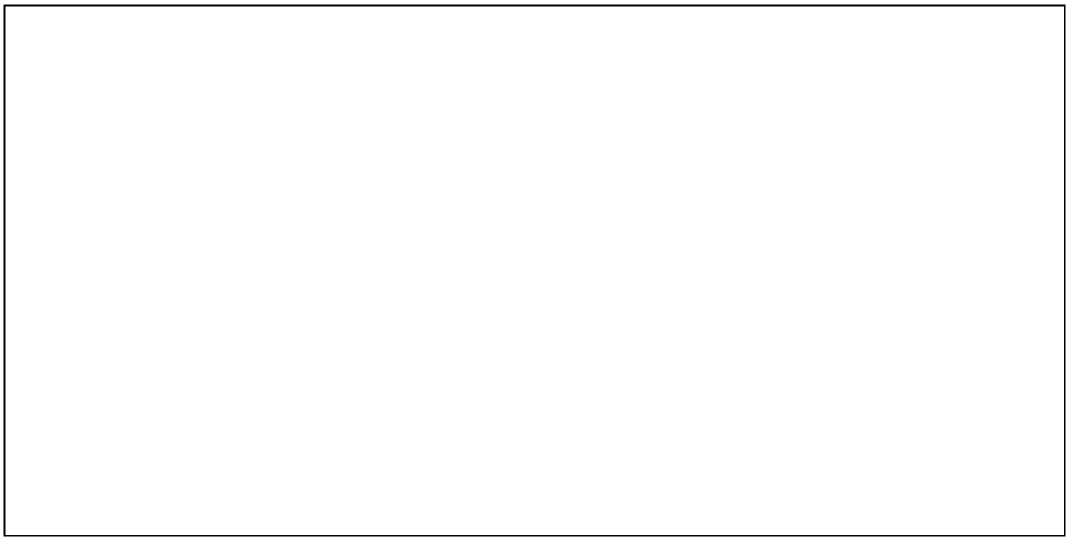

Fig. 2 Modified construction of v_j^k

Proof. Same proof as for Th. 3.7, with the following minor modification: the max procedure $v_j^k := \max\{u_j^k, A\psi\}$ is applied only on a neighborhood ω of $\operatorname{Supp} T \cap (\overline{\Omega} \smallsetminus \Omega_\delta)$ with $\delta > 0$ small, and u_j^k is left unchanged near $\operatorname{Supp} T \cap \overline{\Omega}_\delta$. Observe that the integration by part process requires the functions u_j^k and $u_j^{k,\varepsilon}$ to be defined only near $\overline{\Omega} \cap \operatorname{Supp} T$. □

(4.4) Proposition. *Let $\Omega \subset\subset X$ be a Stein open subset. If V is a plurisubharmonic function on X and $u_1, \ldots, u_q$, $1 \le q \le n-1$, are plurisubharmonic functions such that $\partial\Omega \cap L(u_j) = \emptyset$, then $V dd^c u_1 \wedge \ldots \wedge dd^c u_q$ has locally finite mass in Ω.*

Proof. Same proof as for 3.11, when $\delta > 0$ is taken so small that $\Omega_\delta \supset L(u_j)$ for all $1 \le j \le q$. □

Finally, we show that Monge-Ampère operators can also be defined in the case of plurisubharmonic functions with non compact pole sets, provided that the mutual intersections of the pole sets are of sufficiently small Hausdorff dimension with respect to the dimension p of T.

(4.5) Theorem. *Let $u_1,\ldots,u_q$ be plurisubharmonic functions on X. The currents $u_1 dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T$ and $dd^c u_1\wedge\ldots\wedge dd^c u_q\wedge T$ are well defined and have locally finite mass in X as soon as $q\le p$ and*

$$\mathcal{H}_{2p-2m+1}\big(L(u_{j_1})\cap\ldots\cap L(u_{j_m})\cap \operatorname{Supp} T\big)=0$$

for all choices of indices $j_1<\ldots<j_m$ in $\{1,\ldots,q\}$.

The proof is an easy induction on q, thanks to the following improved version of the Chern-Levine-Nirenberg inequalities.

(4.6) Proposition. *Let $A_1,\ldots,A_q\subset X$ be closed sets such that*

$$\mathcal{H}_{2p-2m+1}\big(A_{j_1}\cap\ldots\cap A_{j_m}\cap \operatorname{Supp} T\big)=0$$

for all choices of $j_1<\ldots<j_m$ in $\{1,\ldots,q\}$. Then for all compact sets K, L of X with $L\subset K^\circ$, there exist neighborhoods V_j of $K\cap A_j$ and a constant $C=C(K,L,A_j)$ such that the conditions $u_j\le 0$ on K and $L(u_j)\subset A_j$ imply

a) $||u_1 dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T||_L\le C||u_1||_{L^\infty(K\smallsetminus V_1)}\cdots||u_q||_{L^\infty(K\smallsetminus V_q)}||T||_K$

b) $||dd^c u_1\wedge\ldots\wedge dd^c u_q\wedge T||_L\le C||u_1||_{L^\infty(K\smallsetminus V_1)}\cdots||u_q||_{L^\infty(K\smallsetminus V_q)}||T||_K$.

Proof. We need only show that every point $x_0\in K^\circ$ has a neighborhood L such that a), b) hold. Hence it is enough to work in a coordinate open set. We may thus assume that $X\subset\mathbb{C}^n$ is open, and after a regularization process $u_j=\lim u_j\star\rho_\varepsilon$ for $j=q,q-1,\ldots,1$ in this order, that $u_1,\ldots,u_q$ are smooth. We proceed by induction on q in two steps:

Step 1. $(\mathrm{b}_{q-1})\Longrightarrow(\mathrm{b}_q)$,

Step 2. (a_{q-1}) and $(\mathrm{b}_q)\Longrightarrow(\mathrm{a}_q)$,

where (b_0) is the trivial statement $||T||_L\le||T||_K$ and (a_0) is void. Observe that we have $(\mathrm{a}_q)\Longrightarrow(\mathrm{a}_\ell)$ and $(\mathrm{b}_q)\Longrightarrow(\mathrm{b}_\ell)$ for $\ell\le q\le p$ by taking $u_{\ell+1}(z)=\ldots=u_q(z)=|z|^2$. We need the following elementary fact.

(4.7) Lemma. *Let $F\subset\mathbb{C}^n$ be a closed set such that $\mathcal{H}_{2s+1}(F)=0$ for some integer $0\le s<n$. Then for almost all choices of unitary coordinates $(z_1,\ldots,z_n)=(z',z'')$ with $z'=(z_1,\ldots,z_s)$, $z''=(z_{s+1},\ldots,z_n)$ and almost all radii of balls $B''=B(0,r'')\subset\mathbb{C}^{n-s}$, the set $\{0\}\times\partial B''$ does not intersect F.*

Proof. The unitary group $U(n)$ has real dimension n^2. There is a proper submersion

$$\Phi : U(n) \times \big(\mathbb{C}^{n-s} \smallsetminus \{0\}\big) \longrightarrow \mathbb{C}^n \smallsetminus \{0\}, \qquad (g, z'') \longmapsto g(0, z''),$$

whose fibers have real dimension $N = n^2 - 2s$. It follows that the inverse image $\Phi^{-1}(F)$ has zero Hausdorff measure $\mathcal{H}_{N+2s+1} = \mathcal{H}_{n^2+1}$. The set of pairs $(g, r'') \in U(n) \times \mathbb{R}_+^\star$ such that $g(\{0\} \times \partial B'')$ intersects F is precisely the image of $\Phi^{-1}(F)$ in $U(n) \times \mathbb{R}_+^\star$ by the Lipschitz map $(g, z'') \mapsto (g, |z''|)$. Hence this set has zero $\mathcal{H}_{n^2+1}$-measure. □

Proof of step 1. Take $x_0 = 0 \in K^\circ$. Suppose first $0 \in A_1 \cap \ldots \cap A_q$ and set $F = A_1 \cap \ldots \cap A_q \cap \operatorname{Supp} T$. Since $\mathcal{H}_{2p-2q+1}(F) = 0$, Lemma 4.7 implies that there are coordinates $z' = (z_1, \ldots, z_s)$, $z'' = (z_{s+1}, \ldots, z_n)$ with $s = p - q$ and a ball $\overline{B}''$ such that $F \cap \big(\{0\} \times \partial B''\big) = \emptyset$ and $\{0\} \times \overline{B}'' \subset K^\circ$. By compactness of K, we can find neighborhoods W_j of $K \cap A_j$ and a ball $B' = B(0, r') \subset \mathbb{C}^s$ such that $\overline{B}' \times \overline{B}'' \subset K^\circ$ and

$$\text{(4.8)} \quad \overline{W}_1 \cap \ldots \cap \overline{W}_q \cap \operatorname{Supp} T \cap \Big(\overline{B}' \times \big(\overline{B}'' \smallsetminus (1-\delta)B''\big)\Big) = \emptyset$$

for $\delta > 0$ small. If $0 \notin A_j$ for some j, we choose instead W_j to be a small neighborhood of 0 such that $\overline{W}_j \subset (\overline{B}' \times (1-\delta)B'') \smallsetminus A_j$; property (4.8) is then automatically satisfied. Let $\chi_j \geq 0$ be a function with compact support in W_j, equal to 1 near $K \cap A_j$ if $A_j \ni 0$ (resp. equal to 1 near 0 if $A_j \not\ni 0$) and let $\chi(z') \geq 0$ be a function equal to 1 on $1/2\, B'$ with compact support in B'. Then

$$\int_{B' \times B''} dd^c(\chi_1 u_1) \wedge \ldots \wedge dd^c(\chi_q u_q) \wedge T \wedge \chi(z')\, (dd^c|z'|^2)^s = 0$$

because the integrand is dd^c exact and has compact support in $B' \times B''$ thanks to (4.8). If we expand all factors $dd^c(\chi_j u_j)$, we find a term

$$\chi_1 \ldots \chi_q \chi(z') dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T \geq 0$$

which coincides with $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ on a small neighborhood of 0 where $\chi_j = \chi = 1$. The other terms involve

$$d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j + u_j dd^c \chi_j$$

for at least one index j. However $d\chi_j$ and $dd^c\chi_j$ vanish on some neighborhood V_j' of $K \cap A_j$ and therefore u_j is bounded on $\overline{B}' \times \overline{B}'' \smallsetminus V_j'$. We then apply the induction hypothesis (b_{q-1}) to the current

$$\Theta = dd^c u_1 \wedge \ldots \wedge \widehat{dd^c u_j} \wedge \ldots \wedge dd^c u_q \wedge T$$

and the usual Chern-Levine-Nirenberg inequality to the product of Θ with the mixed term $d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j$. Remark 3.5 can be applied because χ_j is smooth and is therefore a difference $\chi_j^{(1)} - \chi_j^{(2)}$ of locally bounded plurisubharmonic functions in $\mathbb{C}^n$. Let K' be a compact neighborhood of

$\overline{B}' \times \overline{B}''$ with $K' \subset K^\circ$, and let V_j be a neighborhood of $K \cap A_j$ with $\overline{V}_j \subset V_j'$. Then with $L' := (\overline{B}' \times \overline{B}'') \smallsetminus V_j' \subset (K' \smallsetminus V_j)^\circ$ we obtain

$$\begin{aligned}||(d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j) \wedge \Theta||_{\overline{B}' \times \overline{B}''} &= ||(d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j) \wedge \Theta||_{L'} \\ &\le C_1 ||u_j||_{L^\infty(K' \smallsetminus V_j)} ||\Theta||_{K' \smallsetminus V_j}, \\ ||\Theta||_{K' \smallsetminus V_j} \le ||\Theta||_{K'} \le C_2 ||u_1||_{L^\infty(K \smallsetminus V_1)} &\ldots \widehat{||u_j||} \ldots ||u_q||_{L^\infty(K \smallsetminus V_q)} ||T||_K.\end{aligned}$$

Now, we may slightly move the unitary basis in $\mathbb{C}^n$ and get coordinate systems $z^m = (z_1^m, \ldots, z_n^m)$ with the same properties as above, such that the forms

$$(dd^c|z^{m\prime}|^2)^s = \frac{s!}{\pi^s} \mathrm{i}\, dz_1^m \wedge d\overline{z}_1^m \wedge \ldots \wedge \mathrm{i}\, dz_s^m \wedge d\overline{z}_s^m, \qquad 1 \le m \le N$$

define a basis of $\bigwedge^{s,s}(\mathbb{C}^n)^\star$. It follows that all measures

$$dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \mathrm{i}\, dz_1^m \wedge d\overline{z}_1^m \wedge \ldots \wedge \mathrm{i}\, dz_s^m \wedge d\overline{z}_s^m$$

satisfy estimate (b_q) on a small neighborhood L of 0.

Proof of Step 2. We argue in a similar way with the integrals

$$\begin{aligned}&\int_{B' \times B''} \chi_1 u_1 dd^c(\chi_2 u_2) \wedge \ldots dd^c(\chi_q u_q) \wedge T \wedge \chi(z')(dd^c|z'|^2)^s \wedge dd^c|z_{s+1}|^2 \\ &\qquad = \int_{B' \times B''} |z_{s+1}|^2 dd^c(\chi_1 u_1) \wedge \ldots dd^c(\chi_q u_q) \wedge T \wedge \chi(z')(dd^c|z'|^2)^s.\end{aligned}$$

We already know by (b_q) and Remark 3.5 that all terms in the right hand integral admit the desired bound. For $q = 1$, this shows that $(\mathrm{b}_1) \Longrightarrow (\mathrm{a}_1)$. Except for $\chi_1 \ldots \chi_q \chi(z')\, u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$, all terms in the left hand integral involve derivatives of χ_j. By construction, the support of these derivatives is disjoint from A_j, thus we only have to obtain a bound for

$$\int_L u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha$$

when $L = \overline{B}(x_0, r)$ is disjoint from A_j for some $j \ge 2$, say $L \cap A_2 = \emptyset$, and α is a constant positive form of type $(p-q, p-q)$. Then $\overline{B}(x_0, r+\varepsilon) \subset K^\circ \smallsetminus \overline{V}_2$ for some $\varepsilon > 0$ and some neighborhood V_2 of $K \cap A_2$. By the max construction used e.g. in Prop. 4.1, we can replace u_2 by a plurisubharmonic function $\widetilde{u}_2$ equal to u_2 in L and to $A(|z-x_0|^2 - r^2) - M$ in $\overline{B}(x_0, r+\varepsilon) \smallsetminus B(x_0, r+\varepsilon/2)$, with $M = ||u_2||_{L^\infty(K \smallsetminus V_2)}$ and $A = M/\varepsilon r$. Let $\chi \ge 0$ be a smooth function equal to 1 on $B(x_0, r+\varepsilon/2)$ with support in $B(x_0, r)$. Then

$$\begin{aligned}&\int_{B(x_0, r+\varepsilon)} u_1 dd^c(\chi \widetilde{u}_2) \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha \\ &\qquad = \int_{B(x_0, r+\varepsilon)} \chi \widetilde{u}_2 dd^c u_1 \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha \\ &\qquad \le O(1)\ ||u_1||_{L^\infty(K \smallsetminus V_1)} \ldots ||u_q||_{L^\infty(K \smallsetminus V_q)} ||T||_K\end{aligned}$$

where the last estimate is obtained by the induction hypothesis (b_{q-1}) applied to $dd^c u_1 \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T$. By construction

$$dd^c(\chi \widetilde{u}_2) = \chi \, dd^c \widetilde{u}_2 + (\text{smooth terms involving } d\chi)$$

coincides with $dd^c u_2$ in L, and (a_{q-1}) implies the required estimate for the other terms in the left hand integral. □

(4.9) Proposition. *With the assumptions of Th.* 4.5, *the analogue of the monotone convergence Theorem* 3.7 (a,b) *holds.*

Proof. By the arguments already used in the proof of Th. 3.7 (e.g. by Lemma 3.9), it is enough to show that

$$\int_{B' \times B''} \chi_1 \ldots \chi_q \, u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha$$
$$\leq \liminf_{k \to +\infty} \int_{B' \times B''} \chi_1 \ldots \chi_q \, u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \alpha$$

where $\alpha = \chi(z')(dd^c|z'|^2)^s$ is closed. Here the functions χ_j, χ are chosen as in the proof of Step 1 in 4.7, especially their product has compact support in $B' \times B''$ and $\chi_j = \chi = 1$ in a neighborhood of the given point x_0. We argue by induction on q and also on the number m of functions $(u_j)_{j \geq 1}$ which are unbounded near x_0. If u_j is bounded near x_0, we take $W_j'' \subset\subset W_j' \subset\subset W_j$ to be small balls of center x_0 on which u_j is bounded and we modify the sequence u_j^k on the corona $W_j \smallsetminus W_j''$ so as to make it constant and equal to a smooth function $A|z - x_0|^2 + B$ on the smaller corona $W_j \smallsetminus W_j'$. In that case, we take χ_j equal to 1 near $\overline{W}_j'$ and $\operatorname{Supp} \chi_j \subset W_j$. For every $\ell = 1, \ldots, q$, we are going to check that

$$\liminf_{k \to +\infty} \int_{B' \times B''} \chi_1 u_1^k dd^c(\chi_2 u_2^k) \wedge \ldots$$
$$dd^c(\chi_{\ell-1} u_{\ell-1}^k) \wedge dd^c(\chi_\ell u_\ell) \wedge dd^c(\chi_{\ell+1} u_{\ell+1}) \ldots dd^c(\chi_q u_q) \wedge T \wedge \alpha$$
$$\leq \liminf_{k \to +\infty} \int_{B' \times B''} \chi_1 u_1^k dd^c(\chi_2 u_2^k) \wedge \ldots$$
$$dd^c(\chi_{\ell-1} u_{\ell-1}^k) \wedge dd^c(\chi_\ell u_\ell^k) \wedge dd^c(\chi_{\ell+1} u_{\ell+1}) \ldots dd^c(\chi_q u_q) \wedge T \wedge \alpha.$$

In order to do this, we integrate by parts $\chi_1 u_1^k dd^c(\chi_\ell u_\ell)$ into $\chi_\ell u_\ell dd^c(\chi_1 u_1^k)$ for $\ell \geq 2$, and we use the inequality $u_\ell \leq u_\ell^k$. Of course, the derivatives $d\chi_j$, $d^c\chi_j$, $dd^c\chi_j$ produce terms which are no longer positive and we have to take care of these. However, $\operatorname{Supp} d\chi_j$ is disjoint from the unbounded locus of u_j when u_j is unbounded, and contained in $W_j \smallsetminus \overline{W}_j'$ when u_j is bounded. The number m of unbounded functions is therefore replaced by $m - 1$ in the first case, whereas in the second case $u_j^k = u_j$ is constant and smooth on $\operatorname{Supp} d\chi_j$, so q can be replaced by $q - 1$. By induction on $q + m$ (and thanks to the polarization technique 3.5), the limit of the terms involving derivatives of χ_j is equal on both sides to the corresponding terms obtained by suppressing all indices k. Hence these terms do not give any contribution in the inequalities. $\square$

We finally quote the following simple consequences of Th. 4.5 when T is arbitrary and $q = 1$, resp. when $T = 1$ has bidegree $(0, 0)$ and q is arbitrary.

(4.10) Corollary. *Let T be a closed positive current of bidimension (p, p) and let u be a plurisubharmonic function on X such that $L(u) \cap \operatorname{Supp} T$ is contained in an analytic set of dimension at most $p-1$. Then uT and $dd^c u \wedge T$ are well defined and have locally finite mass in X.* $\square$

(4.11) Corollary. *Let $u_1, \ldots, u_q$ be plurisubharmonic functions on X such that $L(u_j)$ is contained in an analytic set $A_j \subset X$ for every j. Then $dd^c u_1 \wedge \ldots \wedge dd^c u_q$ is well defined as soon as $A_{j_1} \cap \ldots \cap A_{j_m}$ has codimension at least m for all choices of indices $j_1 < \ldots < j_m$ in $\{1, \ldots, q\}$.* $\square$

In the particular case when $u_j = \log |f_j|$ for some non zero holomorphic function f_j on X, we see that the intersection product of the associated zero divisors $[Z_j] = dd^c u_j$ is well defined as soon as the supports $|Z_j|$ satisfy $\operatorname{codim} |Z_{j_1}| \cap \ldots \cap |Z_{j_m}| = m$ for every m. Similarly, when $T = [A]$ is an analytic p-cycle, Cor. 4.10 shows that $[Z] \wedge [A]$ is well defined for every divisor Z such that $\dim |Z| \cap |A| = p-1$. These observations easily imply the following

(4.12) Proposition. *Suppose that the divisors Z_j satisfy the above codimension condition and let $(C_k)_{k \geq 1}$ be the irreducible components of the point set intersection $|Z_1| \cap \ldots \cap |Z_q|$. Then there exist integers $m_k > 0$ such that*

$$[Z_1] \wedge \ldots \wedge [Z_q] = \sum m_k [C_k].$$

The integer m_k is called the multiplicity of intersection of $Z_1, \ldots, Z_q$ along the component C_k.

Proof. The wedge product has bidegree (q, q) and support in $C = \bigcup C_k$ where $\operatorname{codim} C = q$, so it must be a sum as above with $m_k \in \mathbb{R}_+$. We check by induction on q that m_k is a positive integer. If we denote by A some irreducible component of $|Z_1| \cap \ldots \cap |Z_{q-1}|$, we need only check that $[A] \wedge [Z_q]$ is an integral analytic cycle of codimension q with positive coefficients on each component C_k of the intersection. However $[A] \wedge [Z_q] = dd^c(\log |f_q|\ [A])$. First suppose that no component of $A \cap f_q^{-1}(0)$ is contained in the singular part A_{sing}. Then the Lelong-Poincaré equation applied on A_{reg} shows that $dd^c(\log |f_q|\ [A]) = \sum m_k [C_k]$ on $X \smallsetminus A_{\mathrm{sing}}$, where m_k is the vanishing order of f_q along C_k in A_{reg}. Since $C \cap A_{\mathrm{sing}}$ has codimension $q + 1$ at least, the equality must hold on X. In general, we replace f_q by $f_q - \varepsilon$ so that the divisor of $f_q - \varepsilon$ has no component contained in $A_{\rm sing}$. Then $dd^c(\log|f_q-\varepsilon|\,[A])$ is an integral codimension q cycle with positive multiplicities on each component of $A \cap f_q^{-1}(\varepsilon)$ and we conclude by letting ε tend to zero. $\square$

5. Generalized Lelong Numbers

The concepts we are going to study mostly concern the behaviour of currents or plurisubharmonic functions in a neighborhood of a point at which we have for instance a logarithmic pole. Since the interesting applications are local, we assume from now on (unless otherwise stated) that X is a Stein manifold, i.e. that X has a strictly plurisubharmonic exhaustion function. Let $\varphi : X \longrightarrow [-\infty, +\infty[$ be a continuous plurisubharmonic function (in general φ may have $-\infty$ poles, our continuity assumption means that e^φ is continuous). The sets

$$(5.1)\qquad S(r) = \{x \in X\,;\, \varphi(x) = r\},$$

$$(5.1')\qquad B(r) = \{x \in X\,;\, \varphi(x) < r\},$$

$$(5.1'')\qquad \overline{B}(r) = \{x \in X\,;\, \varphi(x) \le r\}$$

will be called *pseudo-spheres* and *pseudo-balls* associated with φ. Note that $\overline{B}(r)$ is not necessarily equal to the closure of $B(r)$, but this is often true in concrete situations. The most simple example we have in mind is the case of the function $\varphi(z) = \log|z-a|$ on an open subset $X \subset \mathbb{C}^n$; in this case $B(r)$ is the euclidean ball of center a and radius e^r ; moreover, the forms

$$(5.2)\qquad \frac{1}{2} dd^c e^{2\varphi} = \frac{\mathrm{i}}{2\pi} d'd''|z|^2, \qquad dd^c\varphi = \frac{\mathrm{i}}{\pi} d'd'' \log|z-a|$$

can be interpreted respectively as the flat hermitian metric on $\mathbb{C}^n$ and as the pull-back over $\mathbb{C}^n$ of the Fubini-Study metric of $\mathbb{P}^{n-1}$, translated by a.

(5.3) Definition. *We say that φ is semi-exhaustive if there exists a real number R such that $B(R) \subset\subset X$. Similarly, φ is said to be semi-exhaustive on a closed subset $A \subset X$ if there exists R such that $A \cap B(R) \subset\subset X$.*

We are interested especially in the set of poles $S(-\infty) = \{\varphi = -\infty\}$ and in the behaviour of φ near $S(-\infty)$. Let T be a closed positive current of bidimension (p,p) on X. Assume that φ is semi-exhaustive on Supp T and that $B(R) \cap \operatorname{Supp} T \subset\subset X$. Then $P = S(-\infty) \cap \operatorname{SuppT}$ is compact and the results of §2 show that the measure $T \wedge (dd^c\varphi)^p$ is well defined. Following (Demailly 1982b, 1987a), we introduce:

(5.4) Definition. *If φ is semi-exhaustive on* Supp T *and if R is such that* $B(R) \cap \operatorname{Supp} T \subset\subset X$, *we set for all $r \in\,]-\infty, R[$*

$$\nu(T,\varphi,r) = \int_{B(r)} T \wedge (dd^c\varphi)^p,$$
$$\nu(T,\varphi) = \int_{S(-\infty)} T \wedge (dd^c\varphi)^p = \lim_{r\to-\infty} \nu(T,\varphi,r).$$

The number $\nu(T,\varphi)$ will be called the (generalized) Lelong number of T with respect to the weight φ.

If we had not required $T\wedge(dd^c\varphi)^p$ to be defined pointwise on $\varphi^{-1}(-\infty)$, the assumption that X is Stein could have been dropped: in fact, the integral over $B(r)$ always makes sense if we define

$$\nu(T,\varphi,r) = \int_{B(r)} T \wedge \big(dd^c \max\{\varphi,s\}\big)^p \quad \text{with} \quad s<r.$$

Stokes' formula shows that the right hand integral is actually independent of s. The example given after (4.1) shows however that $T\wedge(dd^c\varphi)^p$ need not exist on $\varphi^{-1}(-\infty)$ if $\varphi^{-1}(-\infty)$ contains an exceptional compact analytic subset. We leave the reader consider by himself this more general situation and extend our statements by the $\max\{\varphi,s\}$ technique. Observe that $r\longmapsto \nu(T,\varphi,r)$ is always an increasing function of r. Before giving examples, we need a formula.

(5.5) Formula. *For any convex increasing function $\chi:\mathbb{R}\longrightarrow\mathbb{R}$ we have*

$$\int_{B(r)} T\wedge(dd^c\chi\circ\varphi)^p = \chi'(r-0)^p\,\nu(T,\varphi,r)$$

where $\chi'(r-0)$ denotes the left derivative of χ at r.

Proof. Let χ_ε be the convex function equal to χ on $[r-\varepsilon,+\infty[$ and to a linear function of slope $\chi'(r-\varepsilon-0)$ on $]-\infty,r-\varepsilon]$. We get $dd^c(\chi_\varepsilon\circ\varphi) = \chi'(r-\varepsilon-0)dd^c\varphi$ on $B(r-\varepsilon)$ and Stokes' theorem implies

$$\begin{aligned}\int_{B(r)} T\wedge(dd^c\chi\circ\varphi)^p &= \int_{B(r)} T\wedge(dd^c\chi_\varepsilon\circ\varphi)^p\\ &\geq \int_{B(r-\varepsilon)} T\wedge(dd^c\chi_\varepsilon\circ\varphi)^p\\ &= \chi'(r-\varepsilon-0)^p\nu(T,\varphi,r-\varepsilon).\end{aligned}$$

Similarly, taking $\widetilde\chi_\varepsilon$ equal to χ on $]-\infty,r-\varepsilon]$ and linear on $[r-\varepsilon,r]$, we obtain

$$\int_{B(r-\varepsilon)} T\wedge(dd^c\chi\circ\varphi)^p \leq \int_{B(r)} T\wedge(dd^c\widetilde\chi_\varepsilon\circ\varphi)^p = \chi'(r-\varepsilon-0)^p\nu(T,\varphi,r).$$

The expected formula follows when ε tends to 0. □

We get in particular $\int_{B(r)} T \wedge (dd^c e^{2\varphi})^p = (2e^{2r})^p \nu(T, \varphi, r)$, whence the formula

$$(5.6)\quad \nu(T, \varphi, r) = e^{-2pr} \int_{B(r)} T \wedge \Big(\frac{1}{2} dd^c e^{2\varphi}\Big)^p.$$

Now, assume that X is an open subset of $\mathbb{C}^n$ and that $\varphi(z) = \log|z - a|$ for some $a \in X$. Formula (5.6) gives

$$\nu(T, \varphi, \log r) = r^{-2p} \int_{|z-a|<r} T \wedge \Big(\frac{\mathrm{i}}{2\pi} d'd''|z|^2\Big)^p.$$

The positive measure $\sigma_T = \frac{1}{p!} T \wedge (\frac{\mathrm{i}}{2} d'd''|z|^2)^p = 2^{-p} \sum T_{I,I} \,.\, \mathrm{i}^n dz_1 \wedge \ldots \wedge d\overline{z}_n$ is called the *trace measure* of T. We get

$$(5.7)\quad \nu(T, \varphi, \log r) = \frac{\sigma_T\big(B(a, r)\big)}{\pi^p r^{2p}/p!}$$

and $\nu(T, \varphi)$ is the limit of this ratio as $r \to 0$. This limit is called the (*ordinary*) *Lelong number* of T at point a and is denoted $\nu(T, a)$. This was precisely the original definition of Lelong, see (Lelong 1968). Let us mention a simple but important consequence.

(5.8) Consequence. *The ratio $\sigma_T\big(B(a, r)\big)/r^{2p}$ is an increasing function of r. Moreover, for every compact subset $K \subset X$ and every $r_0 < d(K, \partial X)$ we have*

$$\sigma_T\big(B(a, r)\big) \le C r^{2p} \qquad \text{for} \quad a \in K \text{ and } r \le r_0,$$

where $C = \sigma_T\big(K + \overline{B}(0, r_0)\big)/r_0^{2p}$.

All these results are particularly interesting when $T = [A]$ is the current of integration over an analytic subset $A \subset X$ of pure dimension p. Then $\sigma_T\big(B(a, r)\big)$ is the euclidean area of $A \cap B(a, r)$, while $\pi^p r^{2p}/p!$ is the area of a ball of radius r in a p-dimensional subspace of $\mathbb{C}^n$. Thus $\nu(T, \varphi, \log r)$ is the ratio of these areas and the Lelong number $\nu(T, a)$ is the limit ratio.

(5.9) Remark. It is immediate to check that

$$\nu([A], x) = \begin{cases} 0 & \text{for } \; x \notin A, \\ 1 & \text{when } x \in A \text{ is a regular point.} \end{cases}$$

We will see later that $\nu([A], x)$ is always an integer (Thie's theorem 8.7).

(5.10) Remark. When $X = \mathbb{C}^n$, $\varphi(z) = \log|z - a|$ and $A = X$ (i.e. $T = 1$), we obtain in particular $\int_{B(a,r)} (dd^c \log|z - a|)^n = 1$ for all r. This implies

$$(dd^c \log|z - a|)^n = \delta_a.$$

This fundamental formula can be viewed as a higher dimensional analogue of the usual formula $\Delta \log |z-a| = 2\pi\delta_a$ in $\mathbb{C}$. □

We next prove a result which shows in particular that the Lelong numbers of a closed positive current are zero except on a very small set.

(5.11) Proposition. *If T is a closed positive current of bidimension (p,p), then for each $c>0$ the set $E_c = \{x \in X\,;\, \nu(T,x) \ge c$ is a closed set of locally finite $\mathcal{H}_{2p}$ Hausdorff measure in X.*

Proof. By (5.7), we infer $\nu(T,a) = \lim_{r\to 0} \sigma_T\big(\overline{B}(a,r)\big)p!/\pi^p r^{2p}$. The function $a \mapsto \sigma_T\big(\overline{B}(a,r)\big)$ is clearly upper semicontinuous. Hence the decreasing limit $\nu(T,a)$ as r decreases to 0 is also upper semicontinuous in a. This implies that E_c is closed. Now, let K be a compact subset in X and let $\{a_j\}_{1\le j\le N}$, $N = N(\varepsilon)$, be a maximal collection of points in $E_c \cap K$ such that $|a_j - a_k| \ge 2\varepsilon$ for $j \ne k$. The balls $B(a_j, 2\varepsilon)$ cover $E_c \cap K$, whereas the balls $B(a_j,\varepsilon)$ are disjoint. If $K_{c,\varepsilon}$ is the set of points which are at distance $\le \varepsilon$ of $E_c \cap K$, we get

$$\sigma_T(K_{c,\varepsilon}) \ge \sum \sigma_T\big(B(a_j,\varepsilon)\big) \ge N(\varepsilon)\, c\pi^p\varepsilon^{2p}/p!,$$

since $\nu(T,a_j) \ge c$. By the definition of Hausdorff measure, we infer

$$\begin{aligned}\mathcal{H}_{2p}(E_c \cap K) &\le \liminf_{\varepsilon\to 0} \sum \big(\operatorname{diam} B(a_j,2\varepsilon)\big)^{2p} \\ &\le \liminf_{\varepsilon\to 0} N(\varepsilon)(4\varepsilon)^{2p} \le \frac{p!4^{2p}}{c\pi^p}\sigma_T(E_c \cap K).\end{aligned}$$ □

Finally, we conclude this section by proving two simple semi-continuity results for Lelong numbers.

(5.12) Proposition. *Let T_k be a sequence of closed positive currents of bidimension (p,p) converging weakly to a limit T. Suppose that there is a closed set A such that* Supp $T_k \subset A$ *for all k and such that φ is semi-exhaustive on A with $A \cap B(R) \subset\subset X$. Then for all $r<R$ we have*

$$\begin{aligned}\int_{B(r)} T \wedge (dd^c\varphi)^p &\le \liminf_{k\to+\infty} \int_{B(r)} T_k \wedge (dd^c\varphi)^p \\ &\le \limsup_{k\to+\infty} \int_{\overline{B}(r)} T_k \wedge (dd^c\varphi)^p \le \int_{\overline{B}(r)} T \wedge (dd^c\varphi)^p.\end{aligned}$$

When r tends to $-\infty$, we find in particular

$$\limsup_{k\to+\infty} \nu(T_k,\varphi) \le \nu(T,\varphi).$$

Proof. Let us prove for instance the third inequality. Let φ_ℓ be a sequence of smooth plurisubharmonic approximations of φ with $\varphi \le \varphi_\ell < \varphi + 1/\ell$ on $\{r-\varepsilon \le \varphi \le r+\varepsilon\}$. We set

$$\psi_\ell = \begin{cases} \varphi & \text{on } \overline{B}(r), \\ \max\{\varphi, (1+\varepsilon)(\varphi_\ell - 1/\ell) - r\varepsilon\} & \text{on } X \smallsetminus B(r). \end{cases}$$

This definition is coherent since $\psi_\ell = \varphi$ near $S(r)$, and we have

$$\psi_\ell = (1+\varepsilon)(\varphi_\ell - 1/\ell) - r\varepsilon \quad \text{near} \quad S(r+\varepsilon/2)$$

as soon as ℓ is large enough, i.e. $(1+\varepsilon)/\ell \le \varepsilon^2/2$. Let χ_ε be a cut-off function equal to 1 in $B(r+\varepsilon/2)$ with support in $B(r+\varepsilon)$. Then

$$\begin{aligned} \int_{\overline{B}(r)} T_k \wedge (dd^c\varphi)^p &\le \int_{B(r+\varepsilon/2)} T_k \wedge (dd^c\psi_\ell)^p \\ &= (1+\varepsilon)^p \int_{B(r+\varepsilon/2)} T_k \wedge (dd^c\varphi_\ell)^p \\ &\le (1+\varepsilon)^p \int_{B(r+\varepsilon)} \chi_\varepsilon T_k \wedge (dd^c\varphi_\ell)^p. \end{aligned}$$

As $\chi_\varepsilon (dd^c\varphi_\ell)^p$ is smooth with compact support and as T_k converges weakly to T, we infer

$$\limsup_{k\to+\infty} \int_{\overline{B}(r)} T_k \wedge (dd^c\varphi)^p \le (1+\varepsilon)^p \int_{B(r+\varepsilon)} \chi_\varepsilon T \wedge (dd^c\varphi_\ell)^p.$$

We then let ℓ tend to $+\infty$ and ε tend to 0 to get the desired inequality. The first inequality is obtained in a similar way, we define ψ_ℓ so that $\psi_\ell = \varphi$ on $X \smallsetminus B(r)$ and $\psi_\ell = \max\{(1-\varepsilon)(\varphi_\ell - 1/\ell) + r\varepsilon\}$ on $\overline{B}(r)$, and we take $\chi_\varepsilon = 1$ on $B(r-\varepsilon)$ with $\operatorname{Supp} \chi_\varepsilon \subset B(r-\varepsilon/2)$. Then for ℓ large

$$\begin{aligned} \int_{B(r)} T_k \wedge (dd^c\varphi)^p &\ge \int_{B(r-\varepsilon/2)} T_k \wedge (dd^c\psi_\ell)^p \\ &\ge (1-\varepsilon)^p \int_{B(r-\varepsilon/2)} \chi_\varepsilon T_k \wedge (dd^c\varphi_\ell)^p. \end{aligned}$$ □

(5.13) Proposition. *Let φ_k be a (non necessarily monotone) sequence of continuous plurisubharmonic functions such that e^{φ_k} converges uniformly to e^φ on every compact subset of X. Suppose that $\{\varphi < R\} \cap \operatorname{Supp} T \subset\subset X$. Then for $r < R$ we have*

$$\limsup_{k\to+\infty} \int_{\{\varphi_k \le r\} \cap \{\varphi < R\}} T \wedge (dd^c\varphi_k)^p \le \int_{\{\varphi \le r\}} T \wedge (dd^c\varphi)^p.$$

In particular $\limsup_{k\to+\infty} \nu(T, \varphi_k) \le \nu(T, \varphi)$.

When we take $\varphi_k(z) = \log|z - a_k|$ with $a_k \to a$, Prop. 5.13 implies the upper semicontinuity of $a \mapsto \nu(T, a)$ which was already noticed in the proof of Prop. 5.11.

Proof. Our assumption is equivalent to saying that $\max\{\varphi_k, t\}$ converges locally uniformly to $\max\{\varphi, t\}$ for every t. Then Cor. 3.6 shows that $T \wedge (dd^c \max\{\varphi_k, t\})^p$ converges weakly to $T \wedge (dd^c \max\{\varphi, t\})^p$. If χ_ε is a cut-off function equal to 1 on $\{\varphi \le r + \varepsilon/2\}$ with support in $\{\varphi < r + \varepsilon\}$, we get

$$\lim_{k\to+\infty} \int_X \chi_\varepsilon T \wedge (dd^c \max\{\varphi_k, t\})^p = \int_X \chi_\varepsilon T \wedge (dd^c \max\{\varphi, t\})^p.$$

For k large, we have $\{\varphi_k \le r\} \cap \{\varphi < R\} \subset \{\varphi < r + \varepsilon/2\}$, thus when ε tends to 0 we infer

$$\limsup_{k\to+\infty} \int_{\{\varphi_k \le r\}\cap\{\varphi<R\}} T \wedge (dd^c \max\{\varphi_k, t\})^p \le \int_{\{\varphi \le r\}} T \wedge (dd^c \max\{\varphi, t\})^p.$$

When we choose $t < r$, this is equivalent to the first inequality in statement (5.13). $\square$

6. The Jensen-Lelong Formula

We assume in this section that X is Stein, that φ is *semi-exhaustive* on X and that $B(R) \subset\subset X$. We set for simplicity $\varphi_{\ge r} = \max\{\varphi, r\}$. For every $r \in\,]-\infty, R[$, the measures $dd^c(\varphi_{\ge r})^n$ are well defined. By Cor. 3.6, the map $r \longmapsto (dd^c\varphi_{\ge r})^n$ is continuous on $]-\infty, R[$ with respect to the weak topology. As $(dd^c\varphi_{\ge r})^n = (dd^c\varphi)^n$ on $X \setminus \overline{B}(r)$ and as $\varphi_{\ge r} \equiv r$, $(dd^c\varphi_{\ge r})^n = 0$ on $B(r)$, the left continuity implies $(dd^c\varphi_{\ge r})^n \ge \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n$. Here $\mathbb{1}_A$ denotes the characteristic function of any subset $A \subset X$. According to the definition introduced in (Demailly 1985a), the collection of *Monge-Ampère measures* associated with φ is the family of positive measures μ_r such that

$$(6.1)\quad \mu_r = (dd^c\varphi_{\ge r})^n - \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n, \qquad r \in\,]-\infty, R[.$$

The measure μ_r is supported on $S(r)$ and $r \longmapsto \mu_r$ is weakly continuous on the left by the bounded convergence theorem. Stokes' formula shows that $\int_{B(s)} (dd^c\varphi_{\ge r})^n - (dd^c\varphi)^n = 0$ for $s > r$, hence the total mass $\mu_r(S(r)) = \mu_r(B(s))$ is equal to the difference between the masses of $(dd^c\varphi)^n$ and $\mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n$ over $B(s)$, i.e.

$$(6.2)\quad \mu_r\big(S(r)\big) = \int_{B(r)} (dd^c\varphi)^n.$$

(6.3) Example. When $(dd^c\varphi)^n = 0$ on $X \setminus \varphi^{-1}(-\infty)$, formula (6.1) can be simplified into $\mu_r = (dd^c\varphi_{\ge r})^n$. This is so for $\varphi(z) = \log|z|$. In this case, the invariance of φ under unitary transformations implies that μ_r is also invariant. As the total mass of μ_r is equal to 1 by 5.10 and (6.2), we see that μ_r is the invariant measure of mass 1 on the euclidean sphere of radius e^r.

(6.4) Proposition. *Assume that φ is smooth near $S(r)$ and that $d\varphi \neq 0$ on $S(r)$, i.e. r is a non critical value. Then $S(r) = \partial B(r)$ is a smooth oriented real hypersurface and the measure μ_r is given by the $(2n-1)$-volume form $(dd^c\varphi)^{n-1} \wedge d^c\varphi_{\restriction S(r)}$.*

Proof. Write $\max\{t, r\} = \lim_{k\to+\infty} \chi_k(t)$ where χ is a decreasing sequence of smooth convex functions with $\chi_k(t) = r$ for $t \leq r - 1/k$, $\chi_k(t) = t$ for $t \geq r+1/k$. Theorem 3.6 shows that $(dd^c\chi_k \circ \varphi)^n$ converges weakly to $(dd^c\varphi_{\geq r})^n$. Let h be a smooth function h with compact support near $S(r)$. Let us apply Stokes' theorem with $S(r)$ considered as the boundary of $X \setminus B(r)$:

$$\begin{aligned}
\int_X h(dd^c\varphi_{\geq r})^n &= \lim_{k\to+\infty} \int_X h(dd^c\chi_k \circ \varphi)^n \\
&= \lim_{k\to+\infty} \int_X -dh \wedge (dd^c\chi_k \circ \varphi)^{n-1} \wedge d^c(\chi_k \circ \varphi) \\
&= \lim_{k\to+\infty} \int_X -\chi_k'(t)^n\, dh \wedge (dd^c\varphi)^{n-1} \wedge d^c\varphi \\
&= \int_{X\setminus B(r)} -dh \wedge (dd^c\varphi)^{n-1} \wedge d^c\varphi \\
&= \int_{S(r)} h\,(dd^c\varphi)^{n-1} \wedge d^c\varphi + \int_{X\setminus B(r)} h\,(dd^c\varphi)^{n-1} \wedge d^c\varphi.
\end{aligned}$$

Near $S(r)$ we thus have an equality of measures

$$(dd^c\varphi_{\geq r})^n = (dd^c\varphi)^{n-1} \wedge d^c\varphi_{\restriction S(r)} + \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n. \qquad \square$$

(6.5) Jensen-Lelong formula. *Let V be any plurisubharmonic function on X. Then V is μ_r-integrable for every $r \in\,]-\infty, R[$ and*

$$\mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n = \int_{-\infty}^r \nu(dd^cV, \varphi, t)\, dt.$$

Proof. Proposition 3.11 shows that V is integrable with respect to the measure $(dd^c\varphi_{\geq r})^n$, hence V is μ_r-integrable. By definition

$$\nu(dd^cV, \varphi, t) = \int_{\varphi(z)<t} dd^cV \wedge (dd^c\varphi)^{n-1}$$

and the Fubini theorem gives

$$\begin{aligned}\int_{-\infty}^{r}\nu(dd^cV,\varphi,t)\,dt &= \iint_{\varphi(z)<t<r} dd^cV(z)\wedge(dd^c\varphi(z))^{n-1}\,dt\\ &= \int_{B(r)}(r-\varphi)dd^cV\wedge(dd^c\varphi)^{n-1}.\end{aligned}\tag{6.6}$$

We first show that Formula 6.5 is true when φ and V are smooth. As both members of the formula are left continuous with respect to r and as almost all values of φ are non critical by Sard's theorem, we may assume r non critical. Formula 3.1 applied with $f=(r-\varphi)(dd^c\varphi)^{n-1}$ and $g=V$ shows that integral (6.6) is equal to

$$\int_{S(r)} V(dd^c\varphi)^{n-1}\wedge d^c\varphi-\int_{B(r)} V\,(dd^c\varphi)^n=\mu_r(V)-\int_{B(r)} V\,(dd^c\varphi)^n.$$

Formula 6.5 is thus proved when φ and V are smooth. If V is smooth and φ merely continuous and finite, one can write $\varphi=\lim\varphi_k$ where φ_k is a decreasing sequence of smooth plurisubharmonic functions (because X is Stein). Then $dd^cV\wedge(dd^c\varphi_k)^{n-1}$ converges weakly to $dd^cV\wedge(dd^c\varphi)^{n-1}$ and (6.6) converges, since $\mathbb{1}_{B(r)}(r-\varphi)$ is continuous with compact support on X. The left hand side of Formula 6.5 also converges because the definition of μ_r implies

$$\mu_{k,r}(V)-\int_{\varphi_k<r} V(dd^c\varphi_k)^n=\int_X V\big((dd^c\varphi_{k,\geq r})^n-(dd^c\varphi_k)^n\big)$$

and we can apply again weak convergence on a neighborhood of $\overline{B}(r)$. If φ takes $-\infty$ values, replace φ by $\varphi_{\geq -k}$ where $k\to+\infty$. Then $\mu_r(V)$ is unchanged, $\int_{B(r)}V(dd^c\varphi_{\geq -k})^n$ converges to $\int_{B(r)}V(dd^c\varphi)^n$ and the right hand side of Formula 6.5 is replaced by $\int_{-k}^{r}\nu(dd^cV,\varphi,t)\,dt$. Finally, for V arbitrary, write $V=\lim\downarrow V_k$ with a sequence of smooth functions V_k. Then $dd^cV_k\wedge(dd^c\varphi)^{n-1}$ converges weakly to $dd^cV\wedge(dd^c\varphi)^{n-1}$ by Prop. 4.4, so the integral (6.6) converges to the expected limit and the same is true for the left hand side of 6.5 by the monotone convergence theorem. □

For $r<r_0<R$, the Jensen-Lelong formula implies

$$\mu_r(V)-\mu_{r_0}(V)+\int_{B(r_0)\setminus B(r)} V(dd^c\varphi)^n=\int_{r_0}^{r}\nu(dd^cV,\varphi,t)\,dt.\tag{6.7}$$

(6.8) Corollary. *Assume that $(dd^c\varphi)^n=0$ on $X\setminus S(-\infty)$. Then $r\mapsto\mu_r(V)$ is a convex increasing function of r and the lelong number $\nu(dd^cV,\varphi)$ is given by*

$$\nu(dd^cV,\varphi)=\lim_{r\to-\infty}\frac{\mu_r(V)}{r}.$$

Proof. By (6.7) we have

$$\mu_r(V) = \mu_{r_0}(V) + \int_{r_0}^{r} \nu(dd^cV, \varphi, t)\,dt.$$

As $\nu(dd^cV, \varphi, t)$ is increasing and nonnegative, it follows that $r \longmapsto \mu_r(V)$ is convex and increasing. The formula for $\nu(dd^cV, \varphi) = \lim_{t\to-\infty} \nu(dd^cV, \varphi, t)$ is then obvious. □

(6.9) Example. Let X be an open subset of $\mathbb{C}^n$ equipped with the semi-exhaustive function $\varphi(z) = \log|z - a|$, $a \in X$. Then $(dd^c\varphi)^n = \delta_a$ and the Jensen-Lelong formula becomes

$$\mu_r(V) = V(a) + \int_{-\infty}^{r} \nu(dd^cV, \varphi, t)\,dt.$$

As μ_r is the mean value measure on the sphere $S(a, e^r)$, we make the change of variables $r \mapsto \log r$, $t \mapsto \log t$ and obtain the more familiar formula

$$\text{(6.9 a)}\quad \mu(V, S(a, r)) = V(a) + \int_0^r \nu(dd^cV, a, t)\,\frac{dt}{t}$$

where $\nu(dd^cV, a, t) = \nu(dd^cV, \varphi, \log t)$ is given by (5.7):

$$\text{(6.9 b)}\quad \nu(dd^cV, a, t) = \frac{1}{\pi^{n-1}t^{2n-2}/(n-1)!}\int_{B(a,t)} \frac{1}{2\pi}\Delta V.$$

In this setting, Cor. 6.8 implies

$$\text{(6.9 c)}\quad \nu(dd^cV, a) = \lim_{r\to 0} \frac{\mu\big(V, S(a, r)\big)}{\log r} = \lim_{r\to 0} \frac{\sup_{S(a,r)} V}{\log r}.$$

To prove the last equality, we may assume $V \le 0$ after subtraction of a constant. Inequality $\ge$ follows from the obvious estimate $\mu(V, S(a, r)) \le \sup_{S(a,r)} V$, while inequality $\le$ follows from the standard Harnack estimate

$$\text{(6.9 d)}\quad \sup_{S(a,\varepsilon r)} V \le \frac{1-\varepsilon}{(1+\varepsilon)^{2n-1}}\,\mu\big(V, S(a, r)\big)$$

when ε is small (this estimate follows easily from the Green-Riesz representation formula 1.4.6 and 1.4.7). As $\sup_{S(a,r)} V = \sup_{B(a,r)} V$, Formula (6.9 c) can also be rewritten $\nu(dd^cV, a) = \liminf_{z\to a} V(z)/\log|z - a|$. Since $\sup_{S(a,r)} V$ is a convex (increasing) function of $\log r$, we infer that

$$\text{(6.9 e)}\quad V(z) \le \gamma \log|z - a| + \mathrm{O}(1)$$

with $\gamma = \nu(dd^cV, a)$, and $\nu(dd^cV, a)$ is the largest constant γ which satisfies this inequality. Thus $\nu(dd^cV, a) = \gamma$ is equivalent to V having a logarithmic pole of coefficient γ.

(6.10) Special case Take in particular $V = \log|f|$ where f is a holomorphic function on X. The Lelong-Poincaré formula shows that $dd^c \log|f|$ is equal to the zero divisor $[Z_f] = \sum m_j [H_j]$, where H_j are the irreducible components of $f^{-1}(0)$ and m_j is the multiplicity of f on H_j. The trace $\frac{1}{2\pi}\Delta f$ is then the euclidean area measure of Z_f (with corresponding multiplicities m_j). By Formula (6.9 c), we see that the Lelong number $\nu([Z_f], a)$ is equal to the vanishing order $\mathrm{ord}_a(f)$, that is, the smallest integer m such that $D^\alpha f(a) \neq 0$ for some multiindex α with $|\alpha| = m$. In dimension $n = 1$, we have $\frac{1}{2\pi}\Delta f = \sum m_j \delta_{a_j}$. Then (6.9 a) is the usual Jensen formula

$$\mu\big(\log|f|, S(0,r)\big) - \log|f(0)| = \int_0^r \nu(t)\frac{dt}{t} = \sum m_j \log\frac{r}{|a_j|}$$

where $\nu(t)$ is the number of zeros a_j in the disk $D(0,t)$, counted with multiplicities m_j.

(6.11) Example. Take $\varphi(z) = \log\max|z_j|^{\lambda_j}$ where $\lambda_j > 0$. Then $B(r)$ is the polydisk of radii $(e^{r/\lambda_1}, \ldots, e^{r/\lambda_n})$. If some coordinate z_j is non zero, say z_1, we can write $\varphi(z)$ as $\lambda_1 \log|z_1|$ plus some function depending only on the $(n-1)$ variables $z_j/z_1^{\lambda_1/\lambda_j}$. Hence $(dd^c\varphi)^n = 0$ on $\mathbb{C}^n \setminus \{0\}$. It will be shown later that

(6.11 a) $\quad (dd^c\varphi)^n = \lambda_1 \ldots \lambda_n\, \delta_0.$

We now determine the measures μ_r. At any point z where not all terms $|z_j|^{\lambda_j}$ are equal, the smallest one can be omitted without changing φ in a neighborhood of z. Thus φ depends only on $(n-1)$-variables and $(dd^c\varphi_{\geq r})^n = 0$, $\mu_r = 0$ near z. It follows that μ_r is supported by the distinguished boundary $|z_j| = e^{r/\lambda_j}$ of the polydisk $B(r)$. As φ is invariant by all rotations $z_j \longmapsto e^{\mathrm{i}\theta_j} z_j$, the measure μ_r is also invariant and we see that μ_r is a constant multiple of $d\theta_1 \ldots d\theta_n$. By formula (6.2) and (6.11 a) we get

(6.11 b) $\quad \mu_r = \lambda_1 \ldots \lambda_n\, (2\pi)^{-n} d\theta_1 \ldots d\theta_n.$

In particular, the Lelong number $\nu(dd^cV, \varphi)$ is given by

$$\nu(dd^cV, \varphi) = \lim_{r\to-\infty} \frac{\lambda_1 \ldots \lambda_n}{r} \int_{\theta_j \in [0,2\pi]} V(e^{r/\lambda_1 + \mathrm{i}\theta_1}, \ldots, e^{r/\lambda_n + \mathrm{i}\theta_n})\, \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n}.$$

These numbers have been introduced and studied by (Kiselman 1986). We call them *directional Lelong numbers* with coefficients $(\lambda_1, \ldots, \lambda_n)$. For an arbitrary current T, we define

(6.11 c) $\quad \nu(T, x, \lambda) = \nu\big(T, \log\max|z_j - x_j|^{\lambda_j}\big).$

The above formula for $\nu(dd^cV, \varphi)$ combined with the analogue of Harnack's inequality (6.9 d) for polydisks gives

$$\nu(dd^cV,x,\lambda) = \lim_{r\to 0} \frac{\lambda_1\ldots\lambda_n}{\log r} \int V(r^{1/\lambda_1}e^{\mathrm{i}\theta_1},\ldots,r^{1/\lambda_n}e^{\mathrm{i}\theta_n})\,\frac{d\theta_1\ldots d\theta_n}{(2\pi)^n}$$

$$= \lim_{r\to 0} \frac{\lambda_1\ldots\lambda_n}{\log r} \sup_{\theta_1,\ldots,\theta_n} V(r^{1/\lambda_1}e^{\mathrm{i}\theta_1},\ldots,r^{1/\lambda_n}e^{\mathrm{i}\theta_n}). \tag{6.11 d}$$

7. Comparison Theorems for Lelong Numbers

Let T be a closed positive current of bidimension (p,p) on a Stein manifold X equipped with a semi-exhaustive plurisubharmonic weight φ. We first show that the Lelong numbers $\nu(T,\varphi)$ only depend on the asymptotic behaviour of φ near the polar set $S(-\infty)$. In a precise way:

(7.1) First comparison theorem. *Let $\varphi,\psi : X \longrightarrow [-\infty,+\infty[$ be continuous plurisubharmonic functions. We assume that φ,ψ are semi-exhaustive on* Supp T *and that*

$$\ell := \limsup \frac{\psi(x)}{\varphi(x)} < +\infty \qquad \textit{as} \quad x\in \mathrm{Supp}\,T \quad \textit{and} \quad \varphi(x)\to -\infty.$$

Then $\nu(T,\psi) \le \ell^p\nu(T,\varphi)$, and the equality holds if $\ell = \lim\psi/\varphi$.

Proof. Definition 6.4 shows immediately that $\nu(T,\lambda\varphi) = \lambda^p\nu(T,\varphi)$ for every scalar $\lambda > 0$. It is thus sufficient to verify the inequality $\nu(T,\psi) \le \nu(T,\varphi)$ under the hypothesis $\limsup\psi/\varphi < 1$. For all $c > 0$, consider the plurisubharmonic function

$$u_c = \max(\psi - c,\varphi).$$

Let R_φ and R_ψ be such that $B_\varphi(R_\varphi)\cap \mathrm{Supp}\,T$ and $B_\psi(R_\psi)\cap\mathrm{Supp}\,T$ be relatively compact in X. Let $r < R_\varphi$ and $a < r$ be fixed. For $c > 0$ large enough, we have $u_c = \varphi$ on $\varphi^{-1}([a,r])$ and Stokes' formula gives

$$\nu(T,\varphi,r) = \nu(T,u_c,r) \ge \nu(T,u_c).$$

The hypothesis $\limsup\psi/\varphi < 1$ implies on the other hand that there exists $t_0 < 0$ such that $u_c = \psi - c$ on $\{u_c < t_0\}\cap\mathrm{Supp}\,T$. We infer

$$\nu(T,u_c) = \nu(T,\psi - c) = \nu(T,\psi),$$

hence $\nu(T,\psi) \le \nu(T,\varphi)$. The equality case is obtained by reversing the roles of φ and ψ and observing that $\lim\varphi/\psi = 1/l$. □

Assume in particular that $z^k = (z_1^k,\ldots,z_n^k)$, $k = 1,2$, are coordinate systems centered at a point $x\in X$ and let

$$\varphi_k(z) = \log|z^k| = \log\big(|z_1^k|^2 + \ldots + |z_n^k|^2\big)^{1/2}.$$

We have $\lim_{z\to x} \varphi_2(z)/\varphi_1(z) = 1$, hence $\nu(T,\varphi_1) = \nu(T,\varphi_2)$ by Th. 7.1.

(7.2) Corollary. *The usual Lelong numbers $\nu(T,x)$ are independent of the choice of local coordinates.* □

This result had been originally proved by (Siu 1974) with a much more delicate proof. Another interesting consequence is:

(7.3) Corollary. *On an open subset of $\mathbb{C}^n$, the Lelong numbers and Kiselman numbers are related by*

$$\nu(T,x) = \nu\big(T,x,(1,\ldots,1)\big).$$

Proof. By definition, the Lelong number $\nu(T,x)$ is associated with the weight $\varphi(z) = \log|z-x|$ and the Kiselman number $\nu\big(T,x,(1,\ldots,1)\big)$ to the weight $\psi(z) = \log\max|z_j - x_j|$. It is clear that $\lim_{z\to x}\psi(z)/\varphi(z) = 1$, whence the conclusion. □

Another consequence of Th. 7.1 is that $\nu(T,x,\lambda)$ is an increasing function of each variable λ_j. Moreover, if $\lambda_1 \le \ldots \le \lambda_n$, we get the inequalities

$$\lambda_1^p \nu(T,x) \le \nu(T,x,\lambda) \le \lambda_n^p \nu(T,x).$$

These inequalities will be improved in section 7 (see Cor. 9.16). For the moment, we just prove the following special case.

(7.4) Corollary. *For all $\lambda_1,\ldots,\lambda_n > 0$ we have*

$$\big(dd^c\log\max_{1\le j\le n}|z_j|^{\lambda_j}\big)^n = \Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\lambda_j}\Big)^n = \lambda_1\ldots\lambda_n\,\delta_0.$$

Proof. In fact, our measures vanish on $\mathbb{C}^n\smallsetminus\{0\}$ by the arguments explained in example 6.11. Hence they are equal to $c\,\delta_0$ for some constant $c\ge 0$ which is simply the Lelong number of the bidimension (n,n)-current $T=[X]=1$ with the corresponding weight. The comparison theorem shows that the first equality holds and that

$$\Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\lambda_j}\Big)^n = \ell^{-n}\Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\ell\lambda_j}\Big)^n$$

for all $\ell > 0$. By taking ℓ large and approximating $\ell\lambda_j$ with $2[\ell\lambda_j/2]$, we may assume that $\lambda_j = 2s_j$ is an even integer. Then formula (5.6) gives

$$\int_{\sum|z_j|^{2s_j}<r^2}\Big(dd^c\log\sum|z_j|^{2s_j}\Big)^n=r^{-2n}\int_{\sum|z_j|^{2s_j}<r^2}\Big(dd^c\sum|z_j|^{2s_j}\Big)^n$$
$$=s_1\ldots s_n\,r^{-2n}\int_{\sum|w_j|^2<r^2}2^n\Big(\frac{\mathrm{i}}{2\pi}d'd''|w|^2\Big)^n=\lambda_1\ldots\lambda_n$$

by using the $s_1\ldots s_n$-sheeted change of variables $w_j=z_j^{s_j}$. □

Now, we assume that $T=[A]$ is the current of integration over an analytic set $A\subset X$ of pure dimension p. The above comparison theorem will enable us to give a simple proof of P. Thie's main result (Thie 1967): the Lelong number $\nu([A],x)$ can be interpreted as the multiplicity of the analytic set A at point x. Our starting point is the following consequence of Th. II.3.19 applied simultaneously to all irreducible components of (A,x).

(7.5) Lemma. *For a generic choice of local coordinates $z'=(z_1,\ldots,z_p)$ and $z''=(z_{p+1},\ldots,z_n)$ on (X,x), the germ (A,x) is contained in a cone $|z''|\le C|z'|$. If $B'\subset\mathbb{C}^p$ is a ball of center 0 and radius r' small, and $B''\subset\mathbb{C}^{n-p}$ is the ball of center 0 and radius $r''=Cr'$, then the projection*

$$\mathrm{pr}:A\cap(B'\times B'')\longrightarrow B'$$

is a ramified covering with finite sheet number m. □

We use these properties to compute the Lelong number of $[A]$ at point x. When $z\in A$ tends to x, the functions

$$\varphi(z)=\log|z|=\log(|z'|^2+|z''|^2)^{1/2},\qquad \psi(z)=\log|z'|.$$

are equivalent. As φ,ψ are semi-exhaustive on A, Th. 7.1 implies

$$\nu([A],x)=\nu([A],\varphi)=\nu([A],\psi).$$

Let us apply formula (5.6) to ψ: for every $t<r'$ we get

$$\begin{aligned}\nu([A],\psi,\log t)&=t^{-2p}\int_{\{\psi<\log t\}}[A]\wedge\Big(\frac12dd^ce^{2\psi}\Big)^p\\&=t^{-2p}\int_{A\cap\{|z'|<t\}}\Big(\frac12\mathrm{pr}^\star dd^c|z'|^2\Big)^p\\&=m\,t^{-2p}\int_{\mathbb{C}^p\cap\{|z'|<t\}}\Big(\frac12dd^c|z'|^2\Big)^p=m,\end{aligned}$$

hence $\nu([A],\psi)=m$. Here, we have used the fact that pr is an étale covering with m sheets over the complement of the ramification locus $S\subset B'$, and the fact that S is of zero Lebesgue measure in B'. We have thus obtained simultaneously the following two results:

(7.6) Theorem and Definition. *Let A be an analytic set of dimension p in a complex manifold X of dimension n. For a generic choice of local coordinates $z' = (z_1, \ldots, z_p)$, $z'' = (z_{p+1}, \ldots, z_n)$ near a point $x \in A$ such that the germ (A, x) is contained in a cone $|z''| \le C|z'|$, the sheet number m of the projection $(A, x) \to (\mathbb{C}^p, 0)$ onto the first p coordinates is independent of the choice of z', z''. This number m is called the multiplicity of A at x.*

(7.7) Theorem (Thie 1967). *One has $\nu([A], x) = m$.* □

There is another interesting version of the comparison theorem which compares the Lelong numbers of two currents obtained as intersection products (in that case, we take the same weight for both).

(7.8) Second comparison theorem. *Let $u_1, \ldots, u_q$ and $v_1, \ldots, v_q$ be plurisubharmonic functions such that each q-tuple satisfies the hypotheses of Th. 4.5 with respect to T. Suppose moreover that $u_j = -\infty$ on* Supp $T \cap \varphi^{-1}(-\infty)$ *and that*

$$\ell_j := \limsup \frac{v_j(z)}{u_j(z)} < +\infty \qquad \text{when} \quad z \in \operatorname{Supp} T \smallsetminus u_j^{-1}(-\infty), \quad \varphi(z) \to -\infty.$$

Then

$$\nu(dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge T, \varphi) \le \ell_1 \ldots \ell_q \, \nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, \varphi).$$

Proof. By homogeneity in each factor v_j, it is enough to prove the inequality with constants $\ell_j = 1$ under the hypothesis $\limsup v_j/u_j < 1$. We set

$$w_{j,c} = \max\{v_j - c, u_j\}.$$

Our assumption implies that $w_{j,c}$ coincides with $v_j - c$ on a neighborhood Supp $T \cap \{\varphi < r_0\}$ of Supp $T \cap \{\varphi < -\infty\}$, thus

$$\nu(dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge T, \varphi) = \nu(dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T, \varphi)$$

for every c. Now, fix $r < R_\varphi$. Proposition 4.9 shows that the current $dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T$ converges weakly to $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ when c tends to $+\infty$. By Prop. 5.12 we get

$$\limsup_{c \to +\infty} \nu(dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T, \varphi) \le \nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, \varphi). \square$$

(7.9) Corollary. *If $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ is well defined, then at every point $x \in X$ we have*

$$\nu\big(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, x\big) \ge \nu(dd^c u_1, x) \ldots \nu(dd^c u_q, x) \, \nu(T, x).$$

Proof. Apply (7.8) with $\varphi(z) = v_1(z) = \ldots = v_q(z) = \log|z-x|$ and observe that $\ell_j := \limsup v_j/u_j = 1/\nu(dd^c u_j, x)$ (there is nothing to prove if $\nu(dd^c u_j, x) = 0$). □

Finally, we present an interesting stability property of Lelong numbers due to (Siu 1974): almost all slices of a closed positive current T along linear subspaces passing through a given point have the same Lelong number as T. Before giving a proof of this, we need a useful formula known as *Crofton's formula.*

(7.10) Lemma. *Let α be a closed positive (p,p)-form on $\mathbb{C}^n \smallsetminus \{0\}$ which is invariant under the unitary group $U(n)$. Then α has the form*

$$\alpha = \big(dd^c\chi(\log|z|)\big)^p$$

where χ is a convex increasing function. Moreover α is invariant by homotheties if and only if χ is an affine function, i.e. $\alpha = \lambda\,(dd^c\log|z|)^p$.

Proof. A radial convolution $\alpha_\varepsilon(z) = \int_{\mathbb{R}} \rho(t/\varepsilon)\,\alpha(e^t z)\,dt$ produces a smooth form with the same properties as α and $\lim_{\varepsilon\to 0}\alpha_\varepsilon = \alpha$. Hence we can suppose that α is smooth on $\mathbb{C}^n \smallsetminus \{0\}$. At a point $z = (0,\ldots,0,z_n)$, the (p,p)-form $\alpha(z) \in \bigwedge^{p,p}(\mathbb{C}^n)^\star$ must be invariant by $U(n-1)$ acting on the first $(n-1)$ coordinates. We claim that the subspace of $U(n-1)$-invariants in $\bigwedge^{p,p}(\mathbb{C}^n)^\star$ is generated by $(dd^c|z|^2)^p$ and $(dd^c|z|^2)^{p-1}\wedge \mathrm{i}dz_n\wedge d\overline{z}_n$. In fact, a form $\beta = \sum \beta_{I,J} dz_I \wedge d\overline{z}_J$ is invariant by $U(1)^{n-1} \subset U(n-1)$ if and only if $\beta_{I,J} = 0$ for $I \neq J$, and invariant by the permutation group $\mathcal{S}_{n-1} \subset U(n-1)$ if and only if all coefficients $\beta_{I,I}$ (resp. $\beta_{Jn,Jn}$) with $I, J \subset \{1,\ldots,n-1\}$ are equal. Hence

$$\beta = \lambda \sum_{|I|=p} dz_I \wedge d\overline{z}_I + \mu\Big(\sum_{|J|=p-1} dz_J \wedge d\overline{z}_J\Big)\wedge dz_n \wedge d\overline{z}_n.$$

This proves our claim. As $d|z|^2 \wedge d^c|z|^2 = \frac{\mathrm{i}}{\pi}|z_n|^2 dz_n \wedge d\overline{z}_n$ at $(0,\ldots,0,z_n)$, we conclude that

$$\alpha(z) = f(z)(dd^c|z|^2)^p + g(z)(dd^c|z|^2)^{p-1}\wedge d|z|^2 \wedge d^c|z|^2$$

for some smooth functions f, g on $\mathbb{C}^n \smallsetminus \{0\}$. The $U(n)$-invariance of α shows that f and g are radial functions. We may rewrite the last formula as

$$\alpha(z) = u(\log|z|)(dd^c\log|z|)^p + v(\log|z|)(dd^c\log|z|)^{p-1}\wedge d\log|z| \wedge d^c\log|z|.$$

Here $(dd^c\log|z|)^p$ is a positive (p,p)-form coming from $\mathbb{P}^{n-1}$, hence it has zero contraction in the radial direction, while the contraction of the form $(dd^c\log|z|)^{p-1}\wedge d\log|z|\wedge d^c\log|z|$ by the radial vector field is $(dd^c\log|z|)^{p-1}$. This shows easily that $\alpha(z) \geq 0$ if and only if $u, v \geq 0$. Next, the closedness condition $d\alpha = 0$ gives $u' - v = 0$. Thus u is increasing and we define a convex increasing function χ by $\chi' = u^{1/p}$. Then $v = u' = p\chi'^{p-1}\chi''$ and

$$\alpha(z) = \big(dd^c\chi(\log|z|)\big)^p.$$

If α is invariant by homotheties, the functions u and v must be constant, thus $v = 0$ and $\alpha = \lambda(dd^c \log|z|)^p$. $\square$

(7.11) Corollary (Crofton's formula). *Let dv be the unique $U(n)$-invariant measure of mass 1 on the Grassmannian $G(p,n)$ of p-dimensional subspaces in $\mathbb{C}^n$. Then*

$$\int_{S\in G(p,n)} [S]\,dv(S) = (dd^c \log|z|)^{n-p}.$$

Proof. The left hand integral is a closed positive bidegree $(n-p, n-p)$ current which is invariant by $U(n)$ and by homotheties. By Lemma 7.10, this current must coincide with the form $\lambda(dd^c \log|z|)^{n-p}$ for some $\lambda \geq 0$. The coefficient λ is the Lelong number at 0. As $\nu([S],0) = 1$ for every S, we get $\lambda = \int_{G(p,n)} dv(S) = 1$. $\square$

We now recall a few basic facts of slicing theory; see (Federer 1969) for details. Let $\sigma : M \to M'$ be a submersion of smooth differentiable manifolds and let Θ be a *locally flat* current on M, that is a current which can be written locally as $\Theta = U + dV$ where U, V have locally integrable coefficients. It can be shown that every current Θ such that both Θ and $d\Theta$ have measure coefficients is locally flat; in particular, closed positive currents are locally flats. Then, for almost every $x' \in M'$, there is a well defined slice $\Theta_{x'}$, which is the current on the fiber $\sigma^{-1}(x')$ defined by

$$\Theta_{x'} = U_{\restriction\sigma^{-1}(x')} + dV_{\restriction\sigma^{-1}(x')}.$$

The restrictions of U, V to the fibers exist for almost all x' by the Fubini theorem. It is easy to show by a regularization $\Theta_\varepsilon = \Theta \star \rho_\varepsilon$ that the slices of a closed positive current are again closed and positive: in fact $U_{\varepsilon,x'}$ and $V_{\varepsilon,x'}$ converge to $U_{x'}$ and $V_{x'}$ in L^1_{loc}, thus $\Theta_{\varepsilon,x'}$ converges weakly to $\Theta_{x'}$ for almost every x'. This kind of slicing can be referred to as *parallel slicing* (if we think of σ as being a projection map). The kind of slicing we need (where the slices are taken over linear subspaces passing through a given point) is of a slightly different nature and is called *concurrent slicing.*

The possibility of concurrent slicing is proved as follows. Let T be a closed positive current of bidimension (p,p) in the ball $B(0,R) \subset \mathbb{C}^n$. Let

$$Y = \big\{(x,S) \in \mathbb{C}^n \times G(q,n)\,;\, x \in S\big\}$$

be the total space of the tautological rank q vector bundle over the Grassmannian $G(q,n)$, equipped with the obvious projections

$$\sigma : Y \longrightarrow G(q,n), \qquad \pi : Y \longrightarrow \mathbb{C}^n.$$

We set $Y_R = \pi^{-1}(B(0,R))$ and $Y_R^\star = \pi^{-1}(B(0,R) \smallsetminus \{0\})$. The restriction π_0 of π to $Y_R^\star$ is a submersion onto $B(0,R) \smallsetminus \{0\}$, so we have a well defined pull-back $\pi_0^\star T$ over $Y_R^\star$. We would like to extend it as a pull-back $\pi^\star T$ over Y_R, so as to define slices $T_{\restriction S} = (\pi^\star T)_{\restriction \sigma^{-1}(S)}$; of course, these slices can be non zero only if the dimension of S is at least equal to the degree of T, i.e. if $q \geq n-p$. We first claim that $\pi_0^\star T$ has locally finite mass near the zero section $\pi^{-1}(0)$ of σ. In fact let ω_G be a unitary invariant Kähler metric over $G(q,n)$ and let $\beta = dd^c|z|^2$ in $\mathbb{C}^n$. Then we get a Kähler metric on Y defined by $\omega_Y = \sigma^\star \omega_G + \pi^\star \beta$. If $N = (q-1)(n-q)$ is the dimension of the fibers of π, the projection formula $\pi_\star(u \wedge \pi^\star v) = (\pi_\star u) \wedge v$ gives

$$\pi_\star \omega_Y^{N+p} = \sum_{1 \leq k \leq p} \binom{N+p}{k} \beta^k \wedge \pi_\star(\sigma^\star \omega_G^{N+p-k}).$$

Here $\pi_\star(\sigma^\star \omega_G^{N+p-k})$ is a unitary and homothety invariant $(p-k, p-k)$ closed positive form on $\mathbb{C}^n \smallsetminus \{0\}$, so $\pi_\star(\sigma^\star \omega_G^{N+p-k})$ is proportional to $(dd^c \log|z|)^{n-k}$. With some constants $\lambda_k > 0$, we thus get

$$\begin{aligned} \int_{Y_r^\star} \pi_0^\star T \wedge \omega_Y^{N+p} &= \sum_{0 \leq k \leq p} \lambda_k \int_{B(0,r) \smallsetminus \{0\}} T \wedge \beta^k \wedge (dd^c \log|z|)^{k-p} \\ &= \sum_{0 \leq k \leq p} \lambda_k 2^{-(p-k)} r^{-2(p-k)} \int_{B(0,r) \smallsetminus \{0\}} T \wedge \beta^p < +\infty. \end{aligned}$$

The Skoda-El Mir theorem 2.3 shows that the trivial extension $\widetilde{\pi}_0^\star T$ of $\pi_0^\star T$ is a closed positive current on Y_R. Of course, the zero section $\pi^{-1}(0)$ might also carry some extra mass of the desired current $\pi^\star T$. Since $\pi^{-1}(0)$ has codimension q, this extra mass cannot exist when $q > n-p = \operatorname{codim} \pi^\star T$ and we simply set $\pi^\star T = \widetilde{\pi}_0^\star T$. On the other hand, if $q = n-p$, we set

(7.12) $\quad \pi^\star T := \widetilde{\pi}_0^\star T + \nu(T,0)\,[\pi^{-1}(0)].$

We can now apply parallel slicing with respect to $\sigma : Y_R \to G(q,n)$, which is a submersion: for almost all $S \in G(q,n)$, there is a well defined slice $T_{\restriction S} = (\pi^\star T)_{\restriction \sigma^{-1}(S)}$. These slices coincide with the usual restrictions of T to S if T is smooth.

(7.13) Theorem (Siu 1974)**.** *For almost all $S \in G(q,n)$ with $q \geq n-p$, the slice $T_{\restriction S}$ satisfies $\nu(T_{\restriction S}, 0) = \nu(T,0)$.*

Proof. If $q = n-p$, the slice $T_{\restriction S}$ consists of some positive measure with support in $S \smallsetminus \{0\}$ plus a Dirac measure $\nu(T,0)\,\delta_0$ coming from the slice of $\nu(T,0)\,[\pi^{-1}(0)]$. The equality $\nu(T_{\restriction S}, 0) = \nu(T,0)$ thus follows directly from (7.12).

In the general case $q > n-p$, it is clearly sufficient to prove the following two properties:

a) $\nu(T,0,r) = \int_{S\in G(q,n)} \nu(T_{\restriction S},0,r)\,dv(S)$ for all $r \in]0,R[$;

b) $\nu(T_{\restriction S},0) \ge \nu(T,0)$ for almost all S.

In fact, a) implies that $\nu(T,0)$ is the average of all Lelong numbers $\nu(T_{\restriction S},0)$ and the conjunction with b) implies that these numbers must be equal to $\nu(T,0)$ for almost all S. In order to prove a) and b), we can suppose without loss of generality that T is smooth on $B(0,R)\smallsetminus\{0\}$. Otherwise, we perform a small convolution with respect to the action of $\mathrm{Gl}_n(\mathbb{C})$ on $\mathbb{C}^n$:

$$T_\varepsilon = \int_{g\in \mathrm{Gl}_n(\mathbb{C})} \rho_\varepsilon(g)\, g^\star T\,dv(g)$$

where (ρ_ε) is a regularizing family with support in an ε-neighborhood of the unit element of $\mathrm{Gl}_n(\mathbb{C})$. Then T_ε is smooth in $B(0,(1-\varepsilon)R)\smallsetminus\{0\}$ and converges weakly to T. Moreover, we have $\nu(T_\varepsilon,0) = \nu(T,0)$ by (7.2) and $\nu(T_{\restriction S},0) \ge \limsup_{\varepsilon\to 0}\nu(T_{\varepsilon,\restriction S},0)$ by (5.12), thus a), b) are preserved in the limit. If T is smooth on $B(0,R)\smallsetminus\{0\}$, the slice $T_{\restriction S}$ is defined for all S and is simply the restriction of T to $S\smallsetminus\{0\}$ (carrying no mass at the origin).

a) Here we may even assume that T is smooth at 0 by performing an ordinary convolution. As $T_{\restriction S}$ has bidegree $(n-p,n-p)$, we have

$$\nu(T_{\restriction S},0,r) = \int_{S\cap B(0,r)} T\wedge\alpha_S^{q-(n-p)} = \int_{B(0,r)} T\wedge[S]\wedge\alpha_S^{p+q-n}$$

where $\alpha_S = dd^c\log|w|$ and $w=(w_1,\ldots,w_q)$ are orthonormal coordinates on S. We simply have to check that

$$\int_{S\in G(q,n)} [S]\wedge\alpha_S^{p+q-n}\,dv(S) = (dd^c\log|z|)^p.$$

However, both sides are unitary and homothety invariant (p,p)-forms with Lelong number 1 at the origin, so they must coincide by Lemma 7.11.

b) We prove the inequality when $S=\mathbb{C}^q\times\{0\}$. By the comparison theorem 7.1, for every $r>0$ and $\varepsilon>0$ we have

$$(7.14)\qquad \int_{B(0,r)} T\wedge\gamma_\varepsilon^p \ge \nu(T,0) \quad\text{where}$$
$$\gamma_\varepsilon = \frac{1}{2}dd^c\log(\varepsilon|z_1|^2+\ldots+\varepsilon|z_q|^2+|z_{q+1}|^2+\ldots+|z_n|^2).$$

We claim that the current γ_ε^p converges weakly to

$$[S]\wedge\alpha_S^{p+q-n} = [S]\wedge\Big(\frac{1}{2}dd^c\log(|z_1|^2+\ldots+|z_q|^2)\Big)^{p+q-n}$$

as ε tends to 0. In fact, the Lelong number of γ_ε^p at 0 is 1, hence by homogeneity

$$\int_{B(0,r)} \gamma_\varepsilon^p \wedge (dd^c|z|^2)^{n-p} = (2r^2)^p$$

for all $\varepsilon, r > 0$. Therefore the family (γ_ε^p) is relatively compact in the weak topology. Since $\gamma_0 = \lim \gamma_\varepsilon$ is smooth on $\mathbb{C}^n \setminus S$ and depends only on $n-q$ variables ($n - q \le p$), we have $\lim \gamma_\varepsilon^p = \gamma_0^p = 0$ on $\mathbb{C}^n \setminus S$. This shows that every weak limit of (γ_ε^p) has support in S. Each of these is the direct image by inclusion of a unitary and homothety invariant $(p+q-n, p+q-n)$-form on S with Lelong number equal to 1 at 0. Therefore we must have

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^p = (i_S)_\star(\alpha_S^{p+q-n}) = [S] \wedge \alpha_S^{p+q-n},$$

and our claim is proved (of course, this can also be checked by direct elementary calculations). By taking the limsup in (7.14) we obtain

$$\nu(T_{\restriction S}, 0, r+0) = \int_{\overline{B}(0,r)} T \wedge [S] \wedge \alpha_S^{p+q-n} \ge \nu(T, 0)$$

(the singularity of T at 0 does not create any difficulty because we can modify T by a dd^c-exact form near 0 to make it smooth everywhere). Property b) follows when r tends to 0. □

8. Siu's Semicontinuity Theorem

Let X, Y be complex manifolds of dimension n, m such that X is Stein. Let $\varphi : X \times Y \longrightarrow [-\infty, +\infty[$ be a continuous plurisubharmonic function. We assume that φ is *semi-exhaustive* with respect to Supp T , i.e. that for every compact subset $L \subset Y$ there exists $R = R(L) < 0$ such that

(8.1) $\{(x, y) \in \operatorname{Supp} T \times L\,;\, \varphi(x, y) \le R\} \subset\subset X \times Y.$

Let T be a closed positive current of bidimension (p, p) on X. For every point $y \in Y$, the function $\varphi_y(x) := \varphi(x, y)$ is semi-exhaustive on Supp T; one can therefore associate with y a generalized Lelong number $\nu(T, \varphi_y)$. Proposition 5.13 implies that the map $y \mapsto \nu(T, \varphi_y)$ is upper semi-continuous, hence the upperlevel sets

(8.2) $E_c = E_c(T, \varphi) = \{y \in Y; \nu(T, \varphi_y) \ge c\}\ ,\ c > 0$

are closed. Under mild additional hypotheses, we are going to show that the sets E_c are in fact analytic subsets of Y (Demailly 1987a).

(8.3) Definition. *We say that a function $f(x,y)$ is locally Hölder continuous with respect to y on $X \times Y$ if every point of $X \times Y$ has a neighborhood Ω on which*

$$|f(x,y_1) - f(x,y_2)| \le M|y_1 - y_2|^\gamma$$

for all $(x,y_1) \in \Omega$, $(x,y_2) \in \Omega$, with some constants $M > 0$, $\gamma \in {]0,1]}$, and suitable coordinates on Y.

(8.4) Theorem (Demailly 1987a). *Let T be a closed positive current on X and let*

$$\varphi : X \times Y \longrightarrow [-\infty, +\infty[$$

be a continuous plurisubharmonic function. Assume that φ is semi-exhaustive on $\operatorname{Supp} T$ and that $e^{\varphi(x,y)}$ is locally Hölder continuous with respect to y on $X \times Y$. Then the upperlevel sets

$$E_c(T,\varphi) = \{y \in Y ; \nu(T,\varphi_y) \ge c\}$$

are analytic subsets of Y.

This theorem can be rephrased by saying that $y \longmapsto \nu(T,\varphi_y)$ is upper semi-continuous with respect to the analytic Zariski topology. As a special case, we get the following important result of (Siu 1974):

(8.5) Corollary. *If T is a closed positive current of bidimension (p,p) on a complex manifold X, the upperlevel sets $E_c(T) = \{x \in X ; \nu(T,x) \ge c\}$ of the usual Lelong numbers are analytic subsets of dimension $\le p$.*

Proof. The result is local, so we may assume that $X \subset \mathbb{C}^n$ is an open subset. Theorem 8.4 with $Y = X$ and $\varphi(x,y) = \log|x-y|$ shows that $E_c(T)$ is analytic. Moreover, Prop. 5.11 implies $\dim E_c(T) \le p$. □

(8.6) Generalization. Theorem 8.4 can be applied more generally to weight functions of the type

$$\varphi(x,y) = \max_j \, \log\Big(\sum_k |F_{j,k}(x,y)|^{\lambda_{j,k}}\Big)$$

where $F_{j,k}$ are holomorphic functions on $X \times Y$ and where $\gamma_{j,k}$ are positive real constants; in this case e^φ is Hölder continuous of exponent $\gamma = \min\{\lambda_{j,k}, 1\}$ and φ is semi-exhaustive with respect to the whole space X as soon as the projection $\mathrm{pr}_2 : \bigcap F_{j,k}^{-1}(0) \longrightarrow Y$ is proper and finite.

For example, when $\varphi(x,y) = \log\max|x_j - y_j|_j^\lambda$ on an open subset X of $\mathbb{C}^n$, we see that the upperlevel sets for Kiselman's numbers $\nu(T,x,\lambda)$ are analytic in X (a result first proved in (Kiselman 1986). More generally, set

$\psi_\lambda(z) = \log\max|z_j|^{\lambda_j}$ and $\varphi(x,y,g) = \psi_\lambda\big(g(x-y)\big)$ where $x,y \in \mathbb{C}^n$ and $g \in \mathrm{Gl}(\mathbb{C}^n)$. Then $\nu(T,\varphi_{y,g})$ is the Kiselman number of T at y when the coordinates have been rotated by g. It is clear that φ is plurisubharmonic in (x,y,g) and semi-exhaustive with respect to x, and that e^φ is locally Hölder continuous with respect to (y,g). Thus the upperlevel sets

$$E_c = \{(y,g) \in X \times \mathrm{Gl}(\mathbb{C}^n)\,;\, \nu(T,\varphi_{y,g}) \geq c\}$$

are analytic in $X \times \mathrm{Gl}(\mathbb{C}^n)$. However this result is not meaningful on a manifold, because it is not invariant under coordinate changes. One can obtain an invariant version as follows. Let X be a manifold and let $J^k\mathcal{O}_X$ be the bundle of k-jets of holomorphic functions on X. We consider the bundle S_k over X whose fiber $S_{k,y}$ is the set of n-tuples of k-jets $u = (u_1,\ldots,u_n) \in (J^k\mathcal{O}_{X,y})^n$ such that $u_j(y) = 0$ and $du_1 \wedge \ldots \wedge du_n(y) \neq 0$. Let (z_j) be local coordinates on an open set $\Omega \subset X$. Modulo $O(|z-y|^{k+1})$, we can write

$$u_j(z) = \sum_{1\leq|\alpha|\leq k} a_{j,\alpha}(z-y)^\alpha$$

with $\det(a_{j,(0,\ldots,1_k,\ldots,0)}) \neq 0$. The numbers $((y_j),(a_{j,\alpha}))$ define a coordinate system on the total space of $S_{k\restriction\Omega}$. For $(x,(y,u)) \in X \times S_k$, we introduce the function

$$\varphi(x,y,u) = \log\max|u_j(x)|^{\lambda_j} = \log\max\Big|\sum_{1\leq|\alpha|\leq k} a_{j,\alpha}(x-y)^\alpha\Big|^{\lambda_j}$$

which has all properties required by Th. 8.4 on a neighborhood of the diagonal $x = y$, i.e. a neighborhood of $X \times_X S_k$ in $X \times S_k$. For k large, we claim that Kiselman's directional Lelong numbers

$$\nu(T,y,u,\lambda) := \nu(T,\varphi_{y,u})$$

with respect to the coordinate system (u_j) at y do not depend on the selection of the jet representives and are therefore canonically defined on S_k. In fact, a change of u_j by $O(|z-y|^{k+1})$ adds $O(|z-y|^{(k+1)\lambda_j})$ to e^φ, and we have $e^\varphi \geq O(|z-y|^{\max\lambda_j})$. Hence by (7.1) it is enough to take $k+1 \geq \max\lambda_j/\min\lambda_j$. Theorem 8.4 then shows that the upperlevel sets $E_c(T,\varphi)$ are analytic in S_k. □

Proof of the Semicontinuity Theorem 8.4 As the result is local on Y, we may assume without loss of generality that Y is a ball in $\mathbb{C}^m$. After addition of a constant to φ, we may also assume that there exists a compact subset $K \subset X$ such that

$$\{(x,y) \in X \times Y; \varphi(x,y) \leq 0\} \subset K \times Y.$$

By Th. 7.1, the Lelong numbers depend only on the asymptotic behaviour of φ near the (compact) polar set $\varphi^{-1}(-\infty) \cap (\mathrm{Supp}\,\mathrm{T} \times Y)$. We can add a smooth strictly plurisubharmonic function on $X \times Y$ to make φ strictly plurisuharmonic. Then Richberg's approximation theorem for continuous plurisubharmonic functions shows that there exists a smooth plurisubharmonic function $\widetilde{\varphi}$ such that $\varphi \leq \widetilde{\varphi} \leq \varphi + 1$. We may therefore assume that φ is smooth on $(X \times Y) \setminus \varphi^{-1}(-\infty)$.

• **First step**: *construction of a local plurisubharmonic potential.*

Our goal is to generalize the usual construction of plurisubharmonic potentials associated with a closed positive current (Lelong 1967, Skoda 1972a). We replace here the usual kernel $|z-\zeta|^{-2p}$ arising from the hermitian metric of $\mathbb{C}^n$ by a kernel depending on the weight φ. Let $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ be an increasing function such that $\chi(t) = t$ for $t \leq -1$ and $\chi(t) = 0$ for $t \geq 0$. We consider the half-plane $H = \{z \in \mathbb{C}\,;\ \mathrm{Re} z < -1\}$ and associate with T the potential function V on $Y \times H$ defined by

$$(8.7)\quad V(y,z) = -\int_{\mathrm{Re} z}^{0} \nu(T,\varphi_y,t)\chi'(t)\,dt.$$

For every $t > \mathrm{Re}\, z$, Stokes' formula gives

$$\nu(T,\varphi_y,t) = \int_{\varphi(x,y)<t} T(x) \wedge (dd^c_x \widetilde{\varphi}(x,y,z))^p$$

with $\widetilde{\varphi}(x,y,z) := \max\{\varphi(x,y), \mathrm{Re} z\}$. The Fubini theorem applied to (8.7) gives

$$\begin{aligned} V(y,z) &= -\int_{\substack{x\in X,\varphi(x,y)<t\\ \mathrm{Re}\, z<t<0}} T(x) \wedge (dd^c_x \widetilde{\varphi}(x,y,z))^p\ \chi'(t)dt \\ &= \int_{x\in X} T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))\,(dd^c_x \widetilde{\varphi}(x,y,z))^p. \end{aligned}$$

For all $(n-1,n-1)$-form h of class C^∞ with compact support in $Y \times H$, we get

$$\begin{aligned} \langle dd^c V, h\rangle &= \langle V, dd^c h\rangle \\ &= \int_{X\times Y\times H} T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))(dd^c \widetilde{\varphi}(x,y,z))^p \wedge dd^c h(y,z). \end{aligned}$$

Observe that the replacement of dd^c_x by the total differentiation $dd^c = dd^c_{x,y,z}$ does not modify the integrand, because the terms in dx, $d\overline{x}$ must have total bidegree (n,n). The current $T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))h(y,z)$ has compact support in $X \times Y \times H$. An integration by parts can thus be performed to obtain

$$\langle dd^c V, h\rangle = \int_{X\times Y\times H} T(x) \wedge dd^c(\chi\circ\widetilde{\varphi}(x,y,z)) \wedge (dd^c\widetilde{\varphi}(x,y,z))^p.h(y,z).$$

On the corona $\{-1 \leq \varphi(x,y) \leq 0\}$ we have $\widetilde{\varphi}(x,y,z) = \varphi(x,y)$, whereas for $\varphi(x,y) < -1$ we get $\widetilde{\varphi} < -1$ and $\chi\circ\widetilde{\varphi} = \widetilde{\varphi}$. As $\widetilde{\varphi}$ is plurisubharmonic, we see that $dd^c V(y,z)$ is the sum of the positive $(1,1)$-form

$$(y,z) \longmapsto \int_{\{x\in X;\varphi(x,y)<-1\}} T(x) \wedge (dd^c_{x,y,z}\widetilde{\varphi}(x,y,z))^{p+1}$$

and of the $(1,1)$-form independent of z

$$y \longmapsto \int_{\{x\in X;-1\le\varphi(x,y)\le 0\}} T \wedge dd^c_{x,y}(\chi\circ\varphi) \wedge (dd^c_{x,y}\varphi)^p.$$

As φ is smooth outside $\varphi^{-1}(-\infty)$, this last form has locally bounded coefficients. Hence $dd^cV(y,z)$ is ≥ 0 except perhaps for locally bounded terms. In addition, V is continuous on $Y\times H$ because $T\wedge(dd^c\widetilde{\varphi}_{y,z})^p$ is weakly continuous in the variables (y,z) by Th. 3.5. We therefore obtain the following result.

(8.8) Proposition. *There exists a positive plurisubharmonic function ρ in $C^\infty(Y)$ such that $\rho(y)+V(y,z)$ is plurisubharmonic on $Y\times H$.*

If we let $\mathrm{Re}\,z$ tend to $-\infty$, we see that the function

$$U_0(y) = \rho(y) + V(y,-\infty) = \rho(y) - \int_{-\infty}^{0} \nu(T,\varphi_y,t)\chi'(t)dt$$

is locally plurisubharmonic or $\equiv -\infty$ on Y. Furthermore, it is clear that $U_0(y) = -\infty$ at every point y such that $\nu(T,\varphi_y) > 0$. If Y is connected and $U_0 \not\equiv -\infty$, we already conclude that the density set $\bigcup_{c>0} E_c$ is pluripolar in Y.

• **Second step**: *application of Kiselman's minimum principle.*
Let $a\ge 0$ be arbitrary. The function

$$Y\times H \ni (y,z) \longmapsto \rho(y) + V(y,z) - a\mathrm{Re}z$$

is plurisubharmonic and independent of $\mathrm{Im}\,z$. By Kiselman's theorem 1.7.8, the Legendre transform

$$U_a(y) = \inf_{r<-1}\big\{\rho(y)+V(y,r)-ar\big\}$$

is locally plurisubharmonic or $\equiv -\infty$ on Y.

(8.9) Lemma. *Let $y_0\in Y$ be a given point.*

a) *If $a > \nu(T,\varphi_{y_0})$, then U_a is bounded below on a neighborhood of y_0.*

b) *If $a < \nu(T,\varphi_{y_0})$, then $U_a(y_0) = -\infty$.*

Proof. By definition of V (cf. (8.7)) we have

$$(8.10)\quad V(y,r) \le -\nu(T,\varphi_y,r)\int_r^0 \chi'(t)dt = r\nu(T,\varphi_y,r) \le r\nu(T,\varphi_y).$$

Then clearly $U_a(y_0) = -\infty$ if $a < \nu(T, \varphi_{y_0})$. On the other hand, if $\nu(T, \varphi_{y_0}) < a$, there exists $t_0 < 0$ such that $\nu(T, \varphi_{y_0}, t_0) < a$. Fix $r_0 < t_0$. The semi-continuity property (5.13) shows that there exists a neighborhood ω of y_0 such that $\sup_{y \in \omega} \nu(T, \varphi_y, r_0) < a$. For all $y \in \omega$, we get

$$V(y, r) \geq -C - a \int_r^{r_0} \chi'(t) dt = -C + a(r - r_0),$$

and this implies $U_a(y) \geq -C - ar_0$. □

(8.11) Theorem. *If Y is connected and if $E_c \neq Y$, then E_c is a closed complete pluripolar subset of Y, i.e. there exists a continuous plurisubharmonic function $w : Y \longrightarrow [-\infty, +\infty[$ such that $E_c = w^{-1}(-\infty)$.*

Proof. We first observe that the family (U_a) is increasing in a, that $U_a = -\infty$ on E_c for all $a < c$ and that $\sup_{a<c} U_a(y) > -\infty$ if $y \in Y \setminus E_c$ (apply Lemma 8.9). For any integer $k \geq 1$, let $w_k \in C^\infty(Y)$ be a plurisubharmonic regularization of $U_{c-1/k}$ such that $w_k \geq U_{c-1/k}$ on Y and $w_k \leq -2^k$ on $E_c \cap Y_k$ where $Y_k = \{y \in Y \,;\, d(y, \partial Y) \geq 1/k\}$. Then Lemma 8.9 a) shows that the family (w_k) is uniformly bounded below on every compact subset of $Y \setminus E_c$. We can also choose w_k uniformly bounded above on every compact subset of Y because $U_{c-1/k} \leq U_c$. The function

$$w = \sum_{k=1}^{+\infty} 2^{-k} w_k$$

satifies our requirements. □

• **Third step**: *estimation of the singularities of the potentials U_a.*

(8.12) Lemma. *Let $y_0 \in Y$ be a given point, L a compact neighborhood of y_0, $K \subset X$ a compact subset and r_0 a real number < -1 such that*

$$\{(x, y) \in X \times L; \varphi(x, y) \leq r_0\} \subset K \times L.$$

Assume that $e^{\varphi(x,y)}$ is locally Hölder continuous in y and that

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|^\gamma$$

for all $(x, y_1, y_2) \in K \times L \times L$. Then, for all $\varepsilon \in\,]0, 1[$, there exists a real number $\eta(\varepsilon) > 0$ such that all $y \in Y$ with $|y - y_0| < \eta(\varepsilon)$ satisfy

$$U_a(y) \leq \rho(y) + \big((1 - \varepsilon)^p \nu(T, \varphi_{y_0}) - a\big)\Big(\gamma \log |y - y_0| + \log \frac{2eM}{\varepsilon}\Big).$$

Proof. First, we try to estimate $\nu(T, \varphi_y, r)$ when $y \in L$ is near y_0. Set

$$\begin{cases} \psi(x) = (1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 & \text{if} \qquad \varphi_{y_0}(x) \le r-1 \\ \psi(x) = \max\big(\varphi_y(x), (1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2\big) & \text{if } r-1 \le \varphi_{y_0}(x) \le r \\ \psi(x) = \varphi_y(x) & \text{if} \quad r \le \varphi_{y_0}(x) \le r_0 \end{cases}$$

and verify that this definition is coherent when $|y-y_0|$ is small enough. By hypothesis

$$|e^{\varphi_y(x)} - e^{\varphi_{y_0}(x)}| \le M|y-y_0|^\gamma.$$

This inequality implies

$$\begin{aligned} \varphi_y(x) &\le \varphi_{y_0}(x) + \log\big(1 + M|y-y_0|^\gamma e^{-\varphi_{y_0}(x)}\big) \\ \varphi_y(x) &\ge \varphi_{y_0}(x) + \log\big(1 - M|y-y_0|^\gamma e^{-\varphi_{y_0}(x)}\big). \end{aligned}$$

In particular, for $\varphi_{y_0}(x) = r$, we have $(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 = r - \varepsilon/2$, thus

$$\varphi_y(x) \ge r + \log(1 - M|y-y_0|^\gamma e^{-r}).$$

Similarly, for $\varphi_{y_0}(x) = r-1$, we have $(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 = r - 1 + \varepsilon/2$, thus

$$\varphi_y(x) \le r - 1 + \log(1 + M|y-y_0|^\gamma e^{1-r}).$$

The definition of ψ is thus coherent as soon as $M|y-y_0|^\gamma e^{1-r} \le \varepsilon/2$, i.e.

$$\gamma \log|y-y_0| + \log\frac{2eM}{\varepsilon} \le r.$$

In this case ψ coincides with φ_y on a neighborhood of $\{\psi = r\}$, and with

$$(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2$$

on a neighborhood of the polar set $\psi^{-1}(-\infty)$. By Stokes' formula applied to $\nu(T,\psi,r)$, we infer

$$\nu(T,\varphi_y,r) = \nu(T,\psi,r) \ge \nu(T,\psi) = (1-\varepsilon)^p \nu(T,\varphi_{y_0}).$$

From (8.10) we get $V(y,r) \le r\nu(T,\varphi_y,r)$, hence

$$\begin{aligned} &U_a(y) \le \rho(y) + V(y,r) - ar \le \rho(y) + r\big(\nu(T,\varphi_y,r) - a\big), \\ (8.13)\qquad &U_a(y) \le \rho(y) + r\big((1-\varepsilon)^p\nu(T,\varphi_{y_0}) - a\big). \end{aligned}$$

Suppose $\gamma\log|y-y_0| + \log(2eM/\varepsilon) \le r_0$, i.e. $|y-y_0| \le (\varepsilon/2eM)^{1/\gamma} e^{r_0/\gamma}$; one can then choose $r = \gamma\log|y-y_0| + \log(2eM/\varepsilon)$, and by (8.13) this yields the inequality asserted in Th. 8.12. □

• **Fourth step**: *application of the Hörmander-Bombieri-Skoda theorem.*

The end of the proof relies on the following crucial result, which is a consequence of the Hörmander-Bombieri-Skoda theorem (Bombieri 1970, Skoda 1972a, Skoda 1976); it will be proved in Chapter 8, see Cor. 8.?.?.

(8.14) Proposition. *Let u be a plurisubharmonic function on a complex manifold Y. The set of points in a neighborhood of which e^{-u} is not integrable is an analytic subset of Y.* □

Proof of Theorem 8.4 (end). The main idea in what follows is due to (Kiselman 1979). For $a, b > 0$, we let $Z_{a,b}$ be the set of points in a neighborhood of which $\exp(-U_a/b)$ is not integrable. Then $Z_{a,b}$ is analytic, and as the family (U_a) is increasing in a, we have $Z_{a',b'} \supset Z_{a'',b''}$ if $a' \le a''$, $b' \le b''$.

Let $y_0 \in Y$ be a given point. If $y_0 \notin E_c$, then $\nu(T, \varphi_{y_0}) < c$ by definition of E_c. Choose a such that $\nu(T, \varphi_{y_0}) < a < c$. Lemma 8.9 a) implies that U_a is bounded below in a neighborhood of y_0, thus $\exp(-U_a/b)$ is integrable and $y_0 \notin Z_{a,b}$ for all $b > 0$.

On the other hand, if $y_0 \in E_c$ and if $a < c$, then Lemma 8.12 implies for all $\varepsilon > 0$ that

$$U_a(y) \le (1-\varepsilon)(c-a)\gamma \log|y - y_0| + C(\varepsilon)$$

on a neighborhood of y_0. Hence $\exp(-U_a/b)$ is non integrable at y_0 as soon as $b < (c-a)\gamma/2m$, where $m = \dim Y$. We obtain therefore

$$E_c = \bigcap_{\substack{a<c \\ b<(c-a)\gamma/2m}} Z_{a,b}.$$

This proves that E_c is an analytic subset of Y. □

Finally, we use Cor. 8.5 to derive an important decomposition formula for currents, which is again due to (Siu 1974). We first begin by two simple observations.

(8.15) Lemma. *If T is a closed positive current of bidimension (p, p) and A is an irreducible analytic set in X, we set*

$$m_A = \inf\{\nu(T, x)\,;\, x \in A\}.$$

Then $\nu(T, x) = m_A$ for all $x \in A \smallsetminus \bigcup A'_j$, where (A'_j) is a countable family of proper analytic subsets of A. We say that m_A is the generic Lelong number of T along A.

Proof. By definition of m_A and $E_c(T)$, we have $\nu(T, x) \ge m_A$ for every $x \in A$ and

$$\nu(T, x) = m_A \quad \text{on } A \smallsetminus \bigcup_{c\in\mathbb{Q},\, c>m_A} A \cap E_c(T).$$

However, for $c > m_A$, the intersection $A \cap E_c(T)$ is a proper analytic subset of A. $\square$

(8.16) Proposition. *Let T be a closed positive current of bidimension (p,p) and let A be an irreducible p-dimensional analytic subset of X. Then $\mathbb{1}_A T = m_A[A]$, in particular $T - m_A[A]$ is positive.*

Proof. As the question is local and as a closed positive current of bidimension (p,p) cannot carry any mass on a $(p-1)$-dimensional analytic subset, it is enough to work in a neighborhood of a regular point $x_0 \in A$. Hence, by choosing suitable coordinates, we can suppose that X is an open set in $\mathbb{C}^n$ and that A is the intersection of X with a p-dimensional linear subspace. Then, for every point $a \in A$, the inequality $\nu(T,a) \geq m_A$ implies

$$\sigma_T\big(B(a,r)\big) \geq m_A\, \pi^p r^{2p}/p! = m_A \sigma_{[A]}\big(B(a,r)\big)$$

for all r such that $B(a,r) \subset X$. Now, set $\Theta = T - m_A[A]$ and $\beta = dd^c|z|^2$. Our inequality says that $\int \mathbb{1}_{B(a,r)}\Theta \wedge \beta^p \geq 0$. If we integrate this with respect to some positive continuous function f with compact support in A, we get $\int_X g_r \Theta \wedge \beta^p \geq 0$ where

$$g_r(z) = \int_A \mathbb{1}_{B(a,r)}(z)\, f(a)\, d\lambda(a) = \int_{a \in A \cap B(z,r)} f(a)\, d\lambda(a).$$

Here g_r is continuous on $\mathbb{C}^n$, and as r tends to 0 the function $g_r(z)/(\pi^p r^{2p}/p!)$ converges to f on A and to 0 on $X \smallsetminus A$, with a global uniform bound. Hence we obtain $\int \mathbb{1}_A f\, \Theta \wedge \beta^p \geq 0$. Since this inequality is true for all continuous functions $f \geq 0$ with compact support in A, we conclude that the measure $\mathbb{1}_A \Theta \wedge \beta^p$ is positive. By a linear change of coordinates, we see that

$$\mathbb{1}_A \Theta \wedge \Big(dd^c \sum_{1 \leq j \leq n} \lambda_j |u_j|^2\Big)^n \geq 0$$

for every basis $(u_1, \ldots, u_n)$ of linear forms and for all coefficients $\lambda_j > 0$. Take $\lambda_1 = \ldots = \lambda_p = 1$ and let the other λ_j tend to 0. Then we get $\mathbb{1}_A \Theta \wedge i du_1 \wedge d\overline{u}_1 \wedge \ldots \wedge du_p \wedge d\overline{u}_p \geq 0$. This implies $\mathbb{1}_A \Theta \geq 0$, or equivalently $\mathbb{1}_A T \geq m_A[A]$. By Cor. 2.4 we know that $\mathbb{1}_A T$ is a closed positive current, thus $\mathbb{1}_A T = \lambda[A]$ with $\lambda \geq 0$. We have just seen that $\lambda \geq m_A$. On the other hand, $T \geq \mathbb{1}_A T = \lambda[A]$ clearly implies $m_A \geq \lambda$. $\square$

(8.16) Siu's decomposition formula. *If T is a closed positive current of bidimension (p,p), there is a unique decomposition of T as a (possibly finite) weakly convergent series*

$$T = \sum_{j \geq 1} \lambda_j [A_j] + R, \qquad \lambda_j > 0,$$

where $[A_j]$ *is the current of integration over an irreducible* p*-dimensional analytic set* $A_j \subset X$ *and where* R *is a closed positive current with the property that* $\dim E_c(R) < p$ *for every* $c > 0$.

Proof of uniqueness. If T has such a decomposition, the p-dimensional components of $E_c(T)$ are $(A_j)_{\lambda_j \geq c}$, for $\nu(T,x) = \sum \lambda_j \nu([A_j],x) + \nu(R,x)$ is non zero only on $\bigcup A_j \cup \bigcup E_c(R)$, and is equal to λ_j generically on A_j $\big($more precisely, $\nu(T,x) = \lambda_j$ at every regular point of A_j which does not belong to any intersection $A_j \cup A_k$, $k \neq j$ or to $\bigcup E_c(R)\big)$. In particular A_j and λ_j are unique.

Proof of existence. Let $(A_j)_{j\geq 1}$ be the countable collection of p-dimensional components occurring in one of the sets $E_c(T)$, $c \in \mathbb{Q}_+^\star$, and let $\lambda_j > 0$ be the generic Lelong number of T along A_j. Then Prop. 8.16 shows by induction on N that $R_N = T - \sum_{1\leq j\leq N} \lambda_j [A_j]$ is positive. As R_N is a decreasing sequence, there must be a limit $R = \lim_{N\to+\infty} R_N$ in the weak topology. Thus we have the asserted decomposition. By construction, R has zero generic Lelong number along A_j, so $\dim E_c(R) < p$ for every $c > 0$. □

It is very important to note that some components of lower dimension can actually occur in $E_c(R)$, but they cannot be subtracted because R has bidimension (p,p). A typical case is the case of a bidimension $(n-1,n-1)$ current $T = dd^c u$ with $u = \log(|F_j|^{\gamma_1} + \ldots |F_N|^{\gamma_N})$ and $F_j \in \mathcal{O}(X)$. In general $\bigcup E_c(T) = \bigcap F_j^{-1}(0)$ has dimension $< n-1$. In that case, an important formula due to King plays the role of (8.17). We state it in a somewhat more general form than its original version (King 1970).

(8.18) King's formula. *Let* $F_1, \ldots, F_N$ *be holomorphic functions on a complex manifold* X*, such that the zero variety* $Z = \bigcap F_j^{-1}(0)$ *has codimension* $\geq p$*, and set* $u = \log \sum |F_j|^{\gamma_j}$ *with arbitrary coefficients* $\gamma_j > 0$. *Let* $(Z_k)_{k\geq 1}$ *be the irreducible components of* Z *of codimension* p *exactly. Then there exist multiplicities* $\lambda_k > 0$ *such that*

$$(dd^c u)^p = \sum_{k\geq 1} \lambda_k [Z_k] + R,$$

where R *is a closed positive current such that* $\mathbb{1}_Z R = 0$ *and* $\operatorname{codim} E_c(R) > p$ *for every* $c > 0$. *Moreover the multiplicities* λ_k *are integers if* $\gamma_1, \ldots, \gamma_N$ *are integers, and* $\lambda_k = \gamma_1 \ldots \gamma_p$ *if* $\gamma_1 \leq \ldots \leq \gamma_N$ *and some partial Jacobian determinant of* $(F_1, \ldots, F_p)$ *of order* p *does not vanish identically along* Z_k.

Proof. Observe that $(dd^c u)^p$ is well defined thanks to Cor. 4.11. The comparison theorem 7.8 applied with $\varphi(z) = \log|z-x|$, $v_1 = \ldots = v_p = u$, $u_1 = \ldots = u_p = \varphi$ and $T = 1$ shows that the Lelong number of $(dd^c u)^p$ is equal to 0 at every point of $X \smallsetminus Z$. Hence $E_c((dd^c u)^p)$ is contained in

Z and its $(n-p)$-dimensional components are members of the family (Z_k). The asserted decomposition follows from Siu's formula 8.16. We must have $\mathbb{1}_{Z_k}R = 0$ for all irreducible components of Z: when $\operatorname{codim} Z_k > p$ this is automatically true, and when $\operatorname{codim} Z_k = p$ this follows from (8.16) and the fact that $\operatorname{codim} E_c(R) > p$. If $\det(\partial F_j/\partial z_k)_{1\le j,k\le p} \ne 0$ at some point $x_0 \in Z_k$, then $(Z, x_0) = (Z_k, x_0)$ is a smooth germ defined by the equations $F_1 = \ldots = F_p = 0$. If we denote $v = \log \sum_{j\le p} |F_j|^{\gamma_j}$ with $\gamma_1 \le \ldots \le \gamma_N$, then $u \sim v$ near Z_k and Th. 7.8 implies $\nu((dd^c u)^p, x) = \nu((dd^c v)^p, x)$ for all $x \in Z_k$ near x_0. On the other hand, if $G := (F_1, \ldots, F_p) : X \to \mathbb{C}^p$, Cor. 7.4 gives

$$(dd^c v)^p = G^\star\Big(dd^c \log \sum_{1\le j\le p} |z_j|^{\gamma_j}\Big)^p = \gamma_1 \ldots \gamma_p\, G^\star \delta_0 = \gamma_1 \ldots \gamma_p\, [Z_k]$$

near x_0. This implies that the generic Lelong number of $(dd^c u)^p$ along Z_k is $\lambda_k = \gamma_1 \ldots \gamma_p$. The integrality of λ_k when $\gamma_1, \ldots, \gamma_N$ are integers will be proved in the next section. □

9. Transformation of Lelong Numbers by Direct Images

Let $F : X \to Y$ be a holomorphic map between complex manifolds of respective dimensions $\dim X = n$, $\dim Y = m$, and let T be a closed positive current of bidimension (p, p) on X. If $F_{\restriction \mathrm{Supp}\, T}$ is proper, the direct image $F_\star T$ is defined by

(9.1) $\langle F_\star T, \alpha\rangle = \langle T, F^\star \alpha\rangle$

for every test form α of bidegree (p, p) on Y. This makes sense because $\mathrm{Supp}\, T \cap F^{-1}(\mathrm{Supp}\, \alpha)$ is compact. It is easily seen that $F_\star T$ is a closed positive current of bidimension (p, p) on Y.

(9.2) Example. Let $T = [A]$ where A is a p-dimensional irreducible analytic set in X such that $F_{\restriction A}$ is proper. We know by Remmert's theorem 2.7.8 that $F(A)$ is an analytic set in Y. Two cases may occur. Either $F_{\restriction A}$ is generically finite and F induces an étale covering $A \smallsetminus F^{-1}(Z) \longrightarrow F(A) \smallsetminus Z$ for some nowhere dense analytic subset $Z \subset F(A)$, or $F_{\restriction A}$ has generic fibers of positive dimension and $\dim F(A) < \dim A$. In the first case, let $s < +\infty$ be the covering degree. Then for every test form α of bidegree (p, p) on Y we get

$$\langle F_\star[A], \alpha\rangle = \int_A F^\star\alpha = \int_{A\smallsetminus F^{-1}(Z)} F^\star\alpha = s \int_{F(A)\smallsetminus Z} \alpha = s\,\langle [F(A)], \alpha\rangle$$

because Z and $F^{-1}(Z)$ are negligible sets. Hence $F_\star[A] = s[F(A)]$. On the other hand, if $\dim F(A) < \dim A = p$, the restriction of α to $F(A)_{\mathrm{reg}}$ is zero, and therefore so is this the restriction of $F^\star\alpha$ to A_{reg}. Hence $F_\star[A] = 0$. □

Now, let ψ be a continuous plurisubharmonic function on Y which is semi-exhaustive on $F(\mathrm{Supp}\ T)$ (this set certainly contains $\mathrm{Supp}\, F_\star T$). Since $F_{\restriction \mathrm{Supp}\ T}$ is proper, it follows that $\psi \circ F$ is semi-exhaustive on $\mathrm{Supp}\ T$, for

$$\mathrm{Supp}\ T \cap \{\psi \circ F < R\} = F^{-1}\big(F(\mathrm{Supp}\ T) \cap \{\psi < R\}\big).$$

(9.3) Proposition. *If $F(\mathrm{Supp}\ T) \cap \{\psi < R\} \subset\subset Y$, we have*

$$\nu(F_\star T, \psi, r) = \nu(T, \psi \circ F, r) \qquad \text{for all } r < R,$$

in particular $\nu(F_\star T, \psi) = \nu(T, \psi \circ F)$.

Here, we do not necessarily assume that X or Y are Stein; we thus replace ψ with $\psi_{\geq s} = \max\{\psi, s\}$, $s < r$, in the definition of $\nu(F_\star T, \psi, r)$ and $\nu(T, \psi \circ F, r)$.

Proof. The first equality can be written

$$\int_Y F_\star T \wedge 1\!\!1_{\{\psi < r\}} (dd^c \psi_{\geq s})^p = \int_X T \wedge (1\!\!1_{\{\psi < r\}} \circ F)(dd^c \psi_{\geq s} \circ F)^p.$$

This follows almost immediately from the adjunction formula (9.1) when ψ is smooth and when we write $1\!\!1_{\{\psi < R\}} = \lim \uparrow g_k$ for some sequence of smooth functions g_k. In general, we write $\psi_{\geq s}$ as a decreasing limit of smooth plurisubharmonic functions and we apply our monotone continuity theorems (if Y is not Stein, Richberg's theorem shows that we can obtain a decreasing sequence of almost plurisubharmonic approximations such that the negative part of dd^c converges uniformly to 0; this is good enough to apply the monotone continuity theorem; note that the integration is made on compact subsets, thanks to the semi-exhaustivity assumption on ψ). □

It follows from this that understanding the transformation of Lelong numbers under direct images is equivalent to understanding the effect of F on the weight. We are mostly interested in computing the ordinary Lelong numbers $\nu(F_\star T, y)$ associated with the weight $\psi(w) = \log|w - y|$ in some local coordinates $(w_1, \ldots, w_m)$ on Y near y. Then Prop. 9.3 gives

$$(9.4) \qquad \nu(F_\star T, y) = \nu(T, \log|F - y|) \quad \text{with}$$
$$\log|F(z) - y| = \frac{1}{2}\log\sum |F_j(z) - y_j|^2, \qquad F_j = w_j \circ F.$$

We are going to show that $\nu(T, \log|F - y|)$ is bounded below by a linear combination of the Lelong numbers of T at points x in the fiber $F^{-1}(y)$, with suitable multiplicities attached to F at these points. These multiplicities can be seen as generalizations of the notion of multiplicity of an analytic map introduced by (Stoll 1966).

(9.5) Definition. *Let $x \in X$ and $y = F(x)$. Suppose that the codimension of the fiber $F^{-1}(y)$ at x is $\geq p$. Then we set*

$$\mu_p(F, x) = \nu\big((dd^c \log |F - y|)^p, x\big).$$

Observe that $(dd^c \log |F - y|)^p$ is well defined thanks to Cor. 4.10. The second comparison theorem 7.8 immediately shows that $\mu_p(F, x)$ is independent of the choice of local coordinates on Y (and also on X, since Lelong nombers do not depend on coordinates). By definition, $\mu_p(F, x)$ is the mass carried by $\{x\}$ of the measure

$$(dd^c \log |F(z) - y|)^p \wedge (dd^c \log |z - x|)^{n-p}.$$

We are going to give a more geometric interpretation of this multiplicity, from which it will follow that $\mu_p(F, x)$ is always a positive integer (in particular, the proof of (8.18) will be complete).

(9.6) Example. For $p = n = \dim X$, the assumption $\operatorname{codim}_x F^{-1}(y) \geq p$ means that the germ of map $F : (X, x) \longrightarrow (Y, y)$ is finite. Let U_x be a neighborhood of x such that $\overline{U}_x \cap F^{-1}(y) = \{x\}$, let W_y be a neighborhood of y disjoint from $F(\partial U_x)$ and let $V_x = U_x \cap F^{-1}(W_y)$. Then $F : V_x \to W_y$ is proper and finite, and we have $F_\star[V_x] = s\,[F(V_x)]$ where s is the local covering degree of $F : V_x \to F(V_x)$ at x. Therefore

$$\begin{aligned}\mu_n(F, x) &= \int_{\{x\}} \big(dd^c \log |F - y|\big)^n = \nu\big([V_x], \log |F - y|\big) = \nu\big(F_\star[V_x], y\big)\\ &= s\,\nu\big(F(V_x), y\big).\end{aligned}$$

In the particular case when $\dim Y = \dim X$, we have $(F(V_x), y) = (Y, y)$, so $\mu_n(F, x) = s$. In general, it is a well known fact that the ideal generated by $(F_1 - y_1, \ldots, F_m - y_m)$ in $\mathcal{O}_{X,x}$ has the same integral closure as the ideal generated by n generic linear combinations of the generators, that is, for a generic choice of coordinates $w' = (w_1, \ldots, w_n)$, $w'' = (w_{n+1}, \ldots, w_m)$ on (Y, y), we have $|F(z) - y| \leq C|w' \circ F(z)|$ (this is a simple consequence of Lemma 7.5 applied to $A = F(V_x)$). Hence for $p = n$, the comparison theorem 7.1 gives

$$\mu_n(F, x) = \mu_n(w' \circ F, x) = \text{local covering degree of } w' \circ F \text{ at } x,$$

for a generic choice of coordinates (w', w'') on (Y, y). □

(9.7) Geometric interpretation of $\mu_p(F, x)$. An application of Crofton's formula 7.11 shows, after a translation, that there is a small ball $B(x, r_0)$ on which

$$(dd^c \log|F(z)-y|)^p \wedge (dd^c \log|z-x|)^{n-p} =$$

$$\int_{S\in G(p,n)} (dd^c \log|F(z)-y|)^p \wedge [x+S]\,dv(S). \tag{9.7 a}$$

For a rigorous proof of (9.7 a), we replace $\log|F(z)-y|$ by the smooth function $\frac{1}{2}\log(|F(z)-y|^2+\varepsilon^2)$ and let ε tend to 0 on both sides. By (4.3) (resp. by (4.10)), the wedge product $(dd^c\log|F(z)-y|)^p \wedge [x+S]$ is well defined on a small ball $B(x,r_0)$ as soon as $x+S$ does not intersect $F^{-1}(y)\cap \partial B(x,r_0)$ (resp. intersects $F^{-1}(y)\cap B(x,r_0)$ at finitely many points); thanks to the assumption $\operatorname{codim}(F^{-1}(y),x)\ge p$, Sard's theorem shows that this is the case for all S outside a negligible closed subset E in $G(p,n)$ (resp. by Bertini, an analytic subset A in $G(p,n)$ with $A\subset E$). Fatou's lemma then implies that the inequality $\ge$ holds in (9.7 a). To get equality, we observe that we have bounded convergence on all complements $G(p,n)\smallsetminus V(E)$ of neighborhoods $V(E)$ of E. However the mass of $\int_{V(E)}[x+S]\,dv(S)$ in $B(x,r_0)$ is proportional to $v(V(E))$ and therefore tends to 0 when $V(E)$ is small; this is sufficient to complete the proof, since Prop. 4.6 b) gives

$$\int_{z\in\overline{B}(x,r_0)} \big(dd^c\log(|F(z)-y|^2+\varepsilon^2)\big)^p \wedge \int_{S\in V(E)} [x+S]\,dv(S) \le C\,v(V(E))$$

with a constant C independent of ε. By evaluating (9.7 a) on $\{x\}$, we get

$$\mu_p(F,x) = \int_{S\in G(p,n)\smallsetminus A} \nu\big((dd^c\log|F_{\restriction x+S}-z|)^p, x\big)\,dv(S). \tag{9.7 b}$$

Let us choose a linear parametrization $g_S:\mathbb{C}^p\to S$ depending analytically on local coordinates of S in $G(p,n)$. Then Theorem 8.4 with $T=[\mathbb{C}^p]$ and $\varphi(z,S)=\log|F\circ g_S(z)-y|$ shows that

$$\nu\big((dd^c\log|F_{\restriction x+S}-z|)^p,x\big) = \nu\big([\mathbb{C}^p],\log|F\circ g_S(z)-y|\big)$$

is Zariski upper semicontinuous in S on $G(p,n)\smallsetminus A$. However, (9.6) shows that these numbers are integers, so $S\mapsto \nu\big((dd^c\log|F_{\restriction x+S}-z|)^p,x\big)$ must be constant on a Zariski open subset in $G(p,n)$. By (9.7 b), we obtain

$$\mu_p(F,x)=\mu_p(F_{\restriction x+S},x) = \text{local degree of } w'\circ F_{\restriction x+S} \ \text{ at } x \tag{9.7 c}$$

for generic subspaces $S\in G(p,n)$ and generic coordinates $w'=(w_1,\ldots,w_p)$, $w''=(w_{p+1},\ldots,w_m)$ on (Y,y). □

(9.8) Example. Let $F:\mathbb{C}^n \longrightarrow \mathbb{C}^n$ be defined by

$$F(z_1,\ldots,z_n)=(z_1^{s_1},\ldots,z_n^{s_n}),\qquad s_1\le\ldots\le s_n.$$

We claim that $\mu_p(F,0)=s_1\ldots s_p$. In fact, for a generic p-dimensional subspace $S\subset\mathbb{C}^n$ such that $z_1,\ldots,z_p$ are coordinates on S and $z_{p+1},\ldots,z_n$ are linear forms in $z_1,\ldots,z_p$, and for generic coordinates $w'=(w_1,\ldots,w_p)$,

$w'' = (w_{p+1}, \ldots, w_n)$ on $\mathbb{C}^n$, we can rearrange w' by linear combinations so that $w_j \circ F_{\restriction S}$ is a linear combination of $(z_j^{s_j}, \ldots, z_n^{s_n})$ and has non zero coefficient in $z_j^{s_j}$ as a polynomial in $(z_j, \ldots, z_p)$. It is then an exercise to show that $w' \circ F_{\restriction S}$ has covering degree $s_1 \ldots s_p$ at 0 [compute inductively the roots z_n, $z_{n-1}, \ldots, z_j$ of $w_j \circ F_{\restriction S}(z) = a_j$ and use Lemma II.3.10 to show that the s_j values of z_j lie near 0 when $(a_1, \ldots, a_p)$ are small]. □

We are now ready to prove the main result of this section, which describes the behaviour of Lelong numbers under proper morphisms. A similar weaker result was already proved in (Demailly 1982b) with some other non optimal multiplicities $\mu_p(F, x)$.

(9.9) Theorem. *Let T be a closed positive current of bidimension (p,p) on X and let $F : X \longrightarrow Y$ be an analytic map such that the restriction $F_{\restriction \mathrm{Supp}\, T}$ is proper. Let $I(y)$ be the set of points $x \in \mathrm{Supp}\, T \cap F^{-1}(y)$ such that x is equal to its connected component in* Supp $T \cap F^{-1}(y)$ *and* $\mathrm{codim}(F^{-1}(y), x) \geq p$. *Then we have*

$$\nu(F_\star T, y) \geq \sum_{x \in I(y)} \mu_p(F, x)\, \nu(T, x).$$

In particular, we have $\nu(F_\star T, y) \geq \sum_{x \in I(y)} \nu(T, x)$. This inequality no longer holds if the summation is extended to all points $x \in \mathrm{Supp}\, T \cap F^{-1}(Y)$ and if this set contains positive dimensional connected components: for example, if $F : X \longrightarrow Y$ contracts some exceptional subspace E in X to a point y_0 (e.g. if F is a blow-up map, see § 7.12), then $T = [E]$ has direct image $F_\star[E] = 0$ thanks to (9.2).

Proof. We proceed in three steps.

Step 1. Reduction to the case of a single point x in the fiber. It is sufficient to prove the inequality when the summation is taken over an arbitrary finite subset $\{x_1, \ldots, x_N\}$ of $I(y)$. As x_j is equal to its connected component in Supp $T \cap F^{-1}(y)$, it has a fondamental system of relative open-closed neighborhoods, hence there are disjoint neighborhoods U_j of x_j such that ∂U_j does not intersect Supp $T \cap F^{-1}(y)$. Then the image $F(\partial U_j \cap \mathrm{Supp}\, T)$ is a closed set which does not contain y. Let W be a neighborhood of y disjoint from all sets $F(\partial U_j \cap \mathrm{Supp}\, T)$, and let $V_j = U_j \cap F^{-1}(W)$. It is clear that V_j is a neighborhood of x_j and that $F_{\restriction V_j} : V_j \to W$ has a proper restriction to Supp $T \cap V_j$. Moreover, we obviously have $F_\star T \geq \sum_j (F_{\restriction V_j})_\star T$ on W. Therefore, it is enough to check the inequality $\nu(F_\star T, y) \geq \mu_p(F, x)\, \nu(T, x)$ for a single point $x \in I(y)$, in the case when $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$ are open subsets and $x = y = 0$.

Step 2. Reduction to the case when F is finite. By (9.4), we have

$$\begin{aligned}\nu(F_\star T, 0) &= \inf_{V\ni 0}\int_V T\wedge (dd^c\log|F|)^p \\ &= \inf_{V\ni 0}\lim_{\varepsilon\to 0}\int_V T\wedge \big(dd^c\log(|F|+\varepsilon|z|^N)\big)^p,\end{aligned}$$

and the integrals are well defined as soon as ∂V does not intersect the set $\operatorname{Supp} T\cap F^{-1}(0)$ (may be after replacing $\log|F|$ by $\max\{\log|F|, s\}$ with $s\ll 0$). For every V and ε, the last integral is larger than $\nu(G_\star T, 0)$ where G is the finite morphism defined by

$$G: X\longrightarrow Y\times\mathbb{C}^n,\qquad (z_1,\ldots,z_n)\longmapsto (F_1(z),\ldots,F_m(z), z_1^N,\ldots,z_n^N).$$

We claim that for N large enough we have $\mu_p(F,0)=\mu_p(G,0)$. In fact, $x\in I(y)$ implies by definition $\operatorname{codim}(F^{-1}(0),0)\ge p$. Hence, if $S=\{u_1=\ldots=u_{n-p}=0\}$ is a generic p-dimensional subspace of $\mathbb{C}^n$, the germ of variety $F^{-1}(0)\cap S$ defined by $(F_1,\ldots,F_m,u_1,\ldots,u_{n-p})$ is $\{0\}$. Hilbert's Nullstellensatz implies that some powers of $z_1,\ldots,z_n$ are in the ideal (F_j,u_k). Therefore $|F(z)|+|u(z)|\ge C|z|^a$ near 0 for some integer a independent of S (to see this, take coefficients of the u_k's as additional variables); in particular $|F(z)|\ge C|z|^a$ for $z\in S$ near 0. The comparison theorem 7.1 then shows that $\mu_p(F,0)=\mu_p(G,0)$ for $N\ge a$. If we are able to prove that $\nu(G_\star T,0)\ge\mu_p(G,0)\nu(T,0)$ in case G is finite, the obvious inequality $\nu(F_\star T,0)\ge\nu(G_\star T,0)$ concludes the proof.

Step 3. Proof of the inequality $\nu(F_\star T,y)\ge\mu_p(F,x)\,\nu(T,x)$ when F is finite and $F^{-1}(y)=x$. Then $\varphi(z)=\log|F(z)-y|$ has a single isolated pole at x and we have $\mu_p(F,x)=\nu((dd^c\varphi)^p,x)$. It is therefore sufficient to apply to following Proposition.

(9.10) Proposition. *Let φ be a semi-exhaustive continuous plurisubharmonic function on X with a single isolated pole at x. Then*

$$\nu(T,\varphi)\ge\nu(T,x)\,\nu((dd^c\varphi)^p,x).$$

Proof. Since the question is local, we can suppose that X is the ball $B(0,r_0)$ in $\mathbb{C}^n$ and $x=0$. Set $X'=B(0,r_1)$ with $r_1<r_0$ and $\Phi(z,g)=\varphi\circ g(z)$ for $g\in\mathrm{Gl}_n(\mathbb{C})$. Then there is a small neighborhood Ω of the unitary group $U(n)\subset\mathrm{Gl}_n(\mathbb{C})$ such that Φ is plurisubharmonic on $X'\times\Omega$ and semi-exhaustive with respect to X'. Theorem 8.4 implies that the map $g\mapsto\nu(T,\varphi\circ g)$ is Zariski upper semi-continuous on Ω. In particular, we must have $\nu(T,\varphi\circ g)\le\nu(T,\varphi)$ for all $g\in\Omega\smallsetminus A$ in the complement of a complex analytic set A. Since $\mathrm{Gl}_n(\mathbb{C})$ is the complexification of $U(n)$, the intersection $U(n)\cap A$ must be a nowhere dense real analytic subset of $U(n)$. Therefore, if dv is the Haar measure of mass 1 on $U(n)$, we have

$$\nu(T,\varphi) \geq \int_{g\in U(n)} \nu(T,\varphi\circ g)\,dv(g)$$

$$(9.11) \qquad = \lim_{r\to 0}\int_{g\in U(n)} dv(g)\int_{B(0,r)} T\wedge (dd^c\varphi\circ g)^p.$$

Since $\int_{g\in U(n)}(dd^c\varphi\circ g)^p dv(g)$ is a unitary invariant (p,p)-form on B, Lemma 7.10 implies

$$\int_{g\in U(n)} (dd^c\varphi\circ g)^p dv(g) = \big(dd^c\chi(\log|z|)\big)^p$$

where χ is a convex increasing function. The Lelong number at 0 of the left hand side is equal to $\nu((dd^c\varphi)^p,0)$, and must be equal to the Lelong number of the right hand side, which is $\lim_{t\to-\infty}\chi'(t)^p$ (to see this, use either Formula (5.5) or Th. 7.8). Thanks to the last equality, Formulas (9.11) and (5.5) imply

$$\begin{aligned}\nu(T,\varphi) &\geq \lim_{r\to 0}\int_{B(0,r)} T\wedge\big(dd^c\chi(\log|z|)\big)^p\\ &= \lim_{r\to 0}\chi'(\log r-0)^p\nu(T,0,r)\geq \nu((dd^c\varphi)^p,0)\,\nu(T,0).\end{aligned} \qquad \square$$

Another interesting question is to know whether it is possible to get inequalities in the opposite direction, i.e. to find upper bounds for $\nu(F_\star T,y)$ in terms of the Lelong numbers $\nu(T,x)$. The example $T=[\varGamma]$ with the curve $\varGamma: t\mapsto (t^a,t^{a+1},t)$ in $\mathbb{C}^3$ and $F:\mathbb{C}^3\to\mathbb{C}^2$, $(z_1,z_2,z_3)\mapsto(z_1,z_2)$, for which $\nu(T,0)=1$ and $\nu(F_\star T,0)=a$, shows that this may be possible only when F is finite. In this case, we have:

(9.12) Theorem. *Let $F:X\to Y$ be a proper and finite analytic map and let T be a closed positive current of bidimension (p,p) on X. Then*

$$\text{(a)}\quad \nu(F_\star T,y)\leq \sum_{x\in \mathrm{Supp}\ T\cap F^{-1}(y)} \overline{\mu}_p(F,x)\,\nu(T,x)$$

where $\overline{\mu}_p(F,x)$ is the multiplicity defined as follows: if $H:(X,x)\to(\mathbb{C}^n,0)$ is a germ of finite map, we set

$$\text{(b)}\qquad \sigma(H,x)=\inf\big\{\alpha>0\,;\,\exists C>0,\ |H(z)|\geq C|z-x|^\alpha \text{ near } x\big\},$$

$$\text{(c)}\qquad \overline{\mu}_p(F,x)=\inf_G\frac{\sigma(G\circ F,x)^p}{\mu_p(G,0)},$$

where G runs over all germs of maps $(Y,y)\longrightarrow(\mathbb{C}^n,0)$ such that $G\circ F$ is finite.

Proof. If $F^{-1}(y)=\{x_1,\ldots,x_N\}$, there is a neighborhood W of y and disjoint neighborhoods V_j of x_j such that $F^{-1}(W)=\bigcup V_j$. Then $F_\star T=\sum (F_{\restriction V_j})_\star T$

on W, so it is enough to consider the case when $F^{-1}(y)$ consists of a single point x. Therefore, we assume that $F : V \to W$ is proper and finite, where V, W are neighborhoods of 0 in $\mathbb{C}^n$, $\mathbb{C}^m$ and $F^{-1}(0) = \{0\}$. Let $G : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^n, 0)$ be a germ of map such that $G \circ F$ is finite. Hilbert's Nullstellensatz shows that there exists $\alpha > 0$ and $C > 0$ such that $|G \circ F(z)| \geq C|z|^\alpha$ near 0. Then the comparison theorem 7.1 implies

$$\nu(G_\star F_\star T, 0) = \nu(T, \log|G \circ F|) \leq \alpha^p \nu(T, \log|z|) = \alpha^p \nu(T, 0).$$

On the other hand, Th. 9.9 applied to $\Theta = F_\star T$ on W gives

$$\nu(G_\star F_\star T, 0) \geq \mu_p(G, 0)\, \nu(F_\star T, 0).$$

Therefore

$$\nu(F_\star T, 0) \leq \frac{\alpha^p}{\mu_p(G, 0)} \nu(T, 0).$$

The infimum of all possible values of α is by definition $\sigma(G \circ F, 0)$, thus by taking the infimum over G we obtain

$$\nu(F_\star T, 0) \leq \overline{\mu}_p(F, 0)\, \nu(T, 0). \qquad \square$$

(9.13) Example. Let $F(z_1, \ldots, z_n) = (z_1^{s_1}, \ldots, z_n^{s_n})$, $s_1 \leq \ldots \leq s_n$ as in 9.8. Then we have

$$\mu_p(F, 0) = s_1 \ldots s_p, \qquad \overline{\mu}_p(F, 0) = s_{n-p+1} \ldots s_n.$$

To see this, let s be the lowest common multiple of $s_1, \ldots, s_n$ and let $G(z_1, \ldots, z_n) = (z_1^{s/s_1}, \ldots, z_n^{s/s_n})$. Clearly $\mu_p(G, 0) = (s/s_{n-p+1}) \ldots (s/s_n)$ and $\sigma(G \circ F, 0) = s$, so we get by definition $\overline{\mu}_p(F, 0) \leq s_{n-p+1} \ldots s_n$. Finally, if $T = [A]$ is the current of integration over the p-dimensional subspace $A = \{z_1 = \ldots = z_{n-p} = 0\}$, then $F_\star[A] = s_{n-p+1} \ldots s_n\, [A]$ because $F_{\restriction A}$ has covering degree $s_{n-p+1} \ldots s_n$. Theorem 9.12 shows that we must have $s_{n-p+1} \ldots s_n \leq \overline{\mu}_p(F, 0)$, QED. If $\lambda_1 \leq \ldots \leq \lambda_n$ are positive real numbers and s_j is taken to be the integer part of $k\lambda_j$ as k tends to $+\infty$, Theorems 9.9 and 9.12 imply in the limit the following:

(9.14) Corollary. *For $0 < \lambda_1 \leq \ldots \leq \lambda_n$, Kiselman's directional Lelong numbers satisfy the inequalities*

$$\lambda_1 \ldots \lambda_p\, \nu(T, x) \leq \nu(T, x, \lambda) \leq \lambda_{n-p+1} \ldots \lambda_n\, \nu(T, x). \qquad \square$$

(9.15) Remark. It would be interesting to have a direct geometric interpretation of $\overline{\mu}_p(F, x)$. In fact, we do not even know whether $\overline{\mu}_p(F, x)$ is always an integer.

10. A Schwarz Lemma. Application to Number Theory

In this section, we show how Jensen's formula and Lelong numbers can be used to prove a fairly general Schwarz lemma relating growth and zeros of entire functions in $\mathbb{C}^n$. In order to simplify notations, we denote by $|F|_r$ the supremum of the modulus of a function F on the ball of center 0 and radius r. Then, following (Demailly 1982a), we present some applications with a more arithmetical flavour.

(10.1) Schwarz lemma. *Let $P_1, \ldots, P_N \in \mathbb{C}[z_1, \ldots, z_n]$ be polynomials of degree δ, such that their homogeneous parts of degree δ do not vanish simultaneously except at* 0. *Then there is a constant $C \geq 2$ such that for all entire functions $F \in \mathcal{O}(\mathbb{C}^n)$ and all $R \geq r \geq 1$ we have*

$$\log |F|_r \leq \log |F|_R - \delta^{1-n} \nu([Z_F], \log |P|) \log \frac{R}{Cr}$$

where Z_F is the zero divisor of F and $P = (P_1, \ldots, P_N) : \mathbb{C}^n \longrightarrow \mathbb{C}^N$. Moreover

$$\nu([Z_F], \log |P|) \geq \sum_{w \in P^{-1}(0)} \mathrm{ord}(F, w)\, \mu_{n-1}(P, w)$$

where $\mathrm{ord}(F, w)$ *denotes the vanishing order of F at w and $\mu_{n-1}(P, w)$ is the $(n-1)$-multiplicity of P at w, as defined in* (9.5) *and* (9.7).

Proof. Our assumptions imply that P is a proper and finite map. The last inequality is then just a formal consequence of formula (9.4) and Th. 9.9 applied to $T = [Z_F]$. Let Q_j be the homogeneous part of degree δ in P_j. For $z_0 \in B(0, r)$, we introduce the weight functions

$$\varphi(z) = \log |P(z)|, \qquad \psi(z) = \log |Q(z - z_0)|.$$

Since $Q^{-1}(0) = \{0\}$ by hypothesis, the homogeneity of Q shows that there are constants $C_1, C_2 > 0$ such that

$$(10.2) \quad C_1 |z|^\delta \leq |Q(z)| \leq C_2 |z|^\delta \quad \text{on} \quad \mathbb{C}^n.$$

The homogeneity also implies $(dd^c \psi)^n = \delta^n \, \delta_{z_0}$. We apply the Lelong Jensen formula 6.5 to the measures $\mu_{\psi,s}$ associated with ψ and to $V = \log |F|$. This gives

$$(10.3) \quad \mu_{\psi,s}(\log |F|) - \delta^n \log |F(z_0)| = \int_{-\infty}^{s} dt \int_{\{\psi < t\}} [Z_F] \wedge (dd^c \psi)^{n-1}.$$

By (6.2), $\mu_{\psi,s}$ has total mass δ^n and has support in

$$\{\psi(z) = s\} = \{Q(z - z_0) = e^s\} \subset B\big(0, r + (e^s / C_1)^{1/\delta}\big).$$

Note that the inequality in the Schwarz lemma is obvious if $R \le Cr$, so we can assume $R \ge Cr \ge 2r$. We take $s = \delta \log(R/2) + \log C_1$; then

$$\{\psi(z) = s\} \subset B(0, r + R/2) \subset B(0, R).$$

In particular, we get $\mu_{\psi,s}(\log|F|) \le \delta^n \log|F|_R$ and formula (10.3) gives

$$\text{(10.4)} \quad \log|F|_R - \log|F(z_0)| \ge \delta^{-n} \int_{s_0}^{s} dt \int_{\{\psi<t\}} [Z_F] \wedge (dd^c\psi)^{n-1}$$

for any real number $s_0 < s$. The proof will be complete if we are able to compare the integral in (10.4) to the corresponding integral with φ in place of ψ. The argument for this is quite similar to the proof of the comparison theorem, if we observe that $\psi \sim \varphi$ at infinity. We introduce the auxiliary function

$$w = \begin{cases} \max\{\psi, (1-\varepsilon)\varphi + \varepsilon t - \varepsilon\} & \text{on } \{\psi \ge t-2\}, \\ (1-\varepsilon)\varphi + \varepsilon t - \varepsilon & \text{on } \{\psi \le t-2\}, \end{cases}$$

with a constant ε to be determined later, such that $(1-\varepsilon)\varphi + \varepsilon t - \varepsilon > \psi$ near $\{\psi = t-2\}$ and $(1-\varepsilon)\varphi + \varepsilon t - \varepsilon < \psi$ near $\{\psi = t\}$. Then Stokes' theorem implies

$$\int_{\{\psi<t\}} [Z_F] \wedge (dd^c\psi)^{n-1} = \int_{\{\psi<t\}} [Z_F] \wedge (dd^c w)^{n-1}$$

$$\text{(10.5)} \qquad \ge (1-\varepsilon)^{n-1} \int_{\{\psi<t-2\}} [Z_F] \wedge (dd^c\varphi)^{n-1} \ge (1-\varepsilon)^{n-1} \nu([Z_F], \log|P|).$$

By (10.2) and our hypothesis $|z_0| < r$, the condition $\psi(z) = t$ implies

$$|Q(z-z_0)| = e^t \implies e^{t/\delta}/C_1^{1/\delta} \le |z - z_0| \le e^{t/\delta}/C_2^{1/\delta},$$
$$|P(z) - Q(z-z_0)| \le C_3(1+|z_0|)(1+|z|+|z_0|)^{\delta-1} \le C_4 r(r + e^{t/\delta})^{\delta-1},$$
$$\left|\frac{P(z)}{Q(z-z_0)} - 1\right| \le C_4 r e^{-t/\delta}(re^{-t/\delta}+1)^{\delta-1} \le 2^{\delta-1} C_4 r e^{-t/\delta},$$

provided that $t \ge \delta \log r$. Hence for $\psi(z) = t \ge s_0 \ge \delta \log(2^\delta C_4 r)$, we get

$$|\varphi(z) - \psi(z)| = \left|\log \frac{|P(z)|}{|Q(z-z_0)|}\right| \le C_5 r e^{-t/\delta}.$$

Now, we have

$$\big[(1-\varepsilon)\varphi + \varepsilon t - \varepsilon\big] - \psi = (1-\varepsilon)(\varphi - \psi) + \varepsilon(t - 1 - \psi),$$

so this difference is $< C_5 r e^{-t/\delta} - \varepsilon$ on $\{\psi = t\}$ and $> -C_5 r e^{(2-t)/\delta} + \varepsilon$ on $\{\psi = t-2\}$. Hence it is sufficient to take $\varepsilon = C_5 r e^{(2-t)/\delta}$. This number has to be < 1, so we take $t \ge s_0 \ge 2 + \delta \log(C_5 r)$. Moreover, (10.5) actually holds only if $P^{-1}(0) \subset \{\psi < t-2\}$, so by (10.2) it is enough to take $t \ge s_0 \ge$

$2 + \log(C_2(r + C_6)^\delta)$ where C_6 is such that $P^{-1}(0) \subset \overline{B}(0, C_6)$. Finally, we see that we can choose

$$s = \delta \log R - C_7, \qquad s_0 = \delta \log r + C_8,$$

and inequalities (10.4), (10.5) together imply

$$\log |F|_R - \log |F(z_0)| \geq \delta^{-n} \Big(\int_{s_0}^{s} (1 - C_5 r e^{(2-t)/\delta})^{n-1}\, dt \Big) \nu([Z_F], \log |P|).$$

The integral is bounded below by

$$\int_{C_8}^{\delta \log(R/r) - C_7} (1 - C_9 e^{-t/\delta})\, dt \geq \delta \log(R/Cr).$$

This concludes the proof, by taking the infimum when z_0 runs over $B(0, r)$. □

(10.6) Corollary. *Let S be a finite subset of $\mathbb{C}^n$ and let δ be the minimal degree of algebraic hypersurfaces containing S. Then there is a constant $C \geq 2$ such that for all $F \in \mathcal{O}(\mathbb{C}^n)$ and all $R \geq r \geq 1$ we have*

$$\log |F|_r \leq \log |F|_R - \mathrm{ord}(F, S) \frac{\delta + n(n-1)/2}{n!} \log \frac{R}{Cr}$$

where $\mathrm{ord}(F, S) = \min_{w \in S} \mathrm{ord}(F, w)$.

Proof. In view of Th. 10.1, we only have to select suitable polynomials $P_1, \ldots, P_N$. The vector space $\mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ of polynomials of degree $< \delta$ in $\mathbb{C}^n$ has dimension

$$m(\delta) = \binom{\delta + n - 1}{n} = \frac{\delta(\delta + 1) \ldots (\delta + n - 1)}{n!}.$$

By definition of δ, the linear forms

$$\mathbb{C}[z_1, \ldots, z_n]_{<\delta} \longrightarrow \mathbb{C}, \qquad P \longmapsto P(w), \quad w \in S$$

vanish simultaneously only when $P = 0$. Hence we can find $m = m(\delta)$ points $w_1, \ldots, w_m \in S$ such that the linear forms $P \mapsto P(w_j)$ define a basis of $\mathbb{C}[z_1, \ldots, z_n]^\star_{<\delta}$. This means that there is a unique polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ which takes given values $P(w_j)$ for $1 \leq j \leq m$. In particular, for every multiindex α, $|\alpha| = \delta$, there is a unique polynomial $R_\alpha \in \mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ such that $R_\alpha(w_j) = w_j^\alpha$. Then the polynomials $P_\alpha(z) = z^\alpha - R_\alpha(z)$ have degree δ, vanish at all points w_j and their homogeneous parts of maximum degree $Q_\alpha(z) = z^\alpha$ do not vanish simultaneously except at 0. We simply use the fact that $\mu_{n-1}(P, w_j) \geq 1$ to get

$$\nu([Z_F], \log |P|) \geq \sum_{w \in P^{-1}(0)} \mathrm{ord}(F, w) \geq m(\delta)\, \mathrm{ord}(F, S).$$

Theorem 10.1 then gives the desired inequality, because $m(\delta)$ is a polynomial with positive coefficients and with leading terms

$$\frac{1}{n!}\big(\delta^n + n(n-1)/2\,\delta^{n-1} + \ldots\big). \qquad \square$$

Let S be a finite subset of $\mathbb{C}^n$. According to (Waldschmidt 1976), we introduce for every integer $t > 0$ a number $\omega_t(S)$ equal to the minimal degree of polynomials $P \in \mathbb{C}[z_1, \ldots, z_n]$ which vanish at order $\geq t$ at every point of S. The obvious subadditivity property

$$\omega_{t_1+t_2}(S) \leq \omega_{t_1}(S) + \omega_{t_2}(S)$$

easily shows that

$$\Omega(S) := \inf_{t>0} \frac{\omega_t(S)}{t} = \lim_{t\to+\infty} \frac{\omega_t(S)}{t}.$$

We call $\omega_1(S)$ the *degree* of S (minimal degree of algebraic hypersurfaces containing S) and $\Omega(S)$ the *singular degree* of S. If we apply Cor. 10.6 to a polynomial F vanishing at order t on S and fix $r = 1$, we get

$$\log|F|_R \geq t\frac{\delta + n(n-1)/2}{n!}\log\frac{R}{C} + \log|F|_1$$

with $\delta = \omega_1(S)$, in particular

$$\deg F \geq t\frac{\omega_1(S) + n(n-1)/2}{n!}.$$

The minimum of $\deg F$ over all such F is by definition $\omega_t(S)$. If we divide by t and take the infimum over t, we get the interesting inequality

$$(10.7) \qquad \frac{\omega_t(S)}{t} \geq \Omega(S) \geq \frac{\omega_1(S) + n(n-1)/2}{n!}.$$

(10.8) Remark. The constant $\frac{\omega_1(S)+n(n-1)/2}{n!}$ in (10.6) and (10.7) is optimal for $n = 1, 2$ but not for $n \geq 3$. It can be shown by means of Hörmander's L^2 estimates (Waldschmidt 1978) that for every $\varepsilon > 0$ the Schwarz lemma (10.6) holds with coefficient $\Omega(S) - \varepsilon$:

$$\log|F|_r \leq \log|F|_R - \mathrm{ord}(F, S)(\Omega(S) - \varepsilon)\log\frac{R}{C_\varepsilon r},$$

and that $\Omega(S) \geq (\omega_u(S)+1)/(u+n-1)$ for every $u \geq 1$; this last inequality is due to (Esnault-Viehweg 1983), who used deep tools of algebraic geometry; (Azhari 1990) reproved it recently by means of Hörmander's L^2 estimates. Rather simple examples (Demailly 1982a) lead to the conjecture

$$\Omega(S) \geq \frac{\omega_u(S) + n - 1}{u + n - 1} \quad \text{for every} \quad u \geq 1.$$

The special case $u = 1$ of the conjecture was first stated by (Chudnovsky 1979).

Finally, let us mention that Cor. 10.6 contains Bombieri's theorem on algebraic values of meromorphic maps satisfying algebraic differential equations (Bombieri 1970). Recall that an entire function $F \in \mathcal{O}(\mathbb{C}^n)$ is said to be of order $\leq \rho$ if for every $\varepsilon > 0$ there is a constant C_ε such that $|F(z)| \leq C_\varepsilon \exp(|z|^{\rho+\varepsilon})$. A meromorphic function is said to be of order $\leq \rho$ if it can be written G/H where G, H are entire functions of order $\leq \rho$.

(10.9) Theorem (Bombieri 1970). *Let $F_1, \ldots, F_N$ be meromorphic functions on $\mathbb{C}^n$, such that $F_1, \ldots, F_d$, $n < d \leq N$, are algebraically independent over $\mathbb{Q}$ and have finite orders $\rho_1, \ldots, \rho_d$. Let K be a number field of degree $[K : \mathbb{Q}]$. Suppose that the ring $K[f_1, \ldots, f_N]$ is stable under all derivations $d/dz_1, \ldots, d/dz_n$. Then the set S of points $z \in \mathbb{C}^n$, distinct from the poles of the F_j's, such that $(F_1(z), \ldots, F_N(z)) \in K^N$ is contained in an algebraic hypersurface whose degree δ satisfies*

$$\frac{\delta + n(n-1)/2}{n!} \leq \frac{\rho_1 + \ldots + \rho_d}{d - n}[K : \mathbb{Q}].$$

Proof. If the set S is not contained in any algebraic hypersurface of degree $< \delta$, the linear algebra argument used in the proof of Cor. 10.6 shows that we can find $m = m(\delta)$ points $w_1, \ldots, w_m \in S$ which are not located on any algebraic hypersurface of degree $< \delta$. Let $H_1, \ldots, H_d$ be the denominators of $F_1, \ldots, F_d$. The standard arithmetical methods of transcendental number theory allow us to construct a sequence of entire functions in the following way: we set

$$G = P(F_1, \ldots, F_d)(H_1 \ldots H_d)^s$$

where P is a polynomial of degree $\leq s$ in each variable with integer coefficients. The polynomials P are chosen so that G vanishes at a very high order at each point w_j. This amounts to solving a linear system whose unknowns are the coefficients of P and whose coefficients are polynomials in the derivatives of the F_j's (hence lying in the number field K). Careful estimates of size and denominators and a use of the Dirichlet-Siegel box principle lead to the following lemma, see e.g. (Waldschmidt 1978).

(10.10) Lemma. *For every $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$, $r \geq 1$ and an infinite sequence G_t of entire functions, $t \in T \subset \mathbb{N}$ (depending on m and on the choice of the points w_j), such that*

a) *G_t vanishes at order $\geq t$ at all points $w_1, \ldots, w_m$;*

b) $|G_t|_r \geq (C_1 t)^{-t\,[K:\mathbb{Q}]}$;
c) $|G_t|_{R(t)} \leq C_2^t$ *where* $R(t) = (t^{d-n}/\log t)^{1/(\rho_1+\ldots+\rho_d+\varepsilon)}$.

An application of Cor. 10.6 to $F = G_t$ and $R = R(t)$ gives the desired bound for the degree δ as t tends to $+\infty$ and ε tends to 0. If δ_0 is the largest integer which satisfies the inequality of Th. 10.9, we get a contradiction if we take $\delta = \delta_0 + 1$. This shows that S must be contained in an algebraic hypersurface of degree $\delta \leq \delta_0$. □

Chapter IV
Sheaf Cohomology and Spectral Sequences

One of the main topics of this book is the computation of various cohomology groups arising in algebraic geometry. The theory of sheaves provides a general framework in which many cohomology theories can be treated in a unified way. The cohomology theory of sheaves will be constructed here by means of Godement's simplicial flabby resolution. However, we have emphasized the analogy with Alexander-Spanier cochains in order to give a simple definition of the cup product. In this way, all the basic properties of cohomology groups (long exact sequences, Mayer Vietoris exact sequence, Leray's theorem, relations with Cech cohomology, De Rham-Weil isomorphism theorem) can be derived in a very elementary way from the definitions. Spectral sequences and hypercohomology groups are then introduced, with two principal examples in view: the Leray spectral sequence and the Hodge-Frölicher spectral sequence. The basic results concerning cohomology groups with constant or locally constant coefficients (invariance by homotopy, Poincaré duality, Leray-Hirsch theorem) are also included, in order to present a self-contained approach of algebraic topology.

1. Basic Results of Homological Algebra

Let us first recall briefly some standard notations and results of homological algebra that will be used systematically in the sequel. Let R be a commutative ring with unit. A *differential module* (K,d) is a R-module K together with an endomorphism $d : K \to K$, called the *differential*, such that $d \circ d = 0$. The modules of *cycles* and of *boundaries* of K are defined respectively by

(1.1) $\quad Z(K) = \ker d, \qquad B(K) = \operatorname{Im} d.$

Our hypothesis $d \circ d = 0$ implies $B(K) \subset Z(K)$. The *homology group* of K is by definition the quotient module

(1.2) $\quad H(K) = Z(K)/B(K).$

A *morphism of differential modules* $\varphi : K \longrightarrow L$ is a R-homomorphism $\varphi : K \longrightarrow L$ such that $d \circ \varphi = \varphi \circ d$; here we denote by the same symbol d the differentials of K and L. It is then clear that $\varphi\big(Z(K)\big) \subset Z(L)$ and $\varphi\big(B(K)\big) \subset B(L)$. Therefore, we get an induced morphism on homology groups, denoted

(1.3) $H(\varphi) : H(K) \longrightarrow H(L)$.

It is easily seen that H is a functor, i.e. $H(\psi \circ \varphi) = H(\psi) \circ H(\varphi)$. We say that two morphisms $\varphi, \psi : K \longrightarrow L$ are *homotopic* if there exists a R-linear map $h : K \longrightarrow L$ such that

(1.4) $d \circ h + h \circ d = \psi - \varphi$.

Then h is said to be a *homotopy* between φ and ψ. For every cocycle $z \in Z(K)$, we infer $\psi(z) - \varphi(z) = dh(z)$, hence the maps $H(\varphi)$ and $H(\psi)$ coincide. The module K itself is said to be homotopic to 0 if Id_K is homotopic to 0; then $H(K) = 0$.

(1.5) Snake lemma. *Let*

$$0 \longrightarrow K \xrightarrow{\varphi} L \xrightarrow{\psi} M \longrightarrow 0$$

be a short exact sequence of morphisms of differential modules. Then there exists a homomorphism $\partial : H(M) \longrightarrow H(K)$, called the connecting homomorphism, and a homology exact sequence

$$\begin{tikzcd} H(K) \arrow[r, "H(\varphi)"] & H(L) \arrow[r, "H(\psi)"] & H(M) \arrow[ll, bend left, "\partial"] \end{tikzcd}$$

Moreover, to any commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{K} \arrow[r] & \widetilde{L} \arrow[r] & \widetilde{M} \arrow[r] & 0 \end{tikzcd}$$

is associated a commutative diagram of homology exact sequences

$$\begin{tikzcd} H(K) \arrow[r] \arrow[d] & H(L) \arrow[r] \arrow[d] & H(M) \arrow[r, "\partial"] \arrow[d] & H(K) \arrow[r] \arrow[d] & \cdots \\ H(\widetilde{K}) \arrow[r] & H(\widetilde{L}) \arrow[r] & H(\widetilde{M}) \arrow[r, "\partial"] & H(\widetilde{K}) \arrow[r] & \cdots . \end{tikzcd}$$

Proof. We first define the connecting homomorphism ∂ : let $m \in Z(M)$ represent a given cohomology class $\{m\}$ in $H(M)$. Then

$$\partial\{m\} = \{k\} \in H(K)$$

is the class of any element $k \in \varphi^{-1} d\psi^{-1}(m)$, as obtained through the following construction:

$$\begin{tikzcd} & l \in L \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "d"] & m \in M \arrow[d, mapsto, "d"] \\ k \in K \arrow[r, mapsto, "\varphi"] & dl \in L \arrow[r, mapsto, "\psi"] & 0 \in M. \end{tikzcd}$$

The element l is chosen to be a preimage of m by the surjective map ψ ; as $\psi(dl) = d(m) = 0$, there exists a unique element $k \in K$ such that $\varphi(k) = dl$. The element k is actually a cocycle in $Z(K)$ because φ is injective and

$$\varphi(dk) = d\varphi(k) = d(dl) = 0 \quad \Longrightarrow \quad dk = 0.$$

The map ∂ will be well defined if we show that the cohomology class $\{k\}$ depends only on $\{m\}$ and not on the choices made for the representatives m and l. Consider another representative $m' = m + dm_1$. Let $l_1 \in L$ such that $\psi(l_1) = m_1$. Then l has to be replaced by an element $l' \in L$ such that

$$\psi(l') = m + dm_1 = \psi(l + dl_1).$$

It follows that $l' = l + dl_1 + \varphi(k_1)$ for some $k_1 \in K$, hence

$$dl' = dl + d\varphi(k_1) = \varphi(k) + \varphi(dk_1) = \varphi(k'),$$

therefore $k' = k + dk_1$ and k' has the same cohomology class as k.

Now, let us show that $\ker\partial = \operatorname{Im} H(\psi)$. If $\{m\}$ is in the image of $H(\psi)$, we can take $m = \psi(l)$ with $dl = 0$, thus $\partial\{m\} = 0$. Conversely, if $\partial\{m\} = \{k\} = 0$, we have $k = dk_1$ for some $k_1 \in K$, hence $dl = \varphi(k) = d\varphi(k_1)$, $z := l - \varphi(k_1) \in Z(L)$ and $m = \psi(l) = \psi(z)$ is in $\operatorname{Im} H(\psi)$. We leave the verification of the other equalities $\operatorname{Im} H(\varphi) = \ker H(\psi)$, $\operatorname{Im}\partial = \ker H(\varphi)$ and of the commutation statement to the reader. □

In most applications, the differential modules come with a natural $\mathbb{Z}$-grading. A homological complex is a graded differential module $K_\bullet = \bigoplus_{q\in\mathbb{Z}} K_q$ together with a differential d of degree -1, i.e. $d = \bigoplus d_q$ with $d_q : K_q \longrightarrow K_{q-1}$ and $d_{q-1} \circ d_q = 0$. Similarly, a cohomological complex is a graded differential module $K^\bullet = \bigoplus_{q\in\mathbb{Z}} K^q$ with differentials $d^q : K^q \longrightarrow K^{q+1}$ such that $d^{q+1} \circ d^q = 0$ (superscripts are always used instead of subscripts in that case). The corresponding (co)cycle, (co)boundary and (co)homology modules inherit a natural $\mathbb{Z}$-grading. In the case of cohomology, say, these modules will be denoted

$$Z^\bullet(K^\bullet) = \bigoplus Z^q(K^\bullet), \quad B^\bullet(K^\bullet) = \bigoplus B^q(K^\bullet), \quad H^\bullet(K^\bullet) = \bigoplus H^q(K^\bullet).$$

Unless otherwise stated, morphisms of complexes are assumed to be of degree 0, i.e. of the form $\varphi^\bullet = \bigoplus \varphi^q$ with $\varphi^q : K^q \longrightarrow L^q$. Any short exact sequence

$$0 \longrightarrow K^\bullet \xrightarrow{\varphi^\bullet} L^\bullet \xrightarrow{\psi^\bullet} M^\bullet \longrightarrow 0$$

gives rise to a corresponding *long exact sequence* of cohomology groups

$$(1.6) \quad H^q(K^\bullet) \xrightarrow{H^q(\varphi^\bullet)} H^q(L^\bullet) \xrightarrow{H^q(\psi^\bullet)} H^q(M^\bullet) \xrightarrow{\partial^q} H^{q+1}(K^\bullet) \xrightarrow{H^{q+1}(\varphi^\bullet)} \cdots$$

and there is a similar homology long exact sequence with a connecting homomorphism ∂_q of degree -1. When dealing with commutative diagrams of such sequences, the following simple lemma is often useful; the proof consists in a straightforward diagram chasing.

(1.7) Five lemma. *Consider a commutative diagram of R-modules*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow\varphi_1 & & \downarrow\varphi_2 & & \downarrow\varphi_3 & & \downarrow\varphi_4 & & \downarrow\varphi_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

where the rows are exact sequences. If φ_2 and φ_4 are injective and φ_1 surjective, then φ_3 is injective. If φ_2 and φ_4 is surjective and φ_5 injective, then φ_3 is surjective. In particular, φ_3 is an isomorphism as soon as $\varphi_1, \varphi_2, \varphi_4, \varphi_5$ are isomorphisms.

2. The Simplicial Flabby Resolution of a Sheaf

Let X be a topological space and let $\mathcal{A}$ be a sheaf of abelian groups on X (see § II-2 for the definition). All the sheaves appearing in the sequel are assumed implicitly to be sheaves of *abelian groups*, unless otherwise stated. The first useful notion is that of resolution.

(2.1) Definition. *A (cohomological) resolution of $\mathcal{A}$ is a differential complex of sheaves $(\mathcal{L}^\bullet, d)$ with $\mathcal{L}^q = 0$, $d^q = 0$ for $q < 0$, such that there is an exact sequence*

$$0 \longrightarrow \mathcal{A} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{d^0} \mathcal{L}^1 \longrightarrow \cdots \longrightarrow \mathcal{L}^q \xrightarrow{d^q} \mathcal{L}^{q+1} \longrightarrow \cdots .$$

If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a morphism of sheaves and $(\mathcal{M}^\bullet, d)$ a resolution of $\mathcal{B}$, a morphism of resolutions $\varphi^\bullet : \mathcal{L}^\bullet \longrightarrow \mathcal{M}^\bullet$ is a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{j} & \mathcal{L}^0 & \xrightarrow{d^0} & \mathcal{L}^1 & \longrightarrow \cdots \longrightarrow & \mathcal{L}^q & \xrightarrow{d^q} & \mathcal{L}^{q+1} & \longrightarrow \cdots \\ & & \downarrow\varphi & & \downarrow\varphi^0 & & \downarrow\varphi^1 & & \downarrow\varphi^q & & \downarrow\varphi^{q+1} & \\ 0 & \longrightarrow & \mathcal{B} & \xrightarrow{j} & \mathcal{M}^0 & \xrightarrow{d^0} & \mathcal{M}^1 & \longrightarrow \cdots \longrightarrow & \mathcal{M}^q & \xrightarrow{d^q} & \mathcal{M}^{q+1} & \longrightarrow \cdots . \end{array}$$

(2.2) Example. Let X be a differentiable manifold and $\mathcal{E}^q$ the sheaf of germs of C^∞ differential forms of degree q with real values. The exterior derivative d defines a resolution $(\mathcal{E}^\bullet, d)$ of the sheaf $\mathbb{R}$ of locally constant functions with real values. In fact Poincaré's lemma asserts that d is locally exact in degree $q \geq 1$, and it is clear that the sections of $\ker d^0$ on connected open sets are constants. □

In the sequel, we will be interested by special resolutions in which the sheaves $\mathcal{L}^q$ have no local "rigidity". For that purpose, we introduce flabby sheaves, which have become a standard tool in sheaf theory since the publication of Godement's book (Godement 1957).

(2.3) Definition. *A sheaf $\mathcal{F}$ is called flabby if for every open subset U of X, the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(U)$ is onto, i.e. if every section of $\mathcal{F}$ on U can be extended to X.*

Let $\pi : \mathcal{A} \longrightarrow X$ be a sheaf on X. We denote by $\mathcal{A}^{[0]}$ the sheaf of germs of sections $X \longrightarrow \mathcal{A}$ which are *not necessarily continuous.* In other words, $\mathcal{A}^{[0]}(U)$ is the set of all maps $f : U \longrightarrow \mathcal{A}$ such that $f(x) \in \mathcal{A}_x$ for all $x \in U$, or equivalently $\mathcal{A}^{[0]}(U) = \prod_{x \in U} \mathcal{A}_x$. It is clear that $\mathcal{A}^{[0]}$ is flabby and there is a canonical injection

$$j : \mathcal{A} \longrightarrow \mathcal{A}^{[0]}$$

defined as follows: to any $s \in \mathcal{A}_x$ we associate the germ $\widetilde{s} \in \mathcal{A}_x^{[0]}$ equal to the continuous section $y \longmapsto \widetilde{s}(y)$ near x such that $\widetilde{s}(x) = s$. In the sequel we merely denote $\widetilde{s} : y \longmapsto s(y)$ for simplicity. The sheaf $\mathcal{A}^{[0]}$ is called the *canonical flabby sheaf* associated to $\mathcal{A}$. We define inductively

$$\mathcal{A}^{[q]} = (\mathcal{A}^{[q-1]})^{[0]}.$$

The stalk $\mathcal{A}_x^{[q]}$ can be considered as the set of equivalence classes of maps $f : X^{q+1} \longrightarrow \mathcal{A}$ such that $f(x_0, \ldots, x_q) \in \mathcal{A}_{x_q}$, with two such maps identified if they coincide on a set of the form

$$(2.4) \quad x_0 \in V, \quad x_1 \in V(x_0), \quad \ldots , \quad x_q \in V(x_0, \ldots, x_{q-1}),$$

where V is an open neighborhood of x and $V(x_0, \ldots, x_j)$ an open neighborhood of x_j, depending on $x_0, \ldots, x_j$. This is easily seen by induction on q, if we identify a map $f : X^{q+1} \to \mathcal{A}$ to the map $X \to \mathcal{A}^{[q-1]}$, $x_0 \mapsto f_{x_0}$ such that $f_{x_0}(x_1, \ldots, x_q) = f(x_0, x_1, \ldots, x_q)$. Similarly, $\mathcal{A}^{[q]}(U)$ is the set of equivalence classes of functions $X^{q+1} \ni (x_0, \ldots, x_q) \longmapsto f(x_0, \ldots, x_q) \in \mathcal{A}_{x_q}$, with two such functions identified if they coincide on a set of the form

$$(2.4') \quad x_0 \in U, \quad x_1 \in V(x_0), \quad \ldots , \quad x_q \in V(x_0, \ldots, x_{q-1}).$$

Here, we may of course suppose $V(x_0, \ldots, x_{q-1}) \subset \ldots \subset V(x_0, x_1) \subset V(x_0) \subset U$. We define a differential $d^q : \mathcal{A}^{[q]} \longrightarrow \mathcal{A}^{[q+1]}$ by

$$(2.5) \quad (d^q f)(x_0, \ldots, x_{q+1}) = \sum_{0 \le j \le q} (-1)^j f(x_0, \ldots, \widehat{x_j}, \ldots, x_{q+1}) + (-1)^{q+1} f(x_0, \ldots, x_q)(x_{q+1}).$$

The meaning of the last term is to be understood as follows: the element $s = f(x_0, \ldots, x_q)$ is a germ in $\mathcal{A}_{x_q}$, therefore s defines a continuous section $x_{q+1} \mapsto s(x_{q+1})$ of $\mathcal{A}$ in a neighborhood $V(x_0, \ldots, x_q)$ of x_q. In low degrees, we have the formulas

$$(2.6)\qquad \begin{aligned} (js)(x_0) &= s(x_0), \quad s \in \mathcal{A}_x, \\ (d^0 f)(x_0, x_1) &= f(x_1) - f(x_0)(x_1), \quad f \in \mathcal{A}_x^{[0]}, \\ (d^1 f)(x_0, x_1, x_2) &= f(x_1, x_2) - f(x_0, x_2) + f(x_0, x_1)(x_2), \quad f \in \mathcal{A}_x^{[1]}. \end{aligned}$$

(2.7) Theorem (Godement 1957). *The complex $(\mathcal{A}^{[\bullet]}, d)$ is a resolution of the sheaf $\mathcal{A}$, called the simplicial flabby resolution of $\mathcal{A}$.*

Proof. For $s \in \mathcal{A}_x$, the associated continuous germ obviously satisfies $s(x_0)(x_1) = s(x_1)$ for $x_0 \in V$, $x_1 \in V(x_0)$ small enough. The reader will easily infer from this that $d^0 \circ j = 0$ and $d^{q+1} \circ d^q = 0$. In order to verify that $(\mathcal{A}^{[\bullet]}, d)$ is a resolution of $\mathcal{A}$, we show that the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{A}_x \xrightarrow{\ j\ } \mathcal{A}_x^{[0]} \xrightarrow{\ d^0\ } \cdots \longrightarrow \mathcal{A}_x^{[q]} \xrightarrow{\ d^q\ } \mathcal{A}_x^{[q+1]} \longrightarrow \cdots$$

is homotopic to zero for every point $x \in X$. Set $\mathcal{A}^{[-1]} = \mathcal{A}$, $d^{-1} = j$ and

$$\begin{aligned} h^0 &: \mathcal{A}_x^{[0]} \longrightarrow \mathcal{A}_x, & h^0(f) &= f(x) \in \mathcal{A}_x, \\ h^q &: \mathcal{A}_x^{[q]} \longrightarrow \mathcal{A}_x^{[q-1]}, & h^q(f)(x_0, \ldots, x_{q-1}) &= f(x, x_0, \ldots, x_{q-1}). \end{aligned}$$

A straightforward computation shows that $(h^{q+1} \circ d^q + d^{q-1} \circ h^q)(f) = f$ for all $q \in \mathbb{Z}$ and $f \in \mathcal{A}_x^{[q]}$. □

If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a sheaf morphism, it is clear that φ induces a morphism of resolutions

$$(2.8)\qquad \varphi^{[\bullet]} : \mathcal{A}^{[\bullet]} \longrightarrow \mathcal{B}^{[\bullet]}.$$

For every short exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of sheaves, we get a corresponding short exact sequence of sheaf complexes

$$(2.9)\qquad \mathcal{A}^{[\bullet]} \longrightarrow \mathcal{B}^{[\bullet]} \longrightarrow \mathcal{C}^{[\bullet]}.$$

3. Cohomology Groups with Values in a Sheaf

3.A. Definition and Functorial Properties

If $\pi : \mathcal{A} \to X$ is a sheaf of abelian groups, the *cohomology groups* of $\mathcal{A}$ on X are (in a vague sense) algebraic invariants which describe the rigidity properties of the global sections of $\mathcal{A}$.

(3.1) Definition. *For every $q \in \mathbb{Z}$, the q-th cohomology group of X with values in $\mathcal{A}$ is*

$$\begin{aligned} H^q(X, \mathcal{A}) &= H^q\big(\mathcal{A}^{[\bullet]}(X)\big) = \\ &= \ker\big(d^q : \mathcal{A}^{[q]}(X) \to \mathcal{A}^{[q+1]}(X)\big) / \operatorname{Im}(d^{q-1} : \mathcal{A}^{[q-1]}(X) \to \mathcal{A}^{[q]}(X)\big) \end{aligned}$$

with the convention $\mathcal{A}^{[q]} = 0$, $d^q = 0$, $H^q(X, \mathcal{A}) = 0$ *when* $q < 0$.

For any subset $S \subset X$, we denote by $\mathcal{A}_{\restriction S}$ the *restriction* of $\mathcal{A}$ to S, i.e. the sheaf $\mathcal{A}_{\restriction S} = \pi^{-1}(S)$ equipped with the projection $\pi_{\restriction S}$ onto S. Then we write $H^q(S, \mathcal{A}_{\restriction S}) = H^q(S, \mathcal{A})$ for simplicity. When U is open, we see that $(\mathcal{A}^{[q]})_{\restriction U}$ coincides with $(\mathcal{A}_{\restriction U})^{[q]}$, thus we have $H^q(U, \mathcal{A}) = H^q\big(\mathcal{A}^{[\bullet]}(U)\big)$. It is easy to show that every exact sequence of sheaves $0 \to \mathcal{A} \to \mathcal{L}^0 \to \mathcal{L}^1$ induces an exact sequence

$$(3.2)\quad 0 \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{L}^0(X) \longrightarrow \mathcal{L}^1(X).$$

If we apply this to $\mathcal{L}^q = \mathcal{A}^{[q]}$, $q = 0, 1$, we conclude that

$$(3.3)\quad H^0(X, \mathcal{A}) = \mathcal{A}(X).$$

Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a sheaf morphism; (2.8) shows that there is an induced morphism

$$(3.4)\quad H^q(\varphi) : H^q(X, \mathcal{A}) \longrightarrow H^q(X, \mathcal{B})$$

on cohomology groups. Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. Then we have an exact sequence of *groups*

$$0 \longrightarrow \mathcal{A}^{[0]}(X) \longrightarrow \mathcal{B}^{[0]}(X) \longrightarrow \mathcal{C}^{[0]}(X) \longrightarrow 0$$

because $\mathcal{A}^{[0]}(X) = \prod_{x \in X} \mathcal{A}_x$. Similarly, (2.9) yields for every q an exact sequence of *groups*

$$0 \longrightarrow \mathcal{A}^{[q]}(X) \longrightarrow \mathcal{B}^{[q]}(X) \longrightarrow \mathcal{C}^{[q]}(X) \longrightarrow 0.$$

If we take (3.3) into account, the snake lemma implies:

(3.5) Theorem. *To any exact sequence of sheaves* $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ *is associated a long exact sequence of cohomology groups*

$$\begin{array}{l} 0 \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{B}(X) \longrightarrow \mathcal{C}(X) \longrightarrow H^1(X, \mathcal{A}) \longrightarrow \cdots \\ \cdots \longrightarrow H^q(X, \mathcal{A}) \longrightarrow H^q(X, \mathcal{B}) \longrightarrow H^q(X, \mathcal{C}) \longrightarrow H^{q+1}(X, \mathcal{A}) \longrightarrow \cdots. \end{array}$$

(3.6) Corollary. *Let* $\mathcal{B} \to \mathcal{C}$ *be a surjective sheaf morphism and let* $\mathcal{A}$ *be its kernel. If* $H^1(X, \mathcal{A}) = 0$, *then* $\mathcal{B}(X) \longrightarrow \mathcal{C}(X)$ *is surjective.* □

3.B. Exact Sequence Associated to a Closed Subset

Let S be a closed subset of X and $U = X \smallsetminus S$. For any sheaf $\mathcal{A}$ on X, the presheaf

$$\Omega \longmapsto \mathcal{A}(S \cap \Omega), \quad \Omega \subset X \text{ open}$$

with the obvious restriction maps satisfies axioms (II-2.4′) and (II-2.4″), so it defines a sheaf on X which we denote by $\mathcal{A}^S$. This sheaf should not be confused with the restriction sheaf $\mathcal{A}_{\restriction S}$, which is a sheaf on S. We easily find

$$(3.7) \quad (\mathcal{A}^S)_x = \mathcal{A}_x \quad \text{if } x \in S, \quad (\mathcal{A}^S)_x = 0 \quad \text{if } x \in U.$$

Observe that these relations would completely fail if S were not closed. The restriction morphism $f \mapsto f_{\restriction S}$ induces a surjective sheaf morphism $\mathcal{A} \to \mathcal{A}^S$. We let $\mathcal{A}_U$ be its kernel, so that we have the relations

$$(3.8) \quad (\mathcal{A}_U)_x = 0 \quad \text{if } x \in S, \quad (\mathcal{A}_U)_x = \mathcal{A}_x \quad \text{if } x \in U.$$

From the definition, we obtain in particular

$$(3.9) \quad \mathcal{A}^S(X) = \mathcal{A}(S), \quad \mathcal{A}_U(X) = \{\text{sections of } \mathcal{A}(X) \text{ vanishing on } S\}.$$

Theorem 3.5 applied to the exact sequence $0 \to \mathcal{A}_U \to \mathcal{A} \to \mathcal{A}^S \to 0$ on X gives a long exact sequence

$$(3.9) \quad \begin{array}{l} 0 \longrightarrow \mathcal{A}_U(X) \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{A}(S) \longrightarrow H^1(X, \mathcal{A}_U) \cdots \\ \longrightarrow H^q(X, \mathcal{A}_U) \longrightarrow H^q(X, \mathcal{A}) \longrightarrow H^q(X, \mathcal{A}^S) \longrightarrow H^{q+1}(X, \mathcal{A}_U) \cdots \end{array}$$

3.C. Mayer-Vietoris Exact Sequence

Let U_1, U_2 be open subsets of X and $U = U_1 \cup U_2$, $V = U_1 \cap U_2$. For any sheaf $\mathcal{A}$ on X and any q we have an exact sequence

$$0 \longrightarrow \mathcal{A}^{[q]}(U) \longrightarrow \mathcal{A}^{[q]}(U_1) \oplus \mathcal{A}^{[q]}(U_2) \longrightarrow \mathcal{A}^{[q]}(V) \longrightarrow 0$$

where the injection is given by $f \longmapsto (f_{\restriction U_1}, f_{\restriction U_2})$ and the surjection by $(g_1, g_2) \longmapsto g_{2\restriction V} - g_{1\restriction V}$; the surjectivity of this map follows immediately from the fact that $\mathcal{A}^{[q]}$ is flabby. An application of the snake lemma yields:

(3.11) Theorem. *For any sheaf $\mathcal{A}$ on X and any open sets $U_1, U_2 \subset X$, set $U = U_1 \cup U_2$, $V = U_1 \cap U_2$. Then there is an exact sequence*

$$H^q(U, \mathcal{A}) \longrightarrow H^q(U_1, \mathcal{A}) \oplus H^q(U_2, \mathcal{A}) \longrightarrow H^q(V, \mathcal{A}) \longrightarrow H^{q+1}(U, \mathcal{A}) \cdots \square$$

4. Acyclic Sheaves

Given a sheaf $\mathcal{A}$ on X, it is usually very important to decide whether the cohomology groups $H^q(U, \mathcal{A})$ vanish for $q \geq 1$, and if this is the case, for which type of open sets U. Note that one cannot expect to have $H^0(U, \mathcal{A}) = 0$ in general, since a sheaf always has local sections.

(4.1) Definition. *A sheaf $\mathcal{A}$ is said to be acyclic on an open subset U if $H^q(U, \mathcal{A}) = 0$ for $q \geq 1$.*

4.A. Case of Flabby Sheaves

We are going to show that flabby sheaves are acyclic. First we need the following simple result.

(4.2) Proposition. *Let $\mathcal{A}$ be a sheaf with the following property: for every section f of $\mathcal{A}$ on an open subset $U \subset X$ and every point $x \in X$, there exists a neighborhood Ω of x and a section $h \in \mathcal{A}(\Omega)$ such that $h = f$ on $U \cap \Omega$. Then $\mathcal{A}$ is flabby.*

A consequence of this proposition is that flabbiness is a local property: a sheaf $\mathcal{A}$ is flabby on X if and only if it is flabby on a neighborhood of every point of X.

Proof. Let $f \in \mathcal{A}(U)$ be given. Consider the set of pairs (v, V) where v in $\mathcal{B}(V)$ is an extension of f on an open subset $V \supset U$. This set is inductively ordered, so there exists a maximal extension (v, V) by Zorn's lemma. The assumption shows that V must be equal to X. □

(4.3) Proposition. *Let $0 \longrightarrow \mathcal{A} \xrightarrow{j} \mathcal{B} \xrightarrow{p} \mathcal{C} \longrightarrow 0$ be an exact sequence of sheaves. If $\mathcal{A}$ is flabby, the sequence of groups*

$$0 \longrightarrow \mathcal{A}(U) \xrightarrow{j} \mathcal{B}(U) \xrightarrow{p} \mathcal{C}(U) \longrightarrow 0$$

is exact for every open set U. If $\mathcal{A}$ and $\mathcal{B}$ are flabby, then $\mathcal{C}$ is flabby.

Proof. Let $g \in \mathcal{C}(U)$ be given. Consider the set E of pairs (v, V) where V is an open subset of U and $v \in \mathcal{B}(V)$ is such that $p(v) = g$ on V. It is clear that E is inductively ordered, so E has a maximal element (v, V), and we will prove that $V = U$. Otherwise, let $x \in U \smallsetminus V$ and let h be a section of $\mathcal{B}$ in a neighborhood of x such that $p(h_x) = g_x$. Then $p(h) = g$ on a neighborhood Ω of x, thus $p(v - h) = 0$ on $V \cap \Omega$ and $v - h = j(u)$ with $u \in \mathcal{A}(V \cap \Omega)$. If $\mathcal{A}$ is flabby, u has an extension $\widetilde{u} \in \mathcal{A}(X)$ and we can define a section $w \in \mathcal{B}(V \cup \Omega)$ such that $p(w) = g$ by

$$w = v \ \text{ on } \ V, \qquad w = h + j(\widetilde{u}) \ \text{ on } \ \Omega,$$

contradicting the maximality of (v, V). Therefore $V = U$, $v \in \mathcal{B}(U)$ and $p(v) = g$ on U. The first statement is proved. If $\mathcal{B}$ is also flabby, v has an extension $\widetilde{v} \in \mathcal{B}(X)$ and $\widetilde{g} = p(\widetilde{v}) \in \mathcal{C}(X)$ is an extension of g. Hence $\mathcal{C}$ is flabby. □

(4.4) Theorem. *A flabby sheaf $\mathcal{A}$ is acyclic on all open sets $U \subset X$.*

Proof. Let $\mathcal{Z}^q = \ker\big(d^q : \mathcal{A}^{[q]} \to \mathcal{A}^{[q+1]}\big)$. Then $\mathcal{Z}^0 = \mathcal{A}$ and we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{Z}^q \longrightarrow \mathcal{A}^{[q]} \xrightarrow{\ d^q\ } \mathcal{Z}^{q+1} \longrightarrow 0$$

because $\operatorname{Im} d^q = \ker d^{q+1} = \mathcal{Z}^{q+1}$. Proposition 4.3 implies by induction on q that all sheaves $\mathcal{Z}^q$ are flabby, and yields exact sequences

$$0 \longrightarrow \mathcal{Z}^q(U) \longrightarrow \mathcal{A}^{[q]}(U) \xrightarrow{\ d^q\ } \mathcal{Z}^{q+1}(U) \longrightarrow 0.$$

For $q \geq 1$, we find therefore

$$\begin{aligned} \ker\big(d^q : \mathcal{A}^{[q]}(U) \to \mathcal{A}^{[q+1]}(U)\big) &= \mathcal{Z}^q(U) \\ &= \operatorname{Im}\big(d^{q-1} : \mathcal{A}^{[q-1]}(U) \to \mathcal{A}^{[q]}(U)\big), \end{aligned}$$

that is, $H^q(U, \mathcal{A}) = H^q\big(\mathcal{A}^{[\bullet]}(U)\big) = 0$. □

4.B. Soft Sheaves over Paracompact Spaces

We now discuss another general situation which produces acyclic sheaves. Recall that a topological space X is said to be *paracompact* if X is Hausdorff and if every open covering of X has a locally finite refinement. For instance, it is well known that every metric space is paracompact. A paracompact space X is always *normal*; in particular, for any locally finite open covering (U_α) of X there exists an open covering (V_α) such that $\overline{V}_\alpha \subset U_\alpha$. We will also need another closely related concept.

(4.5) Definition. *We say that a subspace S is strongly paracompact in X if S is Hausdorff and if the following property is satisfied: for every covering (U_α) of S by open sets in X, there exists another such covering (V_β) and a neighborhood W of S such that each set $W \cap \overline{V}_\beta$ is contained in some U_α, and such that every point of S has a neighborhood intersecting only finitely many sets V_β.*

It is clear that a strongly paracompact subspace S is itself paracompact. Conversely, the following result is easy to check:

(4.6) Lemma. *A subspace S is strongly paracompact in X as soon as one of the following situations occurs:*

a) *X is paracompact and S is closed;*

b) *S has a fundamental family of paracompact neighborhoods in X;*

c) *S is paracompact and has a neighborhood homeomorphic to some product $S \times T$, in which S is embedded as a slice $S \times \{t_0\}$.* □

(4.7) Theorem. *Let $\mathcal{A}$ be a sheaf on X and S a strongly paracompact subspace of X. Then every section f of $\mathcal{A}$ on S can be extended to a section of $\mathcal{A}$ on some open neighborhood Ω of $\mathcal{A}$.*

Proof. Let $f \in \mathcal{A}(S)$. For every point $z \in S$ there exists an open neighborhood U_z and a section $\widetilde{f}_z \in \mathcal{A}(U_z)$ such that $\widetilde{f}_z(z) = f(z)$. After shrinking U_z, we may assume that $\widetilde{f}_z$ and f coincide on $S \cap U_z$. Let (V_α) be an open covering of S that is locally finite near S and W a neighborhood of S such that $W \cap \overline{V}_\alpha \subset U_{z(\alpha)}$ (Def. 4.5). We let

$$\Omega = \big\{x \in W \cap \bigcup V_\alpha\,;\ \widetilde{f}_{z(\alpha)}(x) = \widetilde{f}_{z(\beta)}(x),\ \forall \alpha, \beta \text{ with } x \in \overline{V}_\alpha \cap \overline{V}_\beta\big\}.$$

Then $(\Omega \cap V_\alpha)$ is an open covering of Ω and all pairs of sections $\widetilde{f}_{z(\alpha)}$ coincide in pairwise intersections. Thus there exists a section F of $\mathcal{A}$ on Ω which is equal to $\widetilde{f}_{z(\alpha)}$ on $\Omega \cap V_\alpha$. It remains only to show that Ω is a neighborhood of S. Let $z_0 \in S$. There exists a neighborhood U' of z_0 which meets only finitely many sets $V_{\alpha_1}, \ldots, V_{\alpha_p}$. After shrinking U', we may keep only those V_{α_l} such that $z_0 \in \overline{V}_{\alpha_l}$. The sections $\widetilde{f}_{z(\alpha_l)}$ coincide at z_0, so they coincide on some neighborhood U'' of this point. Hence $W \cap U'' \subset \Omega$, so Ω is a neighborhood of S. □

(4.8) Corollary. *If X is paracompact, every section $f \in \mathcal{A}(S)$ defined on a closed set S extends to a neighborhood Ω of S.* □

(4.9) Definition. *A sheaf $\mathcal{A}$ on X is said to be soft if every section f of $\mathcal{A}$ on a closed set S can be extended to X, i.e. if the restriction map $\mathcal{A}(X) \longrightarrow \mathcal{A}(S)$ is onto for every closed set S.*

(4.10) Example. On a paracompact space, every flabby sheaf $\mathcal{A}$ is soft: this is a consequence of Cor. 4.8.

(4.11) Example. On a paracompact space, the Tietze-Urysohn extension theorem shows that the sheaf $\mathcal{C}_X$ of germs of continuous functions on X is a soft sheaf of rings. However, observe that $\mathcal{C}_X$ is not flabby as soon as X is not discrete.

(4.12) Example. If X is a paracompact differentiable manifold, the sheaf $\mathcal{E}_X$ of germs of C^∞ functions on X is a soft sheaf of rings. □

Until the end of this section, we assume that X is a *paracompact topological space*. We first show that softness is a local property.

(4.13) Proposition. *A sheaf $\mathcal{A}$ is soft on X if and only if it is soft in a neighborhood of every point $x \in X$.*

Proof. If $\mathcal{A}$ is soft on X, it is soft on any closed neighborhood of a given point. Conversely, let $(U_\alpha)_{\alpha \in I}$ be a locally finite open covering of X which refine some covering by neighborhoods on which $\mathcal{A}$ is soft. Let (V_α) be a finer covering such that $\overline{V}_\alpha \subset U_\alpha$, and $f \in \mathcal{A}(S)$ be a section of $\mathcal{A}$ on a closed subset S of X. We consider the set E of pairs (g, J), where $J \subset I$ and where g is a section over $F_J := S \cup \bigcup_{\alpha \in J} \overline{V}_\alpha$, such that $g = f$ on S. As the family $(\overline{V}_\alpha)$ is locally finite, a section of $\mathcal{A}$ over F_J is continuous as soon it is continuous on S and on each $\overline{V}_\alpha$. Then $(f, \emptyset) \in E$ and E is inductively ordered by the relation

$$(g', J') \longrightarrow (g'', J'') \quad \text{if} \quad J' \subset J'' \quad \text{and} \quad g' = g'' \quad \text{on} \quad F_{J'}$$

No element (g, J), $J \neq I$, can be maximal: the assumption shows that $g_{\restriction F_J \cap \overline{V}_\alpha}$ has an extension to $\overline{V}_\alpha$, thus such a g has an extension to $F_{J \cup \{\alpha\}}$ for any $\alpha \notin J$. Hence E has a maximal element (g, I) defined on $F_I = X$. □

(4.14) Proposition. *Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. If $\mathcal{A}$ is soft, the map $\mathcal{B}(S) \to \mathcal{C}(S)$ is onto for any closed subset S of X. If $\mathcal{A}$ and $\mathcal{B}$ are soft, then $\mathcal{C}$ is soft.*

By the above inductive method, this result can be proved in a way similar to its analogue for flabby sheaves. We therefore obtain:

(4.15) Theorem. *On a paracompact space, a soft sheaf is acyclic on all closed subsets.* □

(4.16) Definition. *The support of a section $f \in \mathcal{A}(X)$ is defined by*

$$\operatorname{Supp} f = \{x \in X \,;\, f(x) \neq 0\}.$$

Supp f is always a closed set: as $\mathcal{A} \to X$ is a local homeomorphism, the equality $f(x) = 0$ implies $f = 0$ in a neighborhood of x.

(4.17) Theorem. *Let $(U_\alpha)_{\alpha \in I}$ be an open covering of X. If $\mathcal{A}$ is soft and $f \in \mathcal{A}(X)$, there exists a partition of f subordinate to (U_α), i.e. a family of sections $f_\alpha \in \mathcal{A}(X)$ such that* (Supp f_α) *is locally finite,* Supp $f_\alpha \subset U_\alpha$ *and $\sum f_\alpha = f$ on X.*

Proof. Assume first that (U_α) is locally finite. There exists an open covering (V_α) such that $\overline{V}_\alpha \subset U_\alpha$. Let $(f_\alpha)_{\alpha \in J}$, $J \subset I$, be a maximal family of sections

$f_\alpha \in \mathcal{A}(X)$ such that Supp $f_\alpha \subset U_\alpha$ and $\sum_{\alpha\in J} f_\alpha = f$ on $S = \bigcup_{\alpha\in J} \overline{V}_\alpha$. If $J \neq I$ and $\beta \in I \smallsetminus J$, there exists a section $f_\beta \in \mathcal{A}(X)$ such that

$$f_\beta = 0 \quad \text{on} \quad X \smallsetminus U_\beta \quad \text{and} \quad f_\beta = f - \sum_{\alpha\in J} f_\alpha \quad \text{on} \quad S \cup \overline{V}_\beta$$

because $(X \smallsetminus U_\beta) \cup S \cup \overline{V}_\beta$ is closed and $f - \sum f_\alpha = 0$ on $(X \smallsetminus U_\alpha) \cap S$. This is a contradiction unless $J = I$.

In general, let (V_j) be a locally finite refinement of (U_α), such that $V_j \subset U_{\rho(j)}$, and let (f'_j) be a partition of f subordinate to (V_j). Then $f_\alpha = \sum_{j\in\rho^{-1}(\alpha)} f'_j$ is the required partition of f. □

Finally, we discuss a special situation which occurs very often in practice. Let $\mathcal{R}$ be a sheaf of commutative rings on X ; the rings $\mathcal{R}_x$ are supposed to have a unit element. Assume that $\mathcal{A}$ is a sheaf of modules over $\mathcal{R}$. It is clear that $\mathcal{A}^{[0]}$ is a $\mathcal{R}^{[0]}$-module, and thus also a $\mathcal{R}$-module. Therefore all sheaves $\mathcal{A}^{[q]}$ are $\mathcal{R}$-modules and the cohomology groups $H^q(U, \mathcal{A})$ have a natural structure of $\mathcal{R}(U)$-module.

(4.18) Lemma. *If $\mathcal{R}$ is soft, every sheaf $\mathcal{A}$ of $\mathcal{R}$-modules is soft.*

Proof. Every section $f \in \mathcal{A}(S)$ defined on a closed set S has an extension to some open neighborhood Ω. Let $\psi \in \mathcal{R}(X)$ be such that $\psi = 1$ on S and $\psi = 0$ on $X \smallsetminus \Omega$. Then ψf, defined as 0 on $X \smallsetminus \Omega$, is an extension of f to X. □

(4.19) Corollary. *Let $\mathcal{A}$ be a sheaf of $\mathcal{E}_X$-modules on a paracompact differentiable manifold X. Then $H^q(X, \mathcal{A}) = 0$ for all $q \geq 1$.*

5. Čech Cohomology

5.A. Definitions

In many important circumstances, cohomology groups with values in a sheaf $\mathcal{A}$ can be computed by means of the complex of Čech cochains, which is directly related to the spaces of sections of $\mathcal{A}$ on sufficiently fine coverings of X. This more concrete approach was historically the first one used to define sheaf cohomology (Leray 1950, Cartan 1950); however Čech cohomology does not always coincide with the "good" cohomology on non paracompact spaces. Let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be an open covering of X. For the sake of simplicity, we denote

$$U_{\alpha_0\alpha_1\ldots\alpha_q} = U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_q}.$$

The group $C^q(\mathcal{U}, \mathcal{A})$ of *Čech q-cochains* is the set of families

$$c = (c_{\alpha_0\alpha_1\ldots\alpha_q}) \in \prod_{(\alpha_0,\ldots,\alpha_q)\in I^{q+1}} \mathcal{A}(U_{\alpha_0\alpha_1\ldots\alpha_q}).$$

The group structure on $C^q(\mathfrak{U},\mathcal{A})$ is the obvious one deduced from the addition law on sections of $\mathcal{A}$. The *Čech differential* $\delta^q : C^q(\mathfrak{U},\mathcal{A}) \longrightarrow C^{q+1}(\mathfrak{U},\mathcal{A})$ is defined by the formula

$$(5.1) \quad (\delta^q c)_{\alpha_0\ldots\alpha_{q+1}} = \sum_{0\le j\le q+1} (-1)^j\, c_{\alpha_0\ldots\widehat{\alpha_j}\ldots\alpha_{q+1}\;\restriction U_{\alpha_0\ldots\alpha_{q+1}}},$$

and we set $C^q(\mathfrak{U},\mathcal{A}) = 0$, $\delta^q = 0$ for $q < 0$. In degrees 0 and 1, we get for example

$$(5.2) \qquad q = 0, \quad c = (c_\alpha), \quad (\delta^0 c)_{\alpha\beta} = c_\beta - c_{\alpha\;\restriction U_{\alpha\beta}},$$

$$(5.2') \qquad q = 1, \quad c = (c_{\alpha\beta}), \quad (\delta^1 c)_{\alpha\beta\gamma} = c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta\;\restriction U_{\alpha\beta\gamma}}.$$

Easy verifications left to the reader show that $\delta^{q+1}\circ\delta^q = 0$. We get therefore a cochain complex $\big(C^\bullet(\mathfrak{U},\mathcal{A}),\delta\big)$, called the *complex of Čech cochains* relative to the covering $\mathfrak{U}$.

(5.3) Definition. *The Čech cohomology group of $\mathcal{A}$ relative to $\mathfrak{U}$ is*

$$\check{H}^q(\mathfrak{U},\mathcal{A}) = H^q\big(C^\bullet(\mathfrak{U},\mathcal{A})\big).$$

Formula (5.2) shows that the set of Čech 0-cocycles is the set of families $(c_\alpha) \in \prod \mathcal{A}(U_\alpha)$ such that $c_\beta = c_\alpha$ on $U_\alpha \cap U_\beta$. Such a family defines in a unique way a global section $f \in \mathcal{A}(X)$ with $f_{\restriction U_\alpha} = c_\alpha$. Hence

$$(5.4) \quad \check{H}^0(\mathfrak{U},\mathcal{A}) = \mathcal{A}(X).$$

Now, let $\mathcal{V} = (V_\beta)_{\beta\in J}$ be another open covering of X that is finer than $\mathfrak{U}$; this means that there exists a map $\rho : J \to I$ such that $V_\beta \subset U_{\rho(\beta)}$ for every $\beta \in J$. Then we can define a morphism $\rho^\bullet : C^\bullet(\mathfrak{U},\mathcal{A}) \longrightarrow C^\bullet(\mathcal{V},\mathcal{A})$ by

$$(5.5) \quad (\rho^q c)_{\beta_0\ldots\beta_q} = c_{\rho(\beta_0)\ldots\rho(\beta_q)\;\restriction V_{\beta_0\ldots\beta_q}}\ ;$$

the commutation property $\delta\rho^\bullet = \rho^\bullet\delta$ is immediate. If $\rho' : J \to I$ is another refinement map such that $V_\beta \subset U_{\rho'(\beta)}$ for all β, the morphisms $\rho^\bullet$, $\rho'^\bullet$ are homotopic. To see this, we define a map $h^q : C^q(\mathfrak{U},\mathcal{A}) \longrightarrow C^{q-1}(\mathcal{V},\mathcal{A})$ by

$$(h^q c)_{\beta_0\ldots\beta_{q-1}} = \sum_{0\le j\le q-1} (-1)^j c_{\rho(\beta_0)\ldots\rho(\beta_j)\rho'(\beta_j)\ldots\rho'(\beta_{q-1})\;\restriction V_{\beta_0\ldots\beta_{q-1}}}.$$

The homotopy identity $\delta^{q-1}\circ h^q + h^{q+1}\circ\delta^q = \rho'^q - \rho^q$ is easy to verify. Hence $\rho^\bullet$ and $\rho'^\bullet$ induce a map depending only on $\mathfrak{U}$, $\mathcal{V}$:

$$(5.6) \quad H^q(\rho^\bullet) = H^q(\rho'^\bullet) \ :\ \check{H}^q(\mathfrak{U},\mathcal{A}) \longrightarrow \check{H}^q(\mathcal{V},\mathcal{A}).$$

Now, we want to define a *direct limit* $\check{H}^q(X,\mathcal{A})$ of the groups $\check{H}^q(\mathcal{U},\mathcal{A})$ by means of the refinement mappings (5.6). In order to avoid set theoretic difficulties, the coverings used in this definition will be considered as subsets of the power set $\mathcal{P}(X)$, so that the collection of all coverings becomes actually a set.

(5.7) Definition. *The Čech cohomology group $\check{H}^q(X,\mathcal{A})$ is the direct limit*

$$\check{H}^q(X,\mathcal{A}) = \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U},\mathcal{A})$$

when $\mathcal{U}$ runs over the collection of all open coverings of X. Explicitly, this means that the elements of $\check{H}^q(X,\mathcal{A})$ are the equivalence classes in the disjoint union of the groups $\check{H}^q(\mathcal{U},\mathcal{A})$, with an element in $\check{H}^q(\mathcal{U},\mathcal{A})$ and another in $\check{H}^q(\mathcal{V},\mathcal{A})$ identified if their images in $\check{H}^q(\mathcal{W},\mathcal{A})$ coincide for some refinement $\mathcal{W}$ of the coverings $\mathcal{U}$ and $\mathcal{V}$.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a sheaf morphism, we have an obvious induced morphism $\varphi^\bullet : C^\bullet(\mathcal{U},\mathcal{A}) \longrightarrow C^\bullet(\mathcal{U},\mathcal{B})$, and therefore we find a morphism

$$H^q(\varphi^\bullet) : \check{H}^q(\mathcal{U},\mathcal{A}) \longrightarrow \check{H}^q(\mathcal{U},\mathcal{B}).$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. We have an exact sequence of groups

$$(5.8) \quad 0 \longrightarrow C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^q(\mathcal{U},\mathcal{B}) \longrightarrow C^q(\mathcal{U},\mathcal{C}),$$

but in general the last map is not surjective, because every section in $\mathcal{C}(U_{\alpha_0,\ldots,\alpha_q})$ need not have a lifting in $\mathcal{B}(U_{\alpha_0,\ldots,\alpha_q})$. The image of $C^\bullet(\mathcal{U},\mathcal{B})$ in $C^\bullet(\mathcal{U},\mathcal{C})$ will be denoted $C^\bullet_{\mathcal{B}}(\mathcal{U},\mathcal{C})$ and called the complex of *liftable cochains* of $\mathcal{C}$ in $\mathcal{B}$. By construction, the sequence

$$(5.9) \quad 0 \longrightarrow C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^q(\mathcal{U},\mathcal{B}) \longrightarrow C^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow 0$$

is exact, thus we get a corresponding long exact sequence of cohomology

$$(5.10) \quad \check{H}^q(\mathcal{U},\mathcal{A}) \longrightarrow \check{H}^q(\mathcal{U},\mathcal{B}) \longrightarrow \check{H}^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow \check{H}^{q+1}(\mathcal{U},\mathcal{A}) \longrightarrow \cdots.$$

If $\mathcal{A}$ is flabby, Prop. 4.3 shows that we have $C^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) = C^q(\mathcal{U},\mathcal{C})$, hence $\check{H}^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) = \check{H}^q(\mathcal{U},\mathcal{C})$.

(5.11) Proposition. *Let $\mathcal{A}$ be a sheaf on X. Assume that either*

a) *$\mathcal{A}$ is flabby, or :*

b) *X is paracompact and $\mathcal{A}$ is a sheaf of modules over a soft sheaf of rings $\mathcal{R}$ on X.*

Then $\check{H}^q(\mathcal{U},\mathcal{A}) = 0$ for every $q \geq 1$ and every open covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ of X.

Proof. b) Let $(\psi_\alpha)_{\alpha\in I}$ be a partition of unity in $\mathcal{R}$ subordinate to $\mathcal{U}$ (Prop. 4.17). We define a map $h^q : C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^{q-1}(\mathcal{U},\mathcal{A})$ by

$$(5.12)\quad (h^q c)_{\alpha_0\ldots\alpha_{q-1}} = \sum_{\nu\in I} \psi_\nu\, c_{\nu\alpha_0\ldots\alpha_{q-1}}$$

where $\psi_\nu\, c_{\nu\alpha_0\ldots\alpha_{q-1}}$ is extended by 0 on $U_{\alpha_0\ldots\alpha_{q-1}} \cap \complement U_\nu$. It is clear that

$$(\delta^{q-1}h^q c)_{\alpha_0\ldots\alpha_q} = \sum_{\nu\in I} \psi_\nu \big(c_{\alpha_0\ldots\alpha_q} - (\delta^q c)_{\nu\alpha_0\ldots\alpha_q}\big),$$

i.e. $\delta^{q-1}h^q + h^{q+1}\delta^q = \mathrm{Id}$. Hence $\delta^q c = 0$ implies $\delta^{q-1}h^q c = c$ if $q \geq 1$.

a) First we show that the result is true for the sheaf $\mathcal{A}^{[0]}$. One can find a family of sets $L_\nu \subset U_\nu$ such that (L_ν) is a partition of X. If ψ_ν is the characteristic function of L_ν, Formula (5.12) makes sense for any cochain $c \in C^q(\mathcal{U},\mathcal{A}^{[0]})$ because $\mathcal{A}^{[0]}$ is a module over the ring $\mathbb{Z}^{[0]}$ of germs of arbitrary functions $X \to \mathbb{Z}$. Hence $\check{H}^q(\mathcal{U},\mathcal{A}^{[0]}) = 0$ for $q \geq 1$. We shall prove this property for all flabby sheaves by induction on q. Consider the exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^{[0]} \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{C} = \mathcal{A}^{[0]}/\mathcal{A}$. By the remark after (5.10), we have exact sequences

$$\mathcal{A}^{[0]}(X) \longrightarrow \mathcal{C}(X) \longrightarrow \check{H}^1(\mathcal{U},\mathcal{A}) \longrightarrow \check{H}^1(\mathcal{U},\mathcal{A}^{[0]}) = 0,$$
$$\check{H}^q(\mathcal{U},\mathcal{C}) \longrightarrow \check{H}^{q+1}(\mathcal{U},\mathcal{A}) \longrightarrow \check{H}^{q+1}(\mathcal{U},\mathcal{A}^{[0]}) = 0.$$

Then $\mathcal{A}^{[0]}(X) \longrightarrow \mathcal{C}(X)$ is surjective by Prop. 4.3, thus $\check{H}^1(\mathcal{U},\mathcal{A}) = 0$. By 4.3 again, $\mathcal{C}$ is flabby; the induction hypothesis $\check{H}^q(\mathcal{U},\mathcal{C}) = 0$ implies that $\check{H}^{q+1}(\mathcal{U},\mathcal{A}) = 0$. □

5.B. Leray's Theorem for Acyclic Coverings

We first show the existence of a natural morphism from Čech cohomology to ordinary cohomology. Let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be a covering of X. Select a map $\lambda : X \to I$ such that $x \in U_{\lambda(x)}$ for every $x \in X$. To every cochain $c \in C^q(\mathcal{U},\mathcal{A})$ we associate the section $\lambda^q c = f \in \mathcal{A}^{[q]}(X)$ such that

$$(5.13)\quad f(x_0,\ldots,x_q) = c_{\lambda(x_0)\ldots\lambda(x_q)}(x_q) \in \mathcal{A}_{x_q}\ ;$$

note that the right hand side is well defined as soon as

$$x_0 \in X,\quad x_1 \in U_{\lambda(x_0)},\quad \ldots\ ,\quad x_q \in U_{\lambda(x_0)\ldots\lambda(x_{q-1})}.$$

A comparison of (2.5) and (5.13) immediately shows that the section of $\mathcal{A}^{[q+1]}(X)$ associated to $\delta^q c$ is

$$\sum_{0\leq j\leq q+1} (-1)^j\, c_{\lambda(x_0)\ldots\widehat{\lambda(x_j)}\ldots\lambda(x_{q+1})}(x_{q+1}) = (d^q f)(x_0,\ldots,x_{q+1}).$$

In this way we get a morphism of complexes $\lambda^\bullet : C^\bullet(\mathcal{U},\mathcal{A}) \longrightarrow \mathcal{A}^{[\bullet]}(X)$. There is a corresponding morphism

$$(5.14)\quad H^q(\lambda^\bullet) : \check{H}^q(\mathcal{U},\mathcal{A}) \longrightarrow H^q(X,\mathcal{A}).$$

If $\mathcal{V} = (V_\beta)_{\beta\in J}$ is a refinement of $\mathcal{U}$ such that $V_\beta \subset U_{\rho(\beta)}$ and $x \in V_{\mu(x)}$ for all x,β, we get a commutative diagram

$$\begin{tikzcd} \check{H}^q(\mathcal{U},\mathcal{A}) \arrow[rr, "H^q(\rho^\bullet)"] \arrow[dr, "H^q(\lambda^\bullet)"'] & & \check{H}^q(\mathcal{V},\mathcal{A}) \arrow[dl, "H^q(\mu^\bullet)"] \\ & H^q(X,\mathcal{A}) & \end{tikzcd}$$

with $\lambda = \rho\circ\mu$. In particular, (5.6) shows that the map $H^q(\lambda^\bullet)$ in (5.14) does not depend on the choice of λ : if λ' is another choice, then $H^q(\lambda^\bullet)$ and $H^q(\lambda'^\bullet)$ can be both factorized through the group $\check{H}^q(\mathcal{V},\mathcal{A})$ with $V_x = U_{\lambda(x)} \cap U_{\lambda'(x)}$ and $\mu = \mathrm{Id}_X$. By the universal property of direct limits, we get an induced morphism

$$(5.15)\quad \check{H}^q(X,\mathcal{A}) \longrightarrow H^q(X,\mathcal{A}).$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet(\mathcal{U},\mathcal{A}) \arrow[r] \arrow[d] & C^\bullet(\mathcal{U},\mathcal{B}) \arrow[r] \arrow[d] & C^\bullet_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{A}^{[\bullet]}(X) \arrow[r] & \mathcal{B}^{[\bullet]}(X) \arrow[r] & \mathcal{C}^{[\bullet]}(X) \arrow[r] & 0 \end{tikzcd}$$

where the vertical arrows are given by the morphisms $\lambda^\bullet$. We obtain therefore a commutative diagram

$$(5.16)\quad \begin{tikzcd} \check{H}^q(\mathcal{U},\mathcal{A}) \arrow[r] \arrow[d] & \check{H}^q(\mathcal{U},\mathcal{B}) \arrow[r] \arrow[d] & \check{H}^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \arrow[r] \arrow[d] & \check{H}^{q+1}(\mathcal{U},\mathcal{A}) \arrow[r] \arrow[d] & \check{H}^{q+1}(\mathcal{U},\mathcal{B}) \arrow[d] \\ H^q(X,\mathcal{A}) \arrow[r] & H^q(X,\mathcal{B}) \arrow[r] & H^q(X,\mathcal{C}) \arrow[r] & H^{q+1}(X,\mathcal{A}) \arrow[r] & H^{q+1}(X,\mathcal{B}). \end{tikzcd}$$

(5.17) Theorem (Leray). *Assume that*

$$H^s(U_{\alpha_0\ldots\alpha_t},\mathcal{A}) = 0$$

for all indices $\alpha_0,\ldots,\alpha_t$ *and* $s \geq 1$. *Then* (5.14) *gives an isomorphism* $\check{H}^q(\mathcal{U},\mathcal{A}) \simeq H^q(X,\mathcal{A})$.

We say that the covering $\mathcal{U}$ is *acyclic* (with respect to $\mathcal{A}$) if the hypothesis of Th. 5.17 is satisfied. Leray's theorem asserts that the cohomology groups of $\mathcal{A}$ on X can be computed by means of an arbitrary acyclic covering (if such a covering exists), without using the direct limit procedure.

Proof. By induction on q, the result being obvious for $q = 0$. Consider the exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ with $\mathcal{B} = \mathcal{A}^{[0]}$ and $\mathcal{C} = \mathcal{A}^{[0]}/\mathcal{A}$. As $\mathcal{B}$

is acyclic, the hypothesis on $\mathcal{A}$ and the long exact sequence of cohomology imply $H^s(U_{\alpha_0\ldots\alpha_t}, \mathcal{C}) = 0$ for $s \geq 1$, $t \geq 0$. Moreover $C^\bullet_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) = C^\bullet(\mathcal{U}, \mathcal{C})$ thanks to Cor. 3.6. The induction hypothesis in degree q and diagram (5.16) give

$$\begin{array}{ccccccc}
\check{H}^q(\mathcal{U}, \mathcal{B}) & \longrightarrow & \check{H}^q(\mathcal{U}, \mathcal{C}) & \longrightarrow & \check{H}^{q+1}(\mathcal{U}, \mathcal{A}) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
H^q(X, \mathcal{B}) & \longrightarrow & H^q(X, \mathcal{C}) & \longrightarrow & H^{q+1}(X, \mathcal{A}) & \longrightarrow & 0,
\end{array}$$

hence $\check{H}^{q+1}(\mathcal{U}, \mathcal{A}) \longrightarrow H^{q+1}(X, \mathcal{A})$ is also an isomorphism. □

(5.18) Remark. The morphism $H^1(\lambda^\bullet) : \check{H}^1(\mathcal{U}, \mathcal{A}) \longrightarrow H^1(X, \mathcal{A})$ is always injective. Indeed, we have a commutative diagram

$$\begin{array}{ccccccc}
\check{H}^0(\mathcal{U}, \mathcal{B}) & \longrightarrow & \check{H}^0_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) & \longrightarrow & \check{H}^1(\mathcal{U}, \mathcal{A}) & \longrightarrow & 0 \\
\downarrow = & & \cap \quad \downarrow & & \downarrow & & \\
H^0(X, \mathcal{B}) & \longrightarrow & H^0(X, \mathcal{C}) & \longrightarrow & H^1(X, \mathcal{A}) & \longrightarrow & 0,
\end{array}$$

where $\check{H}^0_{\mathcal{B}}(\mathcal{U}, \mathcal{C})$ is the subspace of $\mathcal{C}(X) = H^0(X, \mathcal{C})$ consisting of sections which can be lifted in $\mathcal{B}$ over each U_α. As a consequence, the refinement mappings

$$H^1(\rho^\bullet) : \check{H}^1(\mathcal{U}, \mathcal{A}) \longrightarrow \check{H}^1(\mathcal{V}, \mathcal{A})$$

are also injective. □

5.C. Čech Cohomology on Paracompact Spaces

We will prove here that Čech cohomology theory coincides with the ordinary one on paracompact spaces.

(5.19) Proposition. *Assume that X is paracompact. If*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

is an exact sequence of sheaves, there is an exact sequence

$$\check{H}^q(X, \mathcal{A}) \longrightarrow \check{H}^q(X, \mathcal{B}) \longrightarrow \check{H}^q(X, \mathcal{C}) \longrightarrow \check{H}^{q+1}(X, \mathcal{A}) \longrightarrow \cdots$$

which is the direct limit of the exact sequences (5.10) *over all coverings* $\mathcal{U}$.

Proof. We have to show that the natural map

$$\varinjlim \check{H}^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \longrightarrow \varinjlim \check{H}^q(\mathcal{U}, \mathcal{C})$$

is an isomorphism. This follows easily from the following lemma, which says essentially that every cochain in $\mathcal{C}$ becomes liftable in $\mathcal{B}$ after a refinement of the covering.

(5.20) Lifting lemma. *Let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be an open covering of X and $c \in C^q(\mathcal{U}, \mathcal{C})$. If X is paracompact, there exists a finer covering $\mathcal{V} = (V_\beta)_{\beta \in J}$ and a refinement map $\rho : J \to I$ such that $\rho^q c \in C^q_{\mathcal{B}}(\mathcal{V}, \mathcal{C})$.*

Proof. Since $\mathcal{U}$ admits a locally finite refinement, we may assume that $\mathcal{U}$ itself is locally finite. There exists an open covering $\mathcal{W} = (W_\alpha)_{\alpha \in I}$ of X such that $\overline{W}_\alpha \subset U_\alpha$. For every point $x \in X$, we can select an open neighborhood V_x of x with the following properties:

a) if $x \in W_\alpha$, then $V_x \subset W_\alpha$;

b) if $x \in U_\alpha$ or if $V_x \cap W_\alpha \neq \emptyset$, then $V_x \subset U_\alpha$;

c) if $x \in U_{\alpha_0 \ldots \alpha_q}$, then $c_{\alpha_0 \ldots \alpha_q} \in C^q(U_{\alpha_0 \ldots \alpha_q}, \mathcal{C})$ admits a lifting in $\mathcal{B}(V_x)$.

Indeed, a) (resp. c)) can be achieved because x belongs to only finitely many sets W_α (resp. U_α), and so only finitely many sections of $\mathcal{C}$ have to be lifted in $\mathcal{B}$. b) can be achieved because x has a neighborhood V'_x that meets only finitely many sets U_α ; then we take

$$V_x \subset V'_x \cap \bigcap_{U_\alpha \ni x} U_\alpha \cap \bigcap_{U_\alpha \not\ni x} (V'_x \smallsetminus \overline{W}_\alpha).$$

Choose $\rho : X \to I$ such that $x \in W_{\rho(x)}$ for every x. Then a) implies $V_x \subset W_{\rho(x)}$, so $\mathcal{V} = (V_x)_{x \in X}$ is finer than $\mathcal{U}$, and ρ defines a refinement map. If $V_{x_0 \ldots x_q} \neq \emptyset$, we have

$$V_{x_0} \cap W_{\rho(x_j)} \supset V_{x_0} \cap V_{x_j} \neq \emptyset \quad \text{for} \quad 0 \le j \le q,$$

thus $V_{x_0} \subset U_{\rho(x_0) \ldots \rho(x_q)}$ by b). Now, c) implies that the section $c_{\rho(x_0) \ldots \rho(x_q)}$ admits a lifting in $\mathcal{B}(V_{x_0})$, and in particular in $\mathcal{B}(V_{x_0 \ldots x_q})$. Therefore $\rho^q c$ is liftable in $\mathcal{B}$. □

(5.21) Theorem. *If X is a paracompact space, the canonical morphism $\check{H}^q(X, \mathcal{A}) \simeq H^q(X, \mathcal{A})$ is an isomorphism.*

Proof. Argue by induction on q as in Leray's theorem, with the Čech cohomology exact sequence over $\mathcal{U}$ replaced by its direct limit in (5.16). □

In the next chapters, we will be concerned only by paracompact spaces, and most often in fact by manifolds that are either compact or countable at infinity. In these cases, we will not distinguish $H^q(X, \mathcal{A})$ and $\check{H}^q(X, \mathcal{A})$.

5.D. Alternate Čech Cochains

For explicit calculations, it is sometimes useful to consider a slightly modified Čech complex which has the advantage of producing much smaller cochain groups. If $\mathcal{A}$ is a sheaf and $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ an open covering of X, we let

$AC^q(\mathcal{U},\mathcal{A}) \subset C^q(\mathcal{U},\mathcal{A})$ be the subgroup of *alternate Čech cochains*, consisting of Čech cochains $c = (c_{\alpha_0\ldots\alpha_q})$ such that

$$(5.22)\qquad \begin{cases} c_{\alpha_0\ldots\alpha_q} = 0 & \text{if } \alpha_i = \alpha_j,\quad i \neq j, \\ c_{\alpha_{\sigma(0)}\ldots\alpha_{\sigma(q)}} = \varepsilon(\sigma)\, c_{\alpha_0\ldots\alpha_q} & \end{cases}$$

for any permutation σ of $\{1,\ldots,q\}$ of signature $\varepsilon(\sigma)$. Then the Čech differential (5.1) of an alternate cochain is still alternate, so $AC^\bullet(\mathcal{U},\mathcal{A})$ is a subcomplex of $C^\bullet(\mathcal{U},\mathcal{A})$. We are going to show that the inclusion induces an isomorphism in cohomology:

$$(5.23)\qquad H^q\big(AC^\bullet(\mathcal{U},\mathcal{A})\big) \simeq H^q\big(C^\bullet(\mathcal{U},\mathcal{A})\big) = \check{H}^q(\mathcal{U},\mathcal{A}).$$

Select a total ordering on the index set I. For each such ordering, we can define a projection $\pi^q : C^q(\mathcal{U},\mathcal{A}) \longrightarrow AC^q(\mathcal{U},\mathcal{A}) \subset C^q(\mathcal{U},\mathcal{A})$ by

$$c \longmapsto \text{alternate } \widetilde{c} \ \text{ such that } \widetilde{c}_{\alpha_0\ldots\alpha_q} = c_{\alpha_0\ldots\alpha_q} \ \text{ whenever } \ \alpha_0 < \ldots < \alpha_q.$$

As $\pi^\bullet$ is a morphism of complexes, it is enough to verify that $\pi^\bullet$ is homotopic to the identity on $C^\bullet(\mathcal{U},\mathcal{A})$. For a given multi-index $\alpha = (\alpha_0,\ldots,\alpha_q)$, which may contain repeated indices, there is a unique permutation $\big(m(0),\ldots,m(q)\big)$ of $(0,\ldots,q)$ such that

$$\alpha_{m(0)} \leq \ldots \leq \alpha_{m(q)} \quad \text{and} \quad m(l) < m(l+1) \quad \text{whenever} \quad \alpha_{m(l)} = \alpha_{m(l+1)}.$$

For $p \leq q$, we let $\varepsilon(\alpha,p)$ be the sign of the permutation

$$(0,\ldots,q) \longmapsto \big(m(0),\ldots,m(p-1),0,1,\ldots,\widehat{m(0)},\ldots,m(\widehat{p-}1),\ldots,q\big)$$

if the elements $\alpha_{m(0)},\ldots,\alpha_{m(p)}$ are all distinct, and $\varepsilon(\alpha,p) = 0$ otherwise. Finally, we set $h^q = 0$ for $q \leq 0$ and

$$(h^q c)_{\alpha_0\ldots\alpha_{q-1}} = \sum_{0\leq p\leq q-1} (-1)^p \varepsilon(\alpha,p)\, c_{\alpha_{m(0)}\ldots\alpha_{m(p)}\alpha_0\alpha_1\ldots\widehat{\alpha_{m(0)}}\ldots\widehat{\alpha_{m(p-1)}}\ldots\alpha_{q-1}}$$

for $q \geq 1$; observe that the index $\alpha_{m(p)}$ is repeated twice in the right hand side. A rather tedious calculation left to the reader shows that

$$(\delta^{q-1}h^q c + h^{q+1}\delta^q c)_{\alpha_0\ldots\alpha_q} = c_{\alpha_0\ldots\alpha_q} - \varepsilon(\alpha,q)\, c_{\alpha_{m(0)}\ldots\alpha_{m(q)}} = (c - \pi^q c)_{\alpha_0\ldots\alpha_q}.$$

An interesting consequence of the isomorphism (5.23) is the following:

(5.24) Proposition. *Let $\mathcal{A}$ be a sheaf on a paracompact space X. If X has arbitrarily fine open coverings or at least one acyclic open covering $\mathcal{U} = (U_\alpha)$ such that more than $n+1$ distinct sets $U_{\alpha_0},\ldots,U_{\alpha_n}$ have empty intersection, then $H^q(X,\mathcal{A}) = 0$ for $q > n$.*

Proof. In fact, we have $AC^q(\mathcal{U},\mathcal{A}) = 0$ for $q > n$. □

6. The De Rham-Weil Isomorphism Theorem

In § 3 we defined cohomology groups by means of the simplicial flabby resolution. We show here that any resolution by acyclic sheaves could have been used instead. Let $(\mathcal{L}^\bullet, d)$ be a resolution of a sheaf $\mathcal{A}$. We assume in addition that all $\mathcal{L}^q$ are acyclic on X, i.e. $H^s(X, \mathcal{L}^q) = 0$ for all $q \geq 0$ and $s \geq 1$. Set $\mathcal{Z}^q = \ker d^q$. Then $\mathcal{Z}^0 = \mathcal{A}$ and for every $q \geq 1$ we get a short exact sequence

$$0 \longrightarrow \mathcal{Z}^{q-1} \longrightarrow \mathcal{L}^{q-1} \xrightarrow{d^{q-1}} \mathcal{Z}^q \longrightarrow 0.$$

Theorem 3.5 yields an exact sequence

$$\text{(6.1)}\quad H^s(X, \mathcal{L}^{q-1}) \xrightarrow{d^{q-1}} H^s(X, \mathcal{Z}^q) \xrightarrow{\partial^{s,q}} H^{s+1}(X, \mathcal{Z}^{q-1}) \to H^{s+1}(X, \mathcal{L}^{q-1}) = 0.$$

If $s \geq 1$, the first group is also zero and we get an isomorphism

$$\partial^{s,q} : H^s(X, \mathcal{Z}^q) \xrightarrow{\simeq} H^{s+1}(X, \mathcal{Z}^{q-1}).$$

For $s = 0$ we have $H^0(X, \mathcal{L}^{q-1}) = \mathcal{L}^{q-1}(X)$ and $H^0(X, \mathcal{Z}^q) = \mathcal{Z}^q(X)$ is the q-cocycle group of $\mathcal{L}^\bullet(X)$, so the connecting map $\partial^{0,q}$ gives an isomorphism

$$H^q\big(\mathcal{L}^\bullet(X)\big) = \mathcal{Z}^q(X)/d^{q-1}\mathcal{L}^{q-1}(x) \xrightarrow{\widetilde{\partial}^{0,q}} H^1(X, \mathcal{Z}^{q-1}).$$

The composite map $\partial^{q-1,1} \circ \cdots \circ \partial^{1,q-1} \circ \widetilde{\partial}^{0,q}$ therefore defines an isomorphism

$$\text{(6.2)}\qquad H^q\big(\mathcal{L}^\bullet(X)\big) \xrightarrow{\widetilde{\partial}^{0,q}} H^1(X, \mathcal{Z}^{q-1}) \xrightarrow{\partial^{1,q-1}} \cdots \xrightarrow{\partial^{q-1,1}} H^q(X, \mathcal{Z}^0) = H^q(X, \mathcal{A}).$$

This isomorphism behaves functorially with respect to morphisms of resolutions. Our assertion means that for every sheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ and every morphism of resolutions $\varphi^\bullet : \mathcal{L}^\bullet \longrightarrow \mathcal{M}^\bullet$, there is a commutative diagram

$$\text{(6.3)}\qquad \begin{array}{ccc} H^s\big(\mathcal{L}^\bullet(X)\big) & \longrightarrow & H^s(X, \mathcal{A}) \\ \downarrow H^s(\varphi^\bullet) & & \downarrow H^s(\varphi) \\ H^s\big(\mathcal{M}^\bullet(X)\big) & \longrightarrow & H^s(X, \mathcal{B}). \end{array}$$

If $\mathcal{W}^q = \ker\big(d^q : \mathcal{M}^q \to \mathcal{M}^{q+1}\big)$, the functoriality comes from the fact that we have commutative diagrams

$$\begin{array}{ccccccccc} 0 \to & \mathcal{Z}^{q-1} & \to & \mathcal{L}^{q-1} & \to & \mathcal{Z}^q & \to 0\,, \\ & \downarrow \varphi^{q-1} & & \downarrow \varphi^{q-1} & & \downarrow \varphi^q & \\ 0 \to & \mathcal{W}^{q-1} & \to & \mathcal{M}^{q-1} & \to & \mathcal{W}^q & \to 0\,, \end{array} \qquad \begin{array}{ccc} H^s(X, \mathcal{Z}^q) & \xrightarrow{\partial^{s,q}} & H^{s+1}(X, \mathcal{Z}^{q-1}) \\ \downarrow H^s(\varphi^q) & & \downarrow H^{s+1}(\varphi^{q-1}) \\ H^s(X, \mathcal{W}^q) & \xrightarrow{\partial^{s,q}} & H^{s+1}(X, \mathcal{W}^{q-1}). \end{array}$$

(6.4) De Rham-Weil isomorphism theorem. *If $(\mathcal{L}^\bullet, d)$ is a resolution of $\mathcal{A}$ by sheaves $\mathcal{L}^q$ which are acyclic on X, there is a functorial isomorphism*

$$H^q\big(\mathcal{L}^\bullet(X)\big) \longrightarrow H^q(X, \mathcal{A}). \qquad \square$$

(6.5) Example: De Rham cohomology. Let X be a n-dimensional paracompact differential manifold. Consider the resolution

$$0 \to \mathbb{R} \to \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \to \cdots \to \mathcal{E}^q \xrightarrow{d} \mathcal{E}^{q+1} \to \cdots \to \mathcal{E}^n \to 0$$

given by the exterior derivative d acting on germs of C^∞ differential q-forms (c.f. Example 2.2). The *De Rham cohomology groups* of X are precisely

$$(6.6) \quad H^q_{\mathrm{DR}}(X, \mathbb{R}) = H^q\big(\mathcal{E}^\bullet(X)\big).$$

All sheaves $\mathcal{E}^q$ are $\mathcal{E}_X$-modules, so $\mathcal{E}^q$ is acyclic by Cor. 4.19. Therefore, we get an isomorphism

$$(6.7) \quad H^q_{\mathrm{DR}}(X, \mathbb{R}) \xrightarrow{\simeq} H^q(X, \mathbb{R})$$

from the De Rham cohomology onto the cohomology with values in the constant sheaf $\mathbb{R}$. Instead of using C^∞ differential forms, one can consider the resolution of $\mathbb{R}$ given by the exterior derivative d acting on currents:

$$0 \to \mathbb{R} \to \mathcal{D}'_n \xrightarrow{d} \mathcal{D}'_{n-1} \to \cdots \to \mathcal{D}'_{n-q} \xrightarrow{d} \mathcal{D}'_{n-q-1} \to \cdots \to \mathcal{D}'_0 \to 0.$$

The sheaves $\mathcal{D}'_q$ are also $\mathcal{E}_X$-modules, hence acyclic. Thanks to (6.3), the inclusion $\mathcal{E}^q \subset \mathcal{D}'_{n-q}$ induces an isomorphism

$$(6.8) \quad H^q\big(\mathcal{E}^\bullet(X)\big) \simeq H^q\big(\mathcal{D}'_{n-\bullet}(X)\big),$$

both groups being isomorphic to $H^q(X, \mathbb{R})$. The isomorphism between cohomology of differential forms and singular cohomology (another topological invariant) was first established by (De Rham 1931). The above proof follows essentially the method given by (Weil 1952), in a more abstract setting. As we will see, the isomorphism (6.7) can be put under a very explicit form in terms of Čech cohomology. We need a simple lemma.

(6.9) Lemma. *Let X be a paracompact differentiable manifold. There are arbitrarily fine open coverings $\mathfrak{U} = (U_\alpha)$ such that all intersections $U_{\alpha_0 \ldots \alpha_q}$ are diffeomorphic to convex sets.*

Proof. Select locally finite coverings $\Omega'_j \subset\subset \Omega_j$ of X by open sets diffeomorphic to concentric euclidean balls in $\mathbb{R}^n$. Let us denote by τ_{jk} the transition diffeomorphism from the coordinates in Ω_k to those in Ω_j. For any point $a \in \Omega'_j$, the function $x \mapsto |x-a|^2$ computed in terms of the coordinates of Ω_j becomes $|\tau_{jk}(x) - \tau_{jk}(a)|^2$ on any patch $\Omega_k \ni a$. It is clear that these functions are strictly convex at a, thus there is a euclidean ball $B(a, \varepsilon) \subset \Omega'_j$ such that all functions are strictly convex on $B(a, \varepsilon) \cap \Omega'_k \subset \Omega_k$ (only a finite number of indices k is involved). Now, choose $\mathfrak{U}$ to be a (locally finite) covering of X by such balls $U_\alpha = B(a_\alpha, \varepsilon_\alpha)$ with $U_\alpha \subset \Omega'_{\rho(\alpha)}$. Then the intersection $U_{\alpha_0 \ldots \alpha_q}$ is defined in Ω_k, $k = \rho(\alpha_0)$, by the equations

$$|\tau_{jk}(x) - \tau_{jk}(a_{\alpha_m})|^2 < \varepsilon_{\alpha_m}^2$$

where $j = \rho(\alpha_m)$, $0 \le m \le q$. Hence the intersection is convex in the open coordinate chart $\Omega_{\rho(\alpha_0)}$. □

Let Ω be an open subset of $\mathbb{R}^n$ which is starshaped with respect to the origin. Then the De Rham complex $\mathbb{R} \longrightarrow \mathcal{E}^\bullet(\Omega)$ is acyclic: indeed, Poincaré's lemma yields a homotopy operator $k^q : \mathcal{E}^q(\Omega) \longrightarrow \mathcal{E}^{q-1}(\Omega)$ such that

$$k^q f_x(\xi_1, \ldots, \xi_{q-1}) = \int_0^1 t^{q-1} f_{tx}(x, \xi_1, \ldots, \xi_{q-1})\, dt, \quad x \in \Omega, \quad \xi_j \in \mathbb{R}^n,$$
$$k^0 f = f(0) \in \mathbb{R} \quad \text{for} \quad f \in \mathcal{E}^0(\Omega).$$

Hence $H^q_{\mathrm{DR}}(\Omega, \mathbb{R}) = 0$ for $q \ge 1$. Now, consider the resolution $\mathcal{E}^\bullet$ of the constant sheaf $\mathbb{R}$ on X, and apply the proof of the De Rham-Weil isomorphism theorem to Čech cohomology groups over a covering $\mathcal{U}$ chosen as in Lemma 6.9. Since the intersections $U_{\alpha_0 \ldots \alpha_s}$ are convex, all Čech cochains in $C^s(\mathcal{U}, \mathcal{Z}^q)$ are liftable in $\mathcal{E}^{q-1}$ by means of k^q. Hence for all $s = 1, \ldots, q$ we have isomorphisms $\partial^{s,q-s} : \check{H}^s(\mathcal{U}, \mathcal{Z}^{q-s}) \longrightarrow \check{H}^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1})$ for $s \ge 1$ and we get a resulting isomorphism

$$\partial^{q-1,1} \circ \cdots \circ \partial^{1,q-1} \circ \widetilde{\partial}^{0,q} : H^q_{\mathrm{DR}}(X, \mathbb{R}) \xrightarrow{\simeq} \check{H}^q(\mathcal{U}, \mathbb{R})$$

We are going to compute the connecting homomorphisms $\partial^{s,q-s}$ and their inverses explicitly.

Let c in $C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ such that $\delta^s c = 0$. As $c_{\alpha_0 \ldots \alpha_s}$ is d-closed, we can write $c = d(k^{q-s} c)$ where the cochain $k^{q-s} c \in C^s(\mathcal{U}, \mathcal{E}^{q-s-1})$ is defined as the family of sections $k^{q-s} c_{\alpha_0 \ldots \alpha_s} \in \mathcal{E}^{q-s-1}(U_{\alpha_0 \ldots \alpha_s})$. Then $d(\delta^s k^{q-s} c) = \delta^s(d k^{q-s} c) = \delta^s c = 0$ and

$$\partial^{s,q-s}\{c\} = \{\delta^s k^{q-s} c\} \in \check{H}^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1}).$$

The isomorphism $H^q_{\mathrm{DR}}(X, \mathbb{R}) \xrightarrow{\simeq} \check{H}^q(\mathcal{U}, \mathbb{R})$ is thus defined as follows: to the cohomology class $\{f\}$ of a closed q-form $f \in \mathcal{E}^q(X)$, we associate the cocycle $(c^0_\alpha) = (f_{\restriction U_\alpha}) \in C^0(\mathcal{U}, \mathcal{Z}^q)$, then the cocycle

$$c^1_{\alpha\beta} = k^q c^0_\beta - k^q c^0_\alpha \in C^1(\mathcal{U}, \mathcal{Z}^{q-1}),$$

and by induction cocycles $(c^s_{\alpha_0 \ldots \alpha_s}) \in C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ given by

$$(6.10) \quad c^{s+1}_{\alpha_0 \ldots \alpha_{s+1}} = \sum_{0 \le j \le s+1} (-1)^j\, k^{q-s} c^s_{\alpha_0 \ldots \widehat{\alpha_j} \ldots \alpha_{s+1}} \qquad \text{on } U_{\alpha_0 \ldots \alpha_{s+1}}.$$

The image of $\{f\}$ in $\check{H}^q(\mathcal{U}, \mathbb{R})$ is the class of the q-cocycle $(c^q_{\alpha_0 \ldots \alpha_q})$ in $C^q(\mathcal{U}, \mathbb{R})$.

Conversely, let (ψ_α) be a C^∞ partition of unity subordinate to $\mathcal{U}$. Any Čech cocycle $c \in C^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1})$ can be written $c = \delta^s \gamma$ with $\gamma \in C^s(\mathcal{U}, \mathcal{E}^{q-s-1})$ given by

$$\gamma_{\alpha_0\ldots\alpha_s} = \sum_{\nu\in I} \psi_\nu\, c_{\nu\alpha_0\ldots\alpha_s},$$

(c.f. Prop. 5.11 b)), thus $\{c'\} = (\partial^{s,q-s})^{-1}\{c\}$ can be represented by the cochain $c' = d\gamma \in C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ such that

$$c'_{\alpha_0\ldots\alpha_s} = \sum_{\nu\in I} d\psi_\nu \wedge c_{\nu\alpha_0\ldots\alpha_s} = (-1)^{q-s-1}\sum_{\nu\in I} c_{\nu\alpha_0\ldots\alpha_s} \wedge d\psi_\nu.$$

For a reason that will become apparent later, we shall in fact modify the sign of our isomorphism $\partial^{s,q-s}$ by the factor $(-1)^{q-s-1}$. Starting from a class $\{c\} \in \check{H}^q(\mathcal{U}, \mathbb{R})$, we obtain inductively $\{b\} \in \check{H}^0(\mathcal{U}, \mathcal{Z}^q)$ such that

$$(6.11)\quad b_{\alpha_0} = \sum_{\nu_0,\ldots,\nu_{q-1}} c_{\nu_0\ldots\nu_{q-1}\alpha_0}\, d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{q-1}} \quad \text{on } U_{\alpha_0},$$

corresponding to $\{f\} \in H^q_{\mathrm{DR}}(X, \mathbb{R})$ given by the explicit formula

$$(6.12)\quad f = \sum_{\nu_q} \psi_{\nu_q} b_{\nu_q} = \sum_{\nu_0,\ldots,\nu_q} c_{\nu_0\ldots\nu_q}\, \psi_{\nu_q} d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{q-1}}.$$

The choice of sign corresponds to (6.2) multiplied by $(-1)^{q(q-1)/2}$.

(6.13) Example: Dolbeault cohomology groups. Let X be a $\mathbb{C}$-analytic manifold of dimension n, and let $\mathcal{E}^{p,q}$ be the sheaf of germs of C^∞ differential forms of type (p,q) with complex values. For every $p = 0, 1, \ldots, n$, the Dolbeault-Grothendieck Lemma I-2.9 shows that $(\mathcal{E}^{p,\bullet}, d'')$ is a resolution of the sheaf Ω^p_X of germs of holomorphic forms of degree p on X. The *Dolbeault cohomology groups* of X already considered in chapter 1 can be defined by

$$(6.14)\quad H^{p,q}(X, \mathbb{C}) = H^q\big(\mathcal{E}^{p,\bullet}(X)\big).$$

The sheaves $\mathcal{E}^{p,q}$ are acyclic, so we get the *Dolbeault isomorphism theorem*, originally proved in (Dolbeault 1953), which relates d''-cohomology and sheaf cohomology:

$$(6.15)\quad H^{p,q}(X, \mathbb{C}) \xrightarrow{\simeq} H^q(X, \Omega^p_X).$$

The case $p = 0$ is especially interesting:

$$(6.16)\quad H^{0,q}(X, \mathbb{C}) \simeq H^q(X, \mathcal{O}_X).$$

As in the case of De Rham cohomology, there is an inclusion $\mathcal{E}^{p,q} \subset \mathcal{D}'_{n-p,n-q}$ and the complex of currents $(\mathcal{D}'_{n-p,n-\bullet}, d'')$ defines also a resolution of Ω^p_X. Hence there is an isomorphism:

$$(6.17)\quad H^{p,q}(X, \mathbb{C}) = H^q\big(\mathcal{E}^{p,\bullet}(X)\big) \simeq H^q\big(\mathcal{D}'_{n-p,n-\bullet}(X)\big).$$

7. Cohomology with Supports

As its name indicates, cohomology with supports deals with sections of sheaves having supports in prescribed closed sets. We first introduce what is an admissible family of supports.

(7.1) Definition. *A family of supports on a topological space X is a collection Φ of closed subsets of X with the following two properties:*

a) *If F, $F' \in \Phi$, then $F \cup F' \in \Phi$;*

b) *If $F \in \Phi$ and $F' \subset F$ is closed, then $F' \in \Phi$.*

(7.2) Example. Let S be an arbitrary subset of X. Then the family of all closed subsets of X contained in S is a family of supports.

(7.3) Example. The collection of all compact (non necessarily Hausdorff) subsets of X is a family of supports, which will be denoted simply c in the sequel. □

(7.4) Definition. *For any sheaf $\mathcal{A}$ and any family of supports Φ on X, $\mathcal{A}_\Phi(X)$ will denote the set of all sections $f \in \mathcal{A}(X)$ such that* $\mathrm{Supp}\, f \in \Phi$.

It is clear that $\mathcal{A}_\Phi(X)$ is a subgroup of $\mathcal{A}(X)$. We can now introduce cohomology groups with arbitrary supports.

(7.5) Definition. *The cohomology groups of $\mathcal{A}$ with supports in Φ are*

$$H^q_\Phi(X, \mathcal{A}) = H^q\big(\mathcal{A}^{[\bullet]}_\Phi(X)\big).$$

The cohomology groups with compact supports will be denoted $H^q_c(X, \mathcal{A})$ and the cohomology groups with supports in a subset S will be denoted $H^q_S(X, \mathcal{A})$.

In particular $H^0_\Phi(X, \mathcal{A}) = \mathcal{A}_\Phi(X)$. If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is an exact sequence, there are corresponding exact sequences

$$(7.6)\qquad \begin{array}{l} 0 \longrightarrow \mathcal{A}^{[q]}_\Phi(X) \longrightarrow \mathcal{B}^{[q]}_\Phi(X) \longrightarrow \mathcal{C}^{[q]}_\Phi(X) \longrightarrow \cdots \\ H^q_\Phi(X, \mathcal{A}) \longrightarrow H^q_\Phi(X, \mathcal{B}) \longrightarrow H^q_\Phi(X, \mathcal{C}) \longrightarrow H^{q+1}_\Phi(X, \mathcal{A}) \longrightarrow \cdots. \end{array}$$

When $\mathcal{A}$ is flabby, there is an exact sequence

$$(7.7)\qquad 0 \longrightarrow \mathcal{A}_\Phi(X) \longrightarrow \mathcal{B}_\Phi(X) \longrightarrow \mathcal{C}_\Phi(X) \longrightarrow 0$$

and every $g \in \mathcal{C}_\Phi(X)$ can be lifted to $v \in \mathcal{B}_\Phi(X)$ without enlarging the support: apply the proof of Prop. 4.3 to a maximal lifting which extends $w = 0$ on $W = \complement(\mathrm{Supp}\, g)$. It follows that a flabby sheaf $\mathcal{A}$ is Φ-acyclic, i.e. $H^q_\Phi(X, \mathcal{A}) = 0$ for all $q \geq 1$. Similarly, assume that X is paracompact and that $\mathcal{A}$ is soft, and suppose that Φ has the following additional property: every set $F \in \Phi$ has a neighborhood $G \in \Phi$. An adaptation of the proofs of Prop. 4.3 and 4.13 shows that (7.7) is again exact. Therefore every soft sheaf is also Φ-acyclic in that case.

As a consequence of (7.6), any resolution $\mathcal{L}^\bullet$ of $\mathcal{A}$ by Φ-acyclic sheaves provides a canonical De Rham-Weil isomorphism

$$(7.8)\quad H^q\big(\mathcal{L}^\bullet_\Phi(X)\big) \longrightarrow H^q_\Phi(X, \mathcal{A}).$$

(7.9) Example: De Rham cohomology with compact support. In the special case of the De Rham resolution $\mathbb{R} \longrightarrow \mathcal{E}^\bullet$ on a paracompact manifold, we get an isomorphism

$$(7.10)\quad H^q_{\mathrm{DR},c}(X, \mathbb{R}) := H^q\big((\mathcal{D}^\bullet(X)\big) \xrightarrow{\simeq} H^q_c(X, \mathbb{R}),$$

where $\mathcal{D}^q(X)$ is the space of smooth differential q-forms with compact support in X. These groups are called the *De Rham cohomology groups* of X with compact support. When X is oriented, $\dim X = n$, we can also consider the complex of compactly supported currents:

$$0 \longrightarrow \mathcal{E}'_n(X) \xrightarrow{d} \mathcal{E}'_{n-1}(X) \longrightarrow \cdots \longrightarrow \mathcal{E}'_{n-q}(X) \xrightarrow{d} \mathcal{E}'_{n-q-1}(X) \longrightarrow \cdots.$$

Note that $\mathcal{D}^\bullet(X)$ and $\mathcal{E}'_{n-\bullet}(X)$ are respectively the subgroups of compactly supported sections in $\mathcal{E}^\bullet$ and $\mathcal{D}'_{n-\bullet}$, both of which are acyclic resolutions of $\mathbb{R}$. Therefore the inclusion $\mathcal{D}^\bullet(X) \subset \mathcal{E}'_{n-\bullet}(X)$ induces an isomorphism

$$(7.11)\quad H^q\big(\mathcal{D}^\bullet(X)\big) \simeq H^q\big(\mathcal{E}'_{n-\bullet}(X)\big),$$

both groups being isomorphic to $H^q_c(X, \mathbb{R})$. □

Now, we concentrate our attention on cohomology groups with compact support. We assume until the end of this section that X is a *locally compact* space.

(7.12) Proposition. *There is an isomorphism*

$$H^q_c(X, \mathcal{A}) = \varinjlim_{U \subset\subset X} H^q(\overline{U}, \mathcal{A}_U)$$

where $\mathcal{A}_U$ is the sheaf of sections of $\mathcal{A}$ vanishing on $X \smallsetminus U$ (c.f. §3).

Proof. By definition

$$H^q_c(X, \mathcal{A}) = H^q\big(\mathcal{A}^{[\bullet]}_c(X)\big) = \varinjlim_{U \subset\subset X} H^q\big((\mathcal{A}^{[\bullet]})_U(\overline{U})\big)$$

since sections of $(\mathcal{A}^{[\bullet]})_U(\overline{U})$ can be extended by 0 on $X \smallsetminus U$. However, $(\mathcal{A}^{[\bullet]})_U$ is a resolution of $\mathcal{A}_U$ and $(\mathcal{A}^{[q]})_U$ is a $\mathbb{Z}^{[q]}$-module, so it is acyclic on $\overline{U}$. The De Rham-Weil isomorphism theorem implies

$$H^q\big((\mathcal{A}^{[\bullet]})_U(\overline{U})\big) = H^q(\overline{U}, \mathcal{A}_U)$$

and the proposition follows. The reader should take care of the fact that $(\mathcal{A}^{[q]})_U$ does not coincide in general with $(\mathcal{A}_U)^{[q]}$. $\square$

The cohomology groups with compact support can also be defined by means of Čech cohomology.

(7.13) Definition. *Assume that X is a separable locally compact space. If $\mathcal{U} = (U_\alpha)$ is a locally finite covering of X by relatively compact open subsets, we let $C^q_c(\mathcal{U}, \mathcal{A})$ be the subgroups of cochains such that only finitely many coefficients $c_{\alpha_0\ldots\alpha_q}$ are non zero. The Čech cohomology groups with compact support are defined by*

$$\check{H}^q_c(\mathcal{U}, \mathcal{A}) = H^q\big(C^\bullet_c(\mathcal{U}, \mathcal{A})\big)$$
$$\check{H}^q_c(X, \mathcal{A}) = \varinjlim_{\mathcal{U}} H^q\big(C^\bullet_c(\mathcal{U}, \mathcal{A})\big)$$

For such coverings $\mathcal{U}$, Formula (5.13) yields canonical morphisms

$$(7.14)\quad H^q(\lambda^\bullet)\ :\ \check{H}^q_c(\mathcal{U}, \mathcal{A}) \longrightarrow H^q_c(X, \mathcal{A}).$$

Now, the lifting Lemma 5.20 is valid for cochains with compact supports, and the same proof as the one given in §5 gives an isomorphism

$$(7.15)\quad \check{H}^q_c(X, \mathcal{A}) \simeq H^q_c(X, \mathcal{A}).$$

8. Cup Product

Let $\mathcal{R}$ be a sheaf of commutative rings and $\mathcal{A}$, $\mathcal{B}$ sheaves of $\mathcal{R}$-modules on a space X. We denote by $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}$ the sheaf on X defined by

$$(8.1)\quad (\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})_x = \mathcal{A}_x \otimes_{\mathcal{R}_x} \mathcal{B}_x,$$

with the weakest topology such that the range of any section given by $\mathcal{A}(U) \otimes_{\mathcal{R}(U)} \mathcal{B}(U)$ is open in $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}$ for any open set $U \subset X$. Given $f \in \mathcal{A}^{[p]}_x$ and $g \in \mathcal{B}^{[q]}_x$, the *cup product* $f \smile g \in (\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})^{[p+q]}_x$ is defined by

$$(8.2)\quad f \smile g(x_0, \ldots, x_{p+q}) = f(x_0, \ldots, x_p)(x_{p+q}) \otimes g(x_p, \ldots, x_{p+q}).$$

A simple computation shows that

$$(8.3)\quad d^{p+q}(f \smile g) = (d^p f) \smile g + (-1)^p\, f \smile (d^q g).$$

In particular, $f \smile g$ is a cocycle if f, g are cocycles, and we have

$$(f+d^{p-1}f')\smile(g+d^{q-1}g')=f\smile g+d^{p+q-1}\big(f'\smile g+(-1)^p f\smile g'+f'\smile dg'\big).$$

Consequently, there is a well defined $\mathcal{R}(X)$-bilinear morphism

$$(8.4)\quad H^p(X,\mathcal{A})\times H^q(X,\mathcal{B})\longrightarrow H^{p+q}(X,\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B})$$

which maps a pair $(\{f\},\{g\})$ to $\{f\smile g\}$.

Let $0\to\mathcal{B}\to\mathcal{B}'\to\mathcal{B}''\to 0$ be an exact sequence of sheaves. Assume that the sequence obtained after taking the tensor product by $\mathcal{A}$ is also exact:

$$0\longrightarrow\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B}\longrightarrow\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B}'\longrightarrow\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B}''\longrightarrow 0.$$

Then we obtain connecting homomorphisms

$$\begin{aligned}\partial^q &: H^q(X,\mathcal{B}'')\longrightarrow H^{q+1}(X,\mathcal{B}),\\ \partial^q &: H^q(X,\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B}'')\longrightarrow H^{q+1}(X,\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B}).\end{aligned}$$

For every $\alpha\in H^p(X,\mathcal{A})$, $\beta''\in H^q(X,\mathcal{B}'')$ we have

$$(8.5)\qquad \partial^{p+q}(\alpha\smile\beta'')=(-1)^p\,\alpha\smile(\partial^q\beta''),$$
$$(8.5')\qquad \partial^{p+q}(\beta''\smile\alpha)=(\partial^q\beta'')\smile\alpha,$$

where the second line corresponds to the tensor product of the exact sequence by $\mathcal{A}$ on the right side. These formulas are deduced from (8.3) applied to a representant $f\in\mathcal{A}^{[p]}(X)$ of α and to a lifting $g'\in\mathcal{B}'^{[q]}(X)$ of a representative g'' of β'' (note that $d^pf=0$).

(8.6) Associativity and anticommutativity. *Let $i:\mathcal{A}\otimes_{\mathcal{R}}\mathcal{B}\longrightarrow\mathcal{B}\otimes_{\mathcal{R}}\mathcal{A}$ be the canonical isomorphism $s\otimes t\mapsto t\otimes s$. For all $\alpha\in H^p(X,\mathcal{A})$, $\beta\in H^q(X,\mathcal{B})$ we have*

$$\beta\smile\alpha=(-1)^{pq}\,i(\alpha\smile\beta).$$

If $\mathcal{C}$ is another sheaf of $\mathcal{R}$-modules and $\gamma\in H^r(X,\mathcal{C})$, then

$$(\alpha\smile\beta)\smile\gamma=\alpha\smile(\beta\smile\gamma).$$

Proof. The associativity property is easily seen to hold already for all cochains

$$(f\smile g)\smile h=f\smile(g\smile h),\quad f\in\mathcal{A}_x^{[p]},\quad g\in\mathcal{B}_x^{[q]},\quad h\in\mathcal{C}_x^{[r]}.$$

The commutation property is obvious for $p=q=0$, and can be proved in general by induction on $p+q$. Assume for example $q\geq 1$. Consider the exact sequence

$$0\longrightarrow\mathcal{B}\longrightarrow\mathcal{B}'\longrightarrow\mathcal{B}''\longrightarrow 0$$

where $\mathcal{B}'=\mathcal{B}^{[0]}$ and $\mathcal{B}''=\mathcal{B}^{[0]}/\mathcal{B}$. This exact sequence splits on each stalk (but not globally, nor even locally): a left inverse $\mathcal{B}_x^{[0]}\to\mathcal{B}_x$ of the inclusion is given by $g \mapsto g(x)$. Hence the sequence remains exact after taking the tensor product with $\mathcal{A}$. Now, as $\mathcal{B}'$ is acyclic, the connecting homomorphism $H^{q-1}(X, \mathcal{B}'') \longrightarrow H^q(X, \mathcal{B})$ is onto, so there is $\beta'' \in H^{q-1}(X, \mathcal{B}'')$ such that $\beta = \partial^{q-1}\beta''$. Using (8.5′), (8.5) and the induction hypothesis, we find

$$\begin{aligned}\beta \smile \alpha &= \partial^{p+q-1}(\beta'' \smile \alpha) = \partial^{p+q-1}\big((-1)^{p(q-1)}\, i(\alpha \smile \beta'')\big)\\ &= (-1)^{p(q-1)}\, i\partial^{p+q-1}(\alpha \smile \beta'') = (-1)^{p(q-1)}(-1)^p\, i(\alpha \smile \beta). \qquad \square\end{aligned}$$

Theorem 8.6 shows in particular that $H^\bullet(X, \mathcal{R})$ is a graded associative and supercommutative algebra, i.e. $\beta \smile \alpha = (-1)^{pq}\, \alpha \smile \beta$ for all classes $\alpha \in H^p(X, \mathcal{R})$, $\beta \in H^q(X, \mathcal{R})$. If $\mathcal{A}$ is a $\mathcal{R}$-module, then $H^\bullet(X, \mathcal{A})$ is a graded $H^\bullet(X, \mathcal{R})$-module.

(8.7) Remark. The cup product can also be defined for Čech cochains. Given $c \in C^p(\mathcal{U}, \mathcal{A})$ and $c' \in C^q(\mathcal{U}, \mathcal{B})$, the cochain $c \smile c' \in C^{p+q}(\mathcal{U}, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})$ is defined by

$$(c \smile c')_{\alpha_0 \ldots \alpha_{p+q}} = c_{\alpha_0 \ldots \alpha_p} \otimes c'_{\alpha_p \ldots \alpha_{p+q}} \quad \text{on} \quad U_{\alpha_0 \ldots \alpha_{p+q}}.$$

Straightforward calculations show that

$$\delta^{p+q}(c \smile c') = (\delta^p c) \smile c' + (-1)^p\, c \smile (\delta^q c')$$

and that there is a commutative diagram

$$\begin{array}{ccc}\check{H}^p(\mathcal{U}, \mathcal{A}) \times \check{H}^q(\mathcal{U}, \mathcal{B}) & \longrightarrow & \check{H}^{p+q}(\mathcal{U}, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}) \\ \downarrow & & \downarrow \\ H^p(X, \mathcal{A}) \times H^q(X, \mathcal{B}) & \longrightarrow & H^{p+q}(X, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B}),\end{array}$$

where the vertical arrows are the canonical morphisms $H^s(\lambda^\bullet)$ of (5.14). Note that the commutativity already holds in fact on cochains.

(8.8) Remark. Let Φ and Ψ be families of supports on X. Then $\Phi \cap \Psi$ is again a family of supports, and Formula (8.2) defines a bilinear map

$$(8.9) \quad H^p_\Phi(X, \mathcal{A}) \times H^q_\Psi(X, \mathcal{B}) \longrightarrow H^{p+q}_{\Phi \cap \Psi}(X, \mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})$$

on cohomology groups with supports. This follows immediately from the fact that $\operatorname{Supp}(f \smile g) \subset \operatorname{Supp} f \cap \operatorname{Supp} g$.

(8.10) Remark. Assume that X is a differentiable manifold. Then the cohomology complex $H^\bullet_{\mathrm{DR}}(X, \mathbb{R})$ has a natural structure of supercommutative algebra given by the wedge product of differential forms. We shall prove the following compatibility statement:

Let $H^q(X, \mathbb{R}) \longrightarrow H^q_{\mathrm{DR}}(X, \mathbb{R})$ be the De Rham-Weil isomorphism given by Formula (6.12)*. Then the cup product $c' \smile c''$ is mapped on the wedge product $f' \wedge f''$ of the corresponding De Rham cohomology classes.*

By remark 8.7, we may suppose that c', c'' are Čech cohomology classes of respective degrees p, q. Formulas (6.11) and (6.12) give

$$f'_{\restriction U_{\nu_p}} = \sum_{\nu_0,\ldots,\nu_{p-1}} c'_{\nu_0\ldots\nu_{p-1}\nu_p}\, d\psi_{\nu_0}\wedge\ldots\wedge d\psi_{\nu_{p-1}},$$
$$f'' = \sum_{\nu_p,\ldots,\nu_{p+q}} c''_{\nu_p\ldots\nu_{p+q}}\, \psi_{\nu_{p+q}} d\psi_{\nu_p}\wedge\ldots\wedge d\psi_{\nu_{p+q-1}}.$$

We get therefore

$$f'\wedge f'' = \sum_{\nu_0,\ldots,\nu_{p+q}} c'_{\nu_0\ldots\nu_p}\, c''_{\nu_p\ldots\nu_{p+q}}\, \psi_{\nu_{p+q}} d\psi_{\nu_0}\wedge\ldots\wedge \psi_{\nu_{p+q-1}},$$

which is precisely the image of $c \smile c'$ in the De Rham cohomology. □

9. Inverse Images and Cartesian Products

9.A. Inverse Image of a Sheaf

Let $F : X \to Y$ be a continuous map between topological spaces X, Y, and let $\pi : \mathcal{A} \to Y$ be a sheaf of abelian groups. Recall that *inverse image* $F^{-1}\mathcal{A}$ is defined as the sheaf-space

$$F^{-1}\mathcal{A} = \mathcal{A}\times_Y X = \big\{(s,x)\,;\,\pi(s) = F(x)\big\}$$

with projection $\pi' = \mathrm{pr}_2 : F^{-1}\mathcal{A} \to X$. The stalks of $F^{-1}\mathcal{A}$ are given by

(9.1) $\quad (F^{-1}\mathcal{A})_x = \mathcal{A}_{F(x)},$

and the sections $\sigma \in F^{-1}\mathcal{A}(U)$ can be considered as continuous mappings $\sigma : U \to \mathcal{A}$ such that $\pi\circ\sigma = F$. In particular, any section $s \in \mathcal{A}(U)$ has a *pull-back*

(9.2) $\quad F^\star s = s\circ F \in F^{-1}\mathcal{A}\big(F^{-1}(U)\big).$

For any $v \in \mathcal{A}^{[q]}_y$, we define $F^\star v \in (F^{-1}\mathcal{A})^{[q]}_x$ by

(9.3) $\quad F^\star v(x_0,\ldots,x_q) = v\big(F(x_0),\ldots,F(x_q)\big) \in (F^{-1}\mathcal{A})_{x_q} = \mathcal{A}_{F(x_q)}$

for $x_0 \in V(x)$, $x_1 \in V(x_0),\ldots,x_q \in V(x_0,\ldots,x_{q-1})$. We get in this way a morphism of complexes $F^\star : \mathcal{A}^{[\bullet]}(Y) \longrightarrow (F^{-1}\mathcal{A})^{[\bullet]}(X)$. On cohomology groups, we thus have an induced morphism

(9.4) $\quad F^\star \;:\; H^q(Y,\mathcal{A}) \longrightarrow H^q(X, F^{-1}\mathcal{A}).$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on X. Thanks to property (9.1), there is an exact sequence

$$0 \longrightarrow F^{-1}\mathcal{A} \longrightarrow F^{-1}\mathcal{B} \longrightarrow F^{-1}\mathcal{C} \longrightarrow 0.$$

It is clear on the definitions that the morphism $F^\star$ in (9.4) commutes with the associated cohomology exact sequences. Also, $F^\star$ preserves the cup product, i.e. $F^\star(\alpha \smile \beta) = F^\star\alpha \smile F^\star\beta$ whenever α, β are cohomology classes with values in sheaves $\mathcal{A}$, $\mathcal{B}$ on X. Furthermore, if $G : Y \to Z$ is a continuous map, we have

$$(9.5) \quad (G \circ F)^\star = F^\star \circ G^\star.$$

(9.6) Remark. Similar definitions can be given for Čech cohomology. If $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ is an open covering of Y, then $F^{-1}\mathcal{U} = \big(F^{-1}(U_\alpha)\big)_{\alpha\in I}$ is an open covering of X. For $c \in C^q(\mathcal{U}, \mathcal{A})$, we set

$$(F^\star c)_{\alpha_0\ldots\alpha_q} = c_{\alpha_0\ldots\alpha_q} \circ F \in C^q(F^{-1}\mathcal{U}, F^{-1}\mathcal{A}).$$

This definition is obviously compatible with the morphism from Čech cohomology to ordinary cohomology.

(9.7) Remark. Let Φ be a family of supports on Y. We define $F^{-1}\Psi$ to be the family of closed sets $K \subset X$ such that $F(K)$ is contained in some set $L \in \Psi$. Then (9.4) can be generalized in the form

$$(9.8) \quad F^\star \ : \ H^q_\Psi(Y, \mathcal{A}) \longrightarrow H^q_{F^{-1}\Psi}(X, F^{-1}\mathcal{A}).$$

(9.9) Remark. Assume that X and Y are paracompact differentiable manifolds and that $F : X \to Y$ is a C^∞ map. If $(\psi_\alpha)_{\alpha\in I}$ is a partition of unity subordinate to $\mathcal{U}$, then $(\psi_\alpha \circ F)_{\alpha\in I}$ is a partition of unity on X subordinate to $F^{-1}\mathcal{U}$. Let $c \in C^q(\mathcal{U}, \mathbb{R})$. The differential form associated to $F^\star c$ in the De Rham cohomology is

$$\begin{aligned} g &= \sum_{\nu_0,\ldots,\nu_q} c_{\nu_0\ldots\nu_q}(\psi_{\nu_q} \circ F) d(\psi_{\nu_0} \circ F) \wedge \ldots \wedge d(\psi_{\nu_{q-1}} \circ F) \\ &= F^\star\Big(\sum_{\nu_0,\ldots,\nu_q} c_{\nu_0\ldots\nu_q}\, \psi_{\nu_q} d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{q-1}}\Big). \end{aligned}$$

Hence we have a commutative diagram

$$\begin{array}{ccccc} H^q_{\mathrm{DR}}(Y, \mathbb{R}) & \xrightarrow{\simeq} & \check{H}^q(Y, \mathbb{R}) & \xrightarrow{\simeq} & H^q(Y, \mathbb{R}) \\ \downarrow F^\star & & \downarrow F^\star & & \downarrow F^\star \\ H^q_{\mathrm{DR}}(X, \mathbb{R}) & \xrightarrow{\simeq} & \check{H}^q(X, \mathbb{R}) & \xrightarrow{\simeq} & H^q(X, \mathbb{R}). \end{array}$$

9.B. Cohomology Groups of a Subspace

Let $\mathcal{A}$ be a sheaf on a topological space X, let S be a subspace of X and $i_S : S \hookrightarrow X$ the inclusion. Then $i_S^{-1}\mathcal{A}$ is the restriction of $\mathcal{A}$ to S, so that $H^q(S,\mathcal{A}) = H^q(S, i_S^{-1}\mathcal{A})$ by definition. For any two subspaces $S' \subset S$, the inclusion of S' in S induces a restriction morphism

$$H^q(S,\mathcal{A}) \longrightarrow H^q(S',\mathcal{A}).$$

(9.10) Theorem. *Let $\mathcal{A}$ be a sheaf on X and S a strongly paracompact subspace in X. When Ω ranges over open neighborhoods of S, we have*

$$H^q(S,\mathcal{A}) = \varinjlim_{\Omega\supset S} H^q(\Omega,\mathcal{A}).$$

Proof. When $q = 0$, the property is equivalent to Prop. 4.7. The general case follows by induction on q if we use the long cohomology exact sequences associated to the short exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^{[0]} \longrightarrow \mathcal{A}^{[0]}/\mathcal{A} \longrightarrow 0$$

on S and on its neighborhoods Ω (note that the restriction of a flabby sheaf to S is soft by Prop. 4.7 and the fact that every closed subspace of a strongly paracompact subspace is strongly paracompact). □

9.C. Cartesian Product

We use here the formalism of inverse images to deduce the cartesian product from the cup product. Let R be a fixed commutative ring and $\mathcal{A} \to X$, $\mathcal{B} \to Y$ sheaves of R-modules. We define the *external tensor product* by

$$(9.11)\quad \mathcal{A} \boxtimes_R \mathcal{B} = \mathrm{pr}_1^{-1}\mathcal{A} \otimes_R \mathrm{pr}_2^{-1}\mathcal{B}$$

where pr_1, pr_2 are the projections of $X \times Y$ onto X, Y respectively. The sheaf $\mathcal{A} \boxtimes_R \mathcal{B}$ is thus the sheaf on $X \times Y$ whose stalks are

$$(9.12)\quad (\mathcal{A} \boxtimes_R \mathcal{B})_{(x,y)} = \mathcal{A}_x \otimes_R \mathcal{B}_y.$$

For all cohomology classes $\alpha \in H^p(X,\mathcal{A})$, $\beta \in H^q(Y,\mathcal{B})$ the *cartesian product* $\alpha \times \beta \in H^{p+q}(X \times Y, \mathcal{A} \boxtimes_R \mathcal{B})$ is defined by

$$(9.13)\quad \alpha \times \beta = (\mathrm{pr}_1^\star\alpha) \smile (\mathrm{pr}_2^\star\beta).$$

More generally, let Φ and Ψ be families of supports in X and Y respectively. If $\Phi \times \Psi$ denotes the family of all closed subsets of $X \times Y$ contained in products $K \times L$ of elements $K \in \Phi$, $L \in \Psi$, the cartesian product defines a R-bilinear map

(9.14) $H^p_\Phi(X,\mathcal{A}) \times H^q_\Psi(Y,\mathcal{B}) \longrightarrow H^{p+q}_{\Phi\times\Psi}(X\times Y, \mathcal{A}\boxtimes_R \mathcal{B})$.

If $\mathcal{A}' \to X$, $\mathcal{B}' \to Y$ are sheaves of abelian groups and if α', β' are cohomology classes of degree p', q' with values in $\mathcal{A}'$, $\mathcal{B}'$, one gets easily

(9.15) $(\alpha\times\beta) \smile (\alpha'\times\beta') = (-1)^{qp'}(\alpha\smile\alpha')\times(\beta\smile\beta')$.

Furthermore, if $F : X' \to X$ and $G : Y' \to Y$ are continuous maps, then

(9.16) $(F\times G)^\star(\alpha\times\beta) = (F^\star\alpha)\times(G^\star\beta)$.

10. Spectral Sequence of a Filtered Complex

10.A. Construction of the Spectral Sequence

The theory of spectral sequences consists essentially in computing the homology groups of a differential module (K, d) by "successive approximations", once a filtration $F_p(K)$ is given in K and the cohomology groups of the graded modules $G_p(K)$ are known. Let us first recall some standard definitions and notations concerning filtrations.

(10.1) Definition. *Let R be a commutative ring. A filtration of a R-module M is a sequence of submodules $M_p \subset M$, $p\in\mathbb{Z}$, also denoted $M_p = F_p(M)$, such that $M_{p+1}\subset M_p$ for all $p\in\mathbb{Z}$, $\bigcup M_p = M$ and $\bigcap M_p = \{0\}$. The associated graded module is*

$$G(M) = \bigoplus_{p\in\mathbb{Z}} G_p(M), \qquad G_p(M) = M_p/M_{p+1}.$$

Let (K, d) be a differential module equipped with a filtration (K_p) by differential submodules (i.e. $dK_p \subset K_p$ for every p). For any number $r\in\mathbb{N}\cup\{\infty\}$, we define Z^p_r, $B^p_r \subset G_p(K) = K_p/K_{p+1}$ by

$$Z^p_r = K_p \cap d^{-1}K_{p+r} \mod K_{p+1}, \quad Z^p_\infty = K_p\cap d^{-1}\{0\} \mod K_{p+1}, \tag{10.2}$$

$$B^p_r = K_p \cap dK_{p-r+1} \mod K_{p+1}, \quad B^p_\infty = K_p \cap dK \mod K_{p+1}. \tag{10.2$'$}$$

(10.3) Lemma. *For every p and r, there are inclusions*

$$\ldots \subset B^p_r \subset B^p_{r+1} \subset \ldots \subset B^p_\infty \subset Z^p_\infty \subset \ldots \subset Z^p_{r+1} \subset Z^p_r \subset \ldots$$

and the differential d induces an isomorphism

$$\widetilde{d} \;:\; Z^p_r/Z^p_{r+1} \longrightarrow B^{p+r}_{r+1}/B^{p+r}_r.$$

Proof. It is clear that (Z_r^p) decreases with r, that (B_r^p) increases with r, and that $B_\infty^p \subset Z_\infty^p$. By definition

$$Z_r^p = (K_p \cap d^{-1}K_{p+r})/(K_{p+1} \cap d^{-1}K_{p+r}),$$
$$B_r^p = (K_p \cap dK_{p-r+1})/(K_{p+1} \cap dK_{p-r+1}).$$

The differential d induces a morphism

$$Z_r^p \longrightarrow (dK_p \cap K_{p+r})/(dK_{p+1} \cap K_{p+r})$$

whose kernel is $(K_p \cap d^{-1}\{0\})/(K_{p+1} \cap d^{-1}\{0\}) = Z_\infty^p$, whence isomorphisms

$$\widehat{d} \;:\; Z_r^p/Z_\infty^p \longrightarrow (K_{p+r} \cap dK_p)/(K_{p+r} \cap dK_{p+1}),$$
$$\widetilde{d} \;:\; Z_r^p/Z_{r+1}^p \longrightarrow (K_{p+r} \cap dK_p)/(K_{p+r} \cap dK_{p+1} + K_{p+r+1} \cap dK_p).$$

The right hand side of the last arrow can be identified to B_{r+1}^{p+r}/B_r^{p+r}, for

$$B_r^{p+r} = (K_{p+r} \cap dK_{p+1})/(K_{p+r+1} \cap dK_{p+1}),$$
$$B_{r+1}^{p+r} = (K_{p+r} \cap dK_p)/(K_{p+r+1} \cap dK_p).$$

□

Now, for each $r \in \mathbb{N}$, we define a complex $E_r^\bullet = \bigoplus_{p\in\mathbb{Z}} E_r^p$ with a differential $d_r : E_r^p \longrightarrow E_r^{p+r}$ of degree r as follows: we set $E_r^p = Z_r^p/B_r^p$ and take

$$(10.4)\quad d_r \;:\; Z_r^p/B_r^p \twoheadrightarrow Z_r^p/Z_{r+1}^p \xrightarrow{\widetilde{d}} B_{r+1}^{p+r}/B_r^{p+r} \hookrightarrow Z_r^{p+r}/B_r^{p+r}$$

where the first arrow is the obvious projection and the third arrow the obvious inclusion. Since d_r is induced by d, we actually have $d_r \circ d_r = 0$; this can also be seen directly by the fact that $B_{r+1}^{p+r} \subset Z_{r+1}^{p+r}$.

(10.5) Theorem and definition. *There is a canonical isomorphism $E_{r+1}^\bullet \simeq H^\bullet(E_r^\bullet)$. The sequence of differential complexes $(E_r^\bullet, d_r^\bullet)$ is called the spectral sequence of the filtered differential module (K, d).*

Proof. Since $\widetilde{d}$ is an isomorphism in (10.4), we have

$$\ker d_r = Z_{r+1}^p/B_r^p, \qquad \operatorname{Im} d_r = B_{r+1}^{p+r}/B_r^{p+r}.$$

Hence the image of $d_r : E_r^{p-r} \longrightarrow E_r^p$ is B_{r+1}^p/B_r^p and

$$H^p(E_r^\bullet) = (Z_{r+1}^p/B_r^p)/(B_{r+1}^p/B_r^p) \simeq Z_{r+1}^p/B_{r+1}^p = E_{r+1}^p.$$

□

(10.6) Theorem. *Consider the filtration of the homology module $H(K)$ defined by*

$$F_p\big(H(K)\big) = \operatorname{Im}\big(H(K_p) \longrightarrow H(K)\big).$$

Then there is a canonical isomorphism

$$E^p_\infty = G_p\big(H(K)\big).$$

Proof. Clearly $F_p\big(H(K)\big) = (K_p \cap d^{-1}\{0\})/(K_p \cap dK)$, whereas

$$Z^p_\infty = (K_p \cap d^{-1}\{0\})/(K_{p+1} \cap d^{-1}\{0\}), \quad B^p_\infty = (K_p \cap dK)/(K_{p+1} \cap dK),$$
$$E^p_\infty = Z^p_\infty / B^p_\infty = (K_p \cap d^{-1}\{0\})/(K_{p+1} \cap d^{-1}\{0\} + K_p \cap dK).$$

It follows that $E^p_\infty \simeq F_p\big(H(K)\big)/F_{p+1}\big(H(K)\big)$. □

In most applications, the differential module K has a natural grading compatible with the filtration. Let us consider for example the case of a cohomology complex $K^\bullet = \bigoplus_{l\in\mathbb{Z}} K^l$. The filtration $K^\bullet_p = F_p(K^\bullet)$ is said to be *compatible* with the differential complex structure if each $K^\bullet_p$ is a subcomplex of $K^\bullet$, i.e.

$$K^\bullet_p = \bigoplus_{l\in\mathbb{Z}} K^l_p$$

where (K^l_p) is a filtration of K^l. Then we define $Z^{p,q}_r$, $B^{p,q}_r$, $E^{p,q}_r$ to be the sets of elements of Z^p_r, B^p_r, E^p_r of total degree $p+q$. Therefore

$$(10.7)\quad Z^{p,q}_r = K^{p+q}_p \cap d^{-1} K^{p+q+1}_{p+r} \mod K^{p+q}_{p+1}, \qquad Z^p_r = \bigoplus Z^{p,q}_r,$$
$$(10.7')\quad B^{p,q}_r = K^{p+q}_p \cap d K^{p+q-1}_{p-r+1} \mod K^{p+q}_{p+1}, \qquad B^p_r = \bigoplus B^{p,q}_r,$$
$$(10.7'')\quad E^{p,q}_r = Z^{p,q}_r / B^{p,q}_r, \qquad E^p_r = \bigoplus E^{p,q}_r,$$

and the differential d_r has bidegree $(r, -r+1)$, i.e.

$$(10.8)\quad d_r \;:\; E^{p,q}_r \longrightarrow E^{p+r,\,q-r+1}_r.$$

For an element of pure bidegree (p,q), p is called the *filtering degree*, q the *complementary degree* and $p+q$ the *total degree*.

(10.9) Definition. *A filtration $(K^\bullet_p)$ of a complex $K^\bullet$ is said to be regular if for each degree l there are indices $\nu(l) \le N(l)$ such that $K^l_p = K^l$ for $p < \nu(l)$ and $K^l_p = 0$ for $p > N(l)$.*

If the filtration is regular, then (10.7) and (10.7′) show that

$$Z^{p,q}_r = Z^{p,q}_{r+1} = \ldots = Z^{p,q}_\infty \quad \text{for } r > N(p+q+1) - p,$$
$$B^{p,q}_r = B^{p,q}_{r+1} = \ldots = B^{p,q}_\infty \quad \text{for } r > p+1-\nu(p+q-1),$$

therefore $E^{p,q}_r = E^{p,q}_\infty$ for $r \ge r_0(p,q)$. We say that the spectral sequence *converges* to its limit term, and we write symbolically

$$(10.10)\quad E^{p,q}_r \Longrightarrow H^{p+q}(K^\bullet)$$

to express the following facts: there is a spectral sequence whose terms of the r-th generation are $E_r^{p,q}$, the sequence converges to a limit term $E_\infty^{p,q}$, and $E_\infty^{p,l-p}$ is the term $G_p\big(H^l(K^\bullet)\big)$ in the graded module associated to some filtration of $H^l(K^\bullet)$.

(10.11) Definition. *The spectral sequence is said to collapse in $E_r^\bullet$ if all terms $Z_k^{p,q}$, $B_k^{p,q}$, $E_k^{p,q}$ are constant for $k \geq r$, or equivalently if $d_k = 0$ in all bidegrees for $k \geq r$.*

(10.12) Special case. Assume that there exists an integer $r \geq 2$ and an index q_0 such that $E_r^{p,q} = 0$ for $q \neq q_0$. Then this property remains true for larger values of r, and we must have $d_r = 0$. It follows that the spectral sequence collapses in $E_r^\bullet$ and that

$$H^l(K^\bullet) = E_r^{l-q_0,q_0}.$$

Similarly, if $E_r^{p,q} = 0$ for $p \neq p_0$ and some $r \geq 1$ then

$$H^l(K^\bullet) = E_r^{p_0,l-p_0}. \qquad \square$$

10.B. Computation of the First Terms

Consider an arbitrary spectral sequence. For $r = 0$, we have $Z_0^p = K_p/K_{p+1}$, $B_0^p = \{0\}$, thus

$$(10.13) \quad E_0^p = K_p/K_{p+1} = G_p(K).$$

The differential d_0 is induced by d on the quotients, and

$$(10.14) \quad E_1^p = H\big(G_p(K)\big).$$

Now, there is a short exact sequence of differential modules

$$0 \longrightarrow G_{p+1}(K) \longrightarrow K_p/K_{p+2} \longrightarrow G_p(K) \longrightarrow 0.$$

We get therefore a connecting homomorphism

$$(10.15) \quad E_1^p = H\big(G_p(K)\big) \xrightarrow{\ \partial\ } H\big(G_{p+1}(K)\big) = E_1^{p+1}.$$

We claim that ∂ coincides with the differential d_1 : indeed, both are induced by d. When $K^\bullet$ is a filtered cohomology complex, d_1 is the connecting homomorphism

$$(10.16) \quad E_1^{p,q} = H^{p+q}\big(G_p(K^\bullet)\big) \xrightarrow{\ \partial\ } H^{p+q+1}\big(G_{p+1}(K^\bullet)\big) = E_1^{p+1,q}.$$

11. Spectral Sequence of a Double Complex

A double complex is a bigraded module $K^{\bullet,\bullet} = \bigoplus K^{p,q}$ together with a differential $d = d' + d''$ such that

(11.1) $\quad d' : K^{p,q} \longrightarrow K^{p+1,q}, \quad d'' : K^{p,q+1} \longrightarrow K^{p,q+1},$

and $d \circ d = 0$. In particular, d' and d'' satisfy the relations

(11.2) $\quad d'^2 = d''^2 = 0, \quad d'd'' + d''d' = 0.$

The *simple complex associated* to $K^{\bullet,\bullet}$ is defined by

$$K^l = \bigoplus_{p+q=l} K^{p,q}$$

together with the differential d. We will suppose here that both graduations of $K^{\bullet,\bullet}$ are positive, i.e. $K^{p,q} = 0$ for $p < 0$ or $q < 0$. The *first filtration* of $K^\bullet$ is defined by

(11.3) $\quad K^l_p = \bigoplus_{i+j=l,\ i\geq p} K^{i,j} = \bigoplus_{p\leq i\leq l} K^{i,l-i}.$

The graded module associated to this filtration is of course $G_p(K^l) = K^{p,l-p}$, and the differential induced by d on the quotient coincides with d'' because d' takes K^l_p to K^{l+1}_{p+1}. Thus we have a spectral sequence beginning by

(11.4) $\quad E_0^{p,q} = K^{p,q}, \quad d_0 = d'', \quad E_1^{p,q} = H^q_{d''}(K^{p,\bullet}).$

By (10.16), d_1 is the connecting homomorphism associated to the short exact sequence

$$0 \longrightarrow K^{p+1,\bullet} \longrightarrow K^{p,\bullet} \oplus K^{p+1,\bullet} \longrightarrow K^{p,\bullet} \longrightarrow 0$$

where the differential is given by (d mod $K^{p+2,\bullet}$) for the central term and by d'' for the two others. The definition of the connecting homomorphism in the proof of Th. 1.5 shows that

$$d_1 = \partial : \quad H^q_{d''}(K^{p,\bullet}) \longrightarrow H^q_{d''}(K^{p+1,\bullet})$$

is induced by d'. Consequently, we find

(11.5) $\quad E_2^{p,q} = H^p_{d'}(E_1^{\bullet,q}) = H^p_{d'}\big(H^q_{d''}(K^{\bullet,\bullet})\big).$

For such a spectral sequence, several interesting additional features can be pointed out. For all r and l, there is an injective homomorphism

$$E^{0,l}_{r+1} \hookrightarrow E^{0,l}_r$$

whose image can be identified with the set of d_r-cocycles in $E_r^{0,l}$; the coboundary group is zero because $E_r^{p,q} = 0$ for $q < 0$. Similarly, $E_r^{l,0}$ is equal to its cocycle submodule, and there is a surjective homomorphism

$$E_r^{l,0} \twoheadrightarrow E_{r+1}^{l,0} \simeq E_r^{l,0}/d_r E_r^{l-r,r-1}.$$

Furthermore, the filtration on $H^l(K^\bullet)$ begins at $p = 0$ and stops at $p = l$, i.e.

$$\text{(11.6)} \quad F_0\big(H^l(K^\bullet)\big) = H^l(K^\bullet), \quad F_p\big(H^l(K^\bullet)\big) = 0 \quad \text{for} \quad p > l.$$

Therefore, there are canonical maps

$$\text{(11.7)} \quad \begin{aligned} &H^l(K^\bullet) \twoheadrightarrow G_0\big(H^l(K^\bullet)\big) = E_\infty^{0,l} \hookrightarrow E_r^{0,l}, \\ &E_r^{l,0} \twoheadrightarrow E_\infty^{l,0} = G_l\big(H^l(K^\bullet)\big) \hookrightarrow H^l(K^\bullet). \end{aligned}$$

These maps are called the *edge homomorphisms* of the spectral sequence.

(11.8) Theorem. *There is an exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(K^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(K^\bullet)$$

where the non indicated arrows are edge homomorphisms.

Proof. By 11.6, the graded module associated to $H^1(K^\bullet)$ has only two components, and we have an exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1(K^\bullet) \longrightarrow E_\infty^{0,1} \longrightarrow 0.$$

However $E_\infty^{1,0} = E_2^{1,0}$ because all differentials d_r starting from $E_r^{1,0}$ or abuting to $E_r^{1,0}$ must be zero for $r \geq 2$. Similarly, $E_\infty^{0,1} = E_3^{0,1}$ and $E_\infty^{2,0} = E_3^{2,0}$, thus there is an exact sequence

$$0 \longrightarrow E_\infty^{0,1} \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow E_\infty^{2,0} \longrightarrow 0.$$

A combination of the two above exact sequences yields

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(K^\bullet) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow E_\infty^{2,0} \longrightarrow 0.$$

Taking into account the injection $E_\infty^{2,0} \hookrightarrow H^2(K^\bullet)$ in (11.7), we get the required exact sequence. □

(11.9) Example. Let X be a complex manifold of dimension n. Consider the double complex $K^{p,q} = C_{p,q}^\infty(X, \mathbb{C})$ together with the exterior derivative $d = d' + d''$. Then there is a spectral sequence which starts from the Dolbeault cohomology groups

$$E_1^{p,q} = H^{p,q}(X, \mathbb{C})$$

and which converges to the graded module associated to a filtration of the De Rham cohomology groups:

$$E_r^{p,q} \Longrightarrow H_{\mathrm{DR}}^{p+q}(X, \mathbb{C}).$$

This spectral sequence is called the *Hodge-Frölicher spectral sequence* (Frölicher 1955). We will study it in much more detail in chapter 6 when X is compact. $\square$

Frequently, the spectral sequence under consideration can be obtained from two distinct double complexes and one needs to compare the final cohomology groups. The following lemma can often be applied.

(11.10) Lemma. *Let $K^{p,q} \longrightarrow L^{p,q}$ be a morphism of double complexes (i.e. a double sequence of maps commuting with d' and d''). Then there are induced morphisms*

$$_K E_r^{\bullet,\bullet} \longrightarrow {}_L E_r^{\bullet,\bullet}, \qquad r \geq 0$$

of the associated spectral sequences. If one of these morphisms is an isomorphism for some r, then $H^l(K^\bullet) \longrightarrow H^l(L^\bullet)$ is an isomorphism.

Proof. If the r-terms are isomorphic, they have the same cohomology groups, thus $_K E_{r+1}^{\bullet,\bullet} \simeq {}_L E_{r+1}^{\bullet,\bullet}$ and $_K E_\infty^{\bullet,\bullet} \simeq {}_L E_\infty^{\bullet,\bullet}$ in the limit. The lemma follows from the fact that if a morphism of graded modules $\varphi : M \longrightarrow M'$ induces an isomorphism $G_\bullet(M) \longrightarrow G_\bullet(M')$, then φ is an isomorphism. $\square$

12. Hypercohomology Groups

Let $(\mathcal{L}^\bullet, \delta)$ be a complex of sheaves

$$0 \longrightarrow \mathcal{L}^0 \xrightarrow{\delta^0} \mathcal{L}^1 \longrightarrow \cdots \longrightarrow \mathcal{L}^q \xrightarrow{\delta^q} \cdots$$

on a topological space X. We denote by $\mathcal{H}^q = \mathcal{H}^q(\mathcal{L}^\bullet)$ the q-th sheaf of cohomology of this complex; thus $\mathcal{H}^q$ is a sheaf of abelian groups over X. Our goal is to define "generalized cohomology groups" attached to $\mathcal{L}^\bullet$ on X, in such a way that these groups only depend on the cohomology sheaves $\mathcal{H}^q$. For this, we associate to $\mathcal{L}^\bullet$ the double complex of groups

$$(12.1) \quad K_{\mathcal{L}}^{p,q} = (\mathcal{L}^q)^{[p]}(X)$$

with differential $d' = d^p$ given by (2.5), and with $d'' = (-1)^p(\delta^q)^{[p]}$. As $(\delta^q)^{[\bullet]} : (\mathcal{L}^q)^{[\bullet]} \longrightarrow (\mathcal{L}^{q+1})^{[\bullet]}$ is a morphism of complexes, we get the expected relation $d'd'' + d''d' = 0$.

(12.2) Definition. *The groups $H^q(K^\bullet_{\mathcal{L}})$ are called the hypercohomology groups of $\mathcal{L}^\bullet$ and are denoted $\mathbb{H}^q(X, \mathcal{L}^\bullet)$.*

Clearly $\mathbb{H}^0(X, \mathcal{L}^\bullet) = \mathcal{H}^0(X)$ where $\mathcal{H}^0 = \ker \delta^0$ is the first cohomology sheaf of $\mathcal{L}^\bullet$. If $\varphi^\bullet : \mathcal{L}^\bullet \longrightarrow \mathcal{M}^\bullet$ is a morphism of sheaf complexes, there is an associated morphism of double complexes $\varphi^{\bullet,\bullet} : K^{\bullet,\bullet}_{\mathcal{L}} \longrightarrow K^{\bullet,\bullet}_{\mathcal{M}}$, hence a natural morphism

$$(12.3) \quad \mathbb{H}^q(\varphi^\bullet) \ : \ \mathbb{H}^q(X, \mathcal{L}^\bullet) \longrightarrow \mathbb{H}^q(X, \mathcal{M}^\bullet).$$

We first list a few immediate properties of hypercohomology groups, whose proofs are left to the reader.

(12.4) Proposition. *The following properties hold:*

a) *If $\mathcal{L}^q = 0$ for $q \neq 0$, then $\mathbb{H}^q(X, \mathcal{L}^\bullet) = H^q(X, \mathcal{L}^0)$.*

b) *If $\mathcal{L}^\bullet[s]$ denotes the complex $\mathcal{L}^\bullet$ shifted of s indices to the right, i.e. $\mathcal{L}^\bullet[s]^q = \mathcal{L}^{q-s}$, then $\mathbb{H}^q(X, \mathcal{L}^\bullet[s]) = \mathbb{H}^{q-s}(X, \mathcal{L}^\bullet)$.*

c) *If $0 \longrightarrow \mathcal{L}^\bullet \longrightarrow \mathcal{M}^\bullet \longrightarrow \mathcal{N}^\bullet \longrightarrow 0$ is an exact sequence of sheaf complexes, there is a long exact sequence*

$$\cdots \mathbb{H}^q(X, \mathcal{L}^\bullet) \longrightarrow \mathbb{H}^q(X, \mathcal{M}^\bullet) \longrightarrow \mathbb{H}^q(X, \mathcal{N}^\bullet) \xrightarrow{\partial} \mathbb{H}^{q+1}(X, \mathcal{L}^\bullet) \cdots . \ \square$$

If $\mathcal{L}^\bullet$ is a sheaf complex, the spectral sequence associated to the first filtration of $K^\bullet_{\mathcal{L}}$ is given by

$$E_1^{p,q} = H^q_{d''}(K^{p,\bullet}_{\mathcal{L}}) = H^q\big((\mathcal{L}^\bullet)^{[p]}(X)\big).$$

However by (2.9) the functor $\mathcal{A} \longmapsto \mathcal{A}^{[p]}(X)$ preserves exact sequences. Therefore, we get

$$(12.5) \quad E_1^{p,q} = \big(\mathcal{H}^q(\mathcal{L}^\bullet)\big)^{[p]}(X),$$

$$(12.5') \quad E_2^{p,q} = H^p\big(X, \mathcal{H}^q(\mathcal{L}^\bullet)\big),$$

since $E_2^{p,q} = H^p_{d'}(E_1^{\bullet,q})$. If $\varphi^\bullet : \mathcal{L}^\bullet \longrightarrow \mathcal{M}^\bullet$ is a morphism, an application of Lemma 11.10 to the E_2-term of the associated first spectral sequences of $K^{\bullet,\bullet}_{\mathcal{L}}$ and $K^{\bullet,\bullet}_{\mathcal{M}}$ yields:

(12.6) Corollary. *If $\varphi^\bullet : \mathcal{L}^\bullet \longrightarrow \mathcal{M}^\bullet$ is a quasi-isomorphism $\big($this means that $\varphi^\bullet$ induces an isomorphism $\mathcal{H}^\bullet(\mathcal{L}^\bullet) \longrightarrow \mathcal{H}^\bullet(\mathcal{M}^\bullet)$ $\big)$, then*

$$\mathbb{H}^l(\varphi^\bullet) \ : \ \mathbb{H}^l(X, \mathcal{L}^\bullet) \longrightarrow \mathbb{H}^l(X, \mathcal{M}^\bullet)$$

is an isomorphism.

Now, we may reverse the roles of the indices p, q and of the differentials d', d''. The *second filtration* $F_p(K^l_{\mathcal{L}}) = \bigoplus_{j \ge p} K^{l-j,j}_{\mathcal{L}}$ is associated to a spectral sequence such that $\widetilde{E}^{p,q}_1 = H^q_{d'}(K^{\bullet,p}_{\mathcal{L}}) = H^q_{d'}\big((\mathcal{L}^p)^{[\bullet]}(X)\big)$, hence

(12.7) $\widetilde{E}^{p,q}_1 = H^q(X, \mathcal{L}^p)$,

(12.7′) $\widetilde{E}^{p,q}_2 = H^p_\delta\big(H^q(X, \mathcal{L}^\bullet)\big)$.

These two spectral sequences converge to limit terms which are the graded modules associated to filtrations of $\mathbb{H}^\bullet(X, \mathcal{L}^\bullet)$; these filtrations are in general different. Let us mention two interesting special cases.

• Assume first that the complex $\mathcal{L}^\bullet$ is a resolution of a sheaf $\mathcal{A}$, so that $\mathcal{H}^0 = \mathcal{A}$ and $\mathcal{H}^q = 0$ for $q \ge 1$. Then we find

$$E^{p,0}_2 = H^p(X, \mathcal{A}), \qquad E^{p,q}_2 = 0 \quad \text{for} \quad q \ge 1.$$

It follows that the first spectral sequence collapses in $E^\bullet_2$, and 10.12 implies

(12.8) $\mathbb{H}^l(X, \mathcal{L}^\bullet) \simeq H^l(X, \mathcal{A})$.

• Now, assume that the sheaves $\mathcal{L}^q$ are acyclic. The second spectral sequence gives

$$\widetilde{E}^{p,0}_2 = H^p\big(\mathcal{L}^\bullet(X)\big), \qquad \widetilde{E}^{p,q}_2 = 0 \quad \text{for} \quad q \ge 1,$$

(12.9) $$\mathbb{H}^l(X, \mathcal{L}^\bullet) \simeq H^l\big(\mathcal{L}^\bullet(X)\big).$$

If both conditions hold, i.e. if $\mathcal{L}^\bullet$ is a resolution of a sheaf $\mathcal{A}$ by acyclic sheaves, then (12.8) and (12.9) can be combined to obtain a new proof of the De Rham-Weil isomorphism $H^l(X, \mathcal{A}) \simeq H^l\big(\mathcal{L}^\bullet(X)\big)$.

13. Direct Images and the Leray Spectral Sequence

13.A. Direct Images of a Sheaf

Let X, Y be topological spaces, $F : X \to Y$ a continuous map and $\mathcal{A}$ a sheaf of abelian groups on X. Recall that the *direct image* $F_\star\mathcal{A}$ is the presheaf on Y defined for any open set $U \subset Y$ by

(13.1) $(F_\star\mathcal{A})(U) = \mathcal{A}\big(F^{-1}(U)\big)$.

Axioms (II-2.4′ and (II-2.4″) are clearly satisfied, thus $F_\star\mathcal{A}$ is in fact a sheaf. The following result is obvious:

(13.2) $\mathcal{A}$ is flabby $\Longrightarrow$ $F_\star\mathcal{A}$ is flabby.

Every sheaf morphism $\varphi : \mathcal{A} \to \mathcal{B}$ induces a corresponding morphism

$$F_\star\varphi \;:\; F_\star\mathcal{A} \longrightarrow F_\star\mathcal{B},$$

so $F_\star$ is a functor on the category of sheaves on X to the category of sheaves on Y. This functor is exact on the left: indeed, to every exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is associated an exact sequence

$$0 \longrightarrow F_\star\mathcal{A} \longrightarrow F_\star\mathcal{B} \longrightarrow F_\star\mathcal{C},$$

but $F_\star\mathcal{B} \to F_\star\mathcal{C}$ need not be onto if $\mathcal{B} \to \mathcal{C}$ is. All this follows immediately from the considerations of §3. In particular, the simplicial flabby resolution $(\mathcal{A}^{[\bullet]}, d)$ yields a complex of sheaves

$$(13.3)\quad 0 \longrightarrow F_\star\mathcal{A}^{[0]} \longrightarrow F_\star\mathcal{A}^{[1]} \longrightarrow \cdots \longrightarrow F_\star\mathcal{A}^{[q]} \xrightarrow{F_\star d^q} F_\star\mathcal{A}^{[q+1]} \longrightarrow \cdots.$$

(13.4) Definition. *The q-th direct image of $\mathcal{A}$ by F is the q-th cohomology sheaf of the sheaf complex* (13.3). *It is denoted*

$$R^qF_\star\mathcal{A} = \mathcal{H}^q(F_\star\mathcal{A}^{[\bullet]}).$$

As $F_\star$ is exact on the left, the sequence $0 \to F_\star\mathcal{A} \to F_\star\mathcal{A}^{[0]} \to F_\star\mathcal{A}^{[1]}$ is exact, thus

$$(13.5)\quad R^0F_\star\mathcal{A} = F_\star\mathcal{A}.$$

We now compute the stalks of $R^qF_\star\mathcal{A}$. As the kernel or cokernel of a sheaf morphism is obtained stalk by stalk, we have

$$(R^qF_\star\mathcal{A})_y = H^q\big((F_\star\mathcal{A}^{[\bullet]})_y\big) = \varinjlim_{U\ni y} H^q\big(F_\star\mathcal{A}^{[\bullet]}(U)\big).$$

The very definition of $F_\star$ and of sheaf cohomology groups implies

$$H^q\big(F_\star\mathcal{A}^{[\bullet]}(U)\big) = H^q\big(\mathcal{A}^{[\bullet]}(F^{-1}(U))\big) = H^q\big(F^{-1}(U), \mathcal{A}\big),$$

hence we find

$$(13.6)\quad (R^qF_\star\mathcal{A})_y = \varinjlim_{U\ni y} H^q\big(F^{-1}(U), \mathcal{A}\big),$$

i.e. $R^qF_\star\mathcal{A}$ is the sheaf associated to the presheaf $U \mapsto H^q\big(F^{-1}(U), \mathcal{A}\big)$. We must stress here that the stronger relation $R^qF_\star\mathcal{A}(U) = H^q\big(F^{-1}(U), \mathcal{A}\big)$ need not be true in general. If the fiber $F^{-1}(y)$ is strongly paracompact in X and if the family of open sets $F^{-1}(U)$ is a fundamental family of neighborhoods of $F^{-1}(y)$ (this situation occurs for example if X and Y are locally compact spaces and F a proper map, or if $F = \mathrm{pr}_1 : X = Y \times S \longrightarrow Y$ where S is compact), Th. 9.10 implies the more natural relation

$$(13.6')\quad (R^qF_\star\mathcal{A})_y = H^q\big(F^{-1}(y), \mathcal{A}\big).$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on X. Apply the long exact sequence of cohomology on every open set $F^{-1}(U)$ and take the direct limit over U. We get an exact sequence of sheaves:

$$\tag{13.7} \begin{array}{c} 0 \longrightarrow \;\; F_\star\mathcal{A} \;\longrightarrow\;\; F_\star\mathcal{B} \;\longrightarrow\;\; F_\star\mathcal{C} \;\longrightarrow\;\; R^1F_\star\mathcal{A} \longrightarrow \cdots \\ \cdots \longrightarrow R^qF_\star\mathcal{A} \longrightarrow R^qF_\star\mathcal{B} \longrightarrow R^qF_\star\mathcal{C} \longrightarrow R^{q+1}F_\star\mathcal{A} \longrightarrow \cdots. \end{array}$$

13.B. Leray Spectral Sequence

For any continuous map $F : X \to Y$, the Leray spectral sequence relates the cohomology groups of a sheaf $\mathcal{A}$ on X and those of its direct images $R^qF_\star\mathcal{A}$ on Y. Consider the two spectral sequences $E_r^\bullet$, $\widetilde{E}_r^\bullet$ associated with the complex of sheaves $\mathcal{L}^\bullet = F_\star\mathcal{A}^{[\bullet]}$ on Y, as in § 12. By definition we have $\mathcal{H}^q(\mathcal{L}^\bullet) = R^qF_\star\mathcal{A}$. By (12.5′) the second term of the first spectral sequence is

$$E_2^{p,q} = H^p(Y, R^qF_\star\mathcal{A}),$$

and this spectral sequence converges to the graded module associated to a filtration of $\mathbb{H}^l(Y, F_\star\mathcal{A}^{[\bullet]})$. On the other hand, (13.2) implies that $F_\star\mathcal{A}^{[q]}$ is flabby. Hence, the second special case (12.9) can be applied:

$$\mathbb{H}^l(Y, F_\star\mathcal{A}^{[\bullet]}) \simeq H^l\big(F_\star\mathcal{A}^{[\bullet]}(Y)\big) = H^l\big(\mathcal{A}^{[\bullet]}(X)\big) = H^l(X, \mathcal{A}).$$

We may conclude this discussion by the following

(13.8) Theorem. *For any continuous map $F : X \to Y$ and any sheaf $\mathcal{A}$ of abelian groups on X, there exists a spectral sequence whose $E_2^\bullet$ term is*

$$E_2^{p,q} = H^p(Y, R^qF_\star\mathcal{A}),$$

which converges to a limit term $E_\infty^{p,l-p}$ equal to the graded module associated with a filtration of $H^l(X, \mathcal{A})$. The edge homomorphism

$$H^l(Y, F_\star\mathcal{A}) \twoheadrightarrow E_\infty^{l,0} \hookrightarrow H^l(X, \mathcal{A})$$

coincides with the composite morphism

$$F^\# : \quad H^l(Y, F_\star\mathcal{A}) \xrightarrow{F^\star} H^l(X, F^{-1}F_\star\mathcal{A}) \xrightarrow{H^l(\mu_F)} H^l(X, \mathcal{A})$$

where $\mu_F : F^{-1}F_\star\mathcal{A} \longrightarrow \mathcal{A}$ is the canonical sheaf morphism.

Proof. Only the last statement remains to be proved. The morphism μ_F is defined as follows: every element $s \in (F^{-1}F_\star\mathcal{A})_x = (F_\star\mathcal{A})_{F(x)}$ is the germ of a section $\widetilde{s} \in F_\star\mathcal{A}(V) = \mathcal{A}\big(F^{-1}(V)\big)$ on a neighborhood V of $F(x)$. Then $F^{-1}(V)$ is a neighborhood of x and we let $\mu_F s$ be the germ of $\widetilde{s}$ at x.

Now, we observe that to any commutative diagram of topological spaces and continuous maps is associated a commutative diagram involving the direct image sheaves and their cohomology groups:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle H} \\ X' & \xrightarrow{F'} & Y' \end{array} \qquad\qquad \begin{array}{ccc} H^l(X,\mathcal{A}) & \xleftarrow{F^\#} & H^l(Y,F_\star\mathcal{A}) \\ {\scriptstyle G^\#}\uparrow & & \uparrow{\scriptstyle H^\#} \\ H^l(X',G_\star\mathcal{A}) & \xleftarrow{F'^\#} & H^l(Y',F'_\star G_\star\mathcal{A}). \end{array}$$

There is a similar commutative diagram in which $F^\#$ and $F'^\#$ are replaced by the edge homomorphisms of the spectral sequences of F and F' : indeed there is a natural morphism $H^{-1}F'_\star\mathcal{B} \longrightarrow F_\star G^{-1}\mathcal{B}$ for any sheaf $\mathcal{B}$ on X', so we get a morphism of sheaf complexes

$$H^{-1}F'_\star(G_\star\mathcal{A})^{[\bullet]} \longrightarrow F_\star G^{-1}(G_\star\mathcal{A})^{[\bullet]} \longrightarrow F_\star(G^{-1}G_\star\mathcal{A})^{[\bullet]} \longrightarrow F_\star\mathcal{A}^{[\bullet]},$$

hence also a morphism of the spectral sequences associated to both ends.

The special case $X' = Y' = Y$, $G = F$, $F' = H = \mathrm{Id}_Y$ then shows that our statement is true for F if it is true for F'. Hence we may assume that F is the identity map; in this case, we need only show that the edge homomorphism of the spectral sequence of $F_\star\mathcal{A}^{[\bullet]} = \mathcal{A}^{[\bullet]}$ is the identity map. This is an immediate consequence of the fact that we have a quasi-isomorphism

$$(\cdots \to 0 \to \mathcal{A} \to 0 \to \cdots) \longrightarrow \mathcal{A}^{[\bullet]}. \qquad \square$$

(13.9) Corollary. *If $R^qF_\star\mathcal{A} = 0$ for $q \geq 1$, there is an isomorphism $H^l(Y,F_\star\mathcal{A}) \simeq H^l(X,\mathcal{A})$ induced by $F^\#$.*

Proof. We are in the special case 10.12 with $E_2^{p,q} = 0$ for $q \neq 0$, so

$$H^l(Y,F_\star\mathcal{A}) = E_2^{l,0} \simeq H^l(X,\mathcal{A}). \qquad \square$$

(13.10) Corollary. *Let $F : X \longrightarrow Y$ be a proper finite-to-one map. For any sheaf $\mathcal{A}$ on X, we have $R^qF_\star\mathcal{A} = 0$ for $q \geq 1$ and there is an isomorphism $H^l(Y,F_\star\mathcal{A}) \simeq H^l(X,\mathcal{A})$.*

Proof. By definition of higher direct images, we have

$$(R^qF_\star\mathcal{A})_y = \varinjlim_{U\ni y} H^q\big(\mathcal{A}^{[\bullet]}\big(F^{-1}(U)\big)\big).$$

If $F^{-1}(y) = \{x_1,\ldots,x_m\}$, the assumptions imply that $\big(F^{-1}(U)\big)$ is a fundamental system of neighborhoods of $\{x_1,\ldots,x_m\}$. Therefore

$$(R^qF_\star\mathcal{A})_y = \bigoplus_{1\leq j\leq m} H^q\big(\mathcal{A}^{[\bullet]}_{x_j}\big) = \begin{cases} \bigoplus \mathcal{A}_{x_j} & \text{for } q = 0, \\ 0 & \text{for } q \geq 1, \end{cases}$$

and we conclude by Cor. 13.9. □

Corollary 13.10 can be applied in particular to the inclusion $j : S \to X$ of a *closed* subspace S. Then $j_\star\mathcal{A}$ coincides with the sheaf $\mathcal{A}^S$ defined in §3 and we get $H^q(S,\mathcal{A}) = H^q(X,\mathcal{A}^S)$. It is very important to observe that the property $R^q j_\star\mathcal{A} = 0$ for $q \geq 1$ need not be true if S is not closed.

13.C. Topological Dimension

As a first application of the Leray spectral sequence, we are going to derive some properties related to the concept of *topological dimension*.

(13.11) Definition. *A non empty space X is said to be of topological dimension $\leq n$ if $H^q(X,\mathcal{A}) = 0$ for any $q > n$ and any sheaf $\mathcal{A}$ on X. We let* $\operatorname{topdim} X$ *be the smallest such integer n if it exists, and $+\infty$ otherwise.*

(13.12) Criterion. *For a paracompact space X, the following conditions are equivalent:*

a) $\operatorname{topdim} X \leq n$;

b) *the sheaf* $\mathcal{Z}^n = \ker(\mathcal{A}^{[n]} \longrightarrow \mathcal{A}^{[n+1]})$ *is soft for every sheaf* $\mathcal{A}$;

c) *every sheaf $\mathcal{A}$ admits a resolution $0 \to \mathcal{L}^0 \to \cdots \to \mathcal{L}^n \to 0$ of length n by soft sheaves.*

Proof. b) $\Longrightarrow$ c). Take $\mathcal{L}^q = \mathcal{A}^{[q]}$ for $q < n$ and $\mathcal{L}^n = \mathcal{Z}^n$.

c) $\Longrightarrow$ a). For every sheaf $\mathcal{A}$, the De Rham-Weil isomorphism implies $H^q(X,\mathcal{A}) = H^q\big(\mathcal{L}^\bullet(X)\big) = 0$ when $q > n$.

a) $\Longrightarrow$ b). Let S be a closed set and $U = X \smallsetminus S$. As in Prop. 7.12, $(\mathcal{A}^{[\bullet]})_U$ is an acyclic resolution of $\mathcal{A}_U$. Clearly $\ker\big((\mathcal{A}^{[n]})_U \to (\mathcal{A}^{[n+1]})_U\big) = \mathcal{Z}^n_U$, so the isomorphisms (6.2) obtained in the proof of the De Rham-Weil theorem imply

$$H^1(X,\mathcal{Z}^n_U) \simeq H^{n+1}(X,\mathcal{A}_U) = 0.$$

By (3.10), the restriction map $\mathcal{Z}^n(X) \longrightarrow \mathcal{Z}^n(S)$ is onto, so $\mathcal{Z}^n$ is soft. □

(13.13) Theorem. *The following properties hold:*

a) *If X is paracompact and if every point of X has a neighborhood of topological dimension $\leq n$, then* $\operatorname{topdim} X \leq n$.

b) *If $S \subset X$, then* $\operatorname{topdim} S \leq \operatorname{topdim} X$ *provided that S is closed or X metrizable.*

c) *If X, Y are metrizable spaces, one of them locally compact, then*

$$\mathrm{topdim}\,(X \times Y) \le \mathrm{topdim}\, X + \mathrm{topdim}\, Y.$$

d) *If X is metrizable and locally homeomorphic to a subspace of $\mathbb{R}^n$, then* $\mathrm{topdim}\, X \le n$.

Proof. a) Apply criterion 13.12 b) and the fact that softness is a local property (Prop. 4.12).

b) When S is closed in X, the property follows from Cor. 13.10. When X is metrizable, any subset S is strongly paracompact. Let $j : S \longrightarrow X$ be the injection and $\mathcal{A}$ a sheaf on S. As $\mathcal{A} = (j_\star\mathcal{A})_{\restriction S}$, we have

$$H^q(S, \mathcal{A}) = H^q(S, j_\star\mathcal{A}) = \varinjlim_{\Omega \supset S} H^q(\Omega, j_\star\mathcal{A})$$

by Th. 9.10. We may therefore assume that S is open in X. Then every point of S has a neighborhood which is closed in X, so we conclude by a) and the first case of b).

c) Thanks to a) and b), we may assume for example that X is compact. Let $\mathcal{A}$ be a sheaf on $X \times Y$ and $\pi : X \times Y \longrightarrow Y$ the second projection. Set $n_X = \mathrm{topdim}\, X$, $n_Y = \mathrm{topdim}\, Y$. In virtue of (13.6′), we have $R^q\pi_\star\mathcal{A} = 0$ for $q > n_X$. In the Leray spectral sequence, we obtain therefore

$$E_2^{p,q} = H^p(Y, R^q\pi_\star\mathcal{A}) = 0 \quad \text{for} \quad p > n_Y \quad \text{or} \quad q > n_X,$$

thus $E_\infty^{p,l-p} = 0$ when $l > n_X + n_Y$ and we infer $H^l(X \times Y, \mathcal{A}) = 0$.

d) The unit interval $[0, 1] \subset \mathbb{R}$ is of topological dimension ≤ 1, because $[0, 1]$ admits arbitrarily fine coverings

$$(13.14) \quad \mathcal{U}_k = \big(\, [0, 1] \cap \,](\alpha - 1)/k, (\alpha + 1)/k[\, \big)_{0 \le \alpha \le k}$$

for which only consecutive open sets U_α, $U_{\alpha+1}$ intersect; we may therefore apply Prop. 5.24. Hence $\mathbb{R}^n \simeq \,]0, 1[^n \subset [0, 1]^n$ is of topological dimension $\le n$ by b) and c). Property d) follows

14. Alexander-Spanier Cohomology

14.A. Invariance by Homotopy

Alexander-Spanier's theory can be viewed as the special case of sheaf cohomology theory with *constant coefficients*, i.e. with values in constant sheaves.

(14.1) Definition. *Let X be a topological space, R a commutative ring and M a R-module. The constant sheaf $X \times M$ is denoted M for simplicity.*

The Alexander-Spanier q-th cohomology group with values in M is the sheaf cohomology group $H^q(X, M)$.

In particular $H^0(X, M)$ is the set of locally constant functions $X \to M$, so $H^0(X, M) \simeq M^E$, where E is the set of connected components of X. We will not repeat here the properties of Alexander-Spanier cohomology groups that are formal consequences of those of general sheaf theory, but we focus our attention instead on new features, such as invariance by homotopy.

(14.2) Lemma. *Let I denote the interval $[0, 1]$ of real numbers. Then $H^0(I, M) = M$ and $H^q(I, M) = 0$ for $q \neq 0$.*

Proof. Let us employ alternate Čech cochains for the coverings $\mathfrak{U}_n$ defined in (13.14). As I is paracompact, we have

$$H^q(I, M) = \varinjlim \check{H}^q(\mathfrak{U}_n, M).$$

However, the alternate Čech complex has only two non zero components and one non zero differential:

$$\begin{aligned}
&AC^0(\mathfrak{U}_n, M) = \big\{(c_\alpha)_{0\le\alpha\le n}\big\} = M^{n+1},\\
&AC^1(\mathfrak{U}_n, M) = \big\{(c_{\alpha\,\alpha+1})_{0\le\alpha\le n-1}\big\} = M^n,\\
&d^0 : (c_\alpha) \longmapsto (c'_{\alpha\,\alpha+1}) = (c_{\alpha+1} - c_\alpha).
\end{aligned}$$

We see that d^0 is surjective, and that $\ker d^0 = \big\{(m, m, \ldots, m)\big\} = M$. $\square$

For any continuous map $f : X \longrightarrow Y$, the inverse image of the constant sheaf M on Y is $f^{-1}M = M$. We get therefore a morphism

$$(14.3) \qquad f^\star : H^q(Y, M) \longrightarrow H^q(X, M),$$

which, as already mentioned in §9, is compatible with cup product.

(14.4) Proposition. *For any space X, the projection $\pi : X \times I \longrightarrow X$ and the injections $i_t : X \longrightarrow X \times I$, $x \longmapsto (x, t)$ induce inverse isomorphisms*

$$H^q(X, M) \quad \underset{i_t^\star}{\overset{\pi^\star}{\rightleftarrows}} \quad H^q(X \times I, M).$$

In particular, $i_t^\star$ does not depend on t.

Proof. As $\pi \circ i_t = \mathrm{Id}$, we have $i_t^\star \circ \pi^\star = \mathrm{Id}$, so it is sufficient to check that $\pi^\star$ is an isomorphism. However $(R^q\pi_\star M)_x = H^q(I, M)$ in virtue of (13.6′), so we get

$$R^0\pi_\star M = M, \qquad R^q\pi_\star M = 0 \quad \text{for} \quad q \neq 0$$

and conclude by Cor. 13.9. □

(14.4) Theorem. *If $f, g : X \longrightarrow Y$ are homotopic maps, then*

$$f^\star = g^\star : H^q(Y, M) \longrightarrow H^q(X, M).$$

Proof. Let $H : X \times I \longrightarrow Y$ be a homotopy between f and g, with $f = H \circ i_0$ and $g = H \circ i_1$. Proposition 14.3 implies

$$f^\star = i_0^\star \circ H^\star = i_1^\star \circ H^\star = g^\star. \qquad \square$$

We denote $f \sim g$ the homotopy equivalence relation. Two spaces X, Y are said to be homotopically equivalent $(X \sim Y)$ if there exist continuous maps $u : X \longrightarrow Y$, $v : Y \longrightarrow X$ such that $v \circ u \sim \mathrm{Id}_X$ and $u \circ v \sim \mathrm{Id}_Y$. Then $H^q(X, M) \simeq H^q(Y, M)$ and $u^\star, v^\star$ are inverse isomorphisms.

(14.5) Example. A subspace $S \subset X$ is said to be a (strong) deformation retract of X if there exists a *retraction* of X onto S, i.e. a map $r : X \longrightarrow S$ such that $r \circ j = \mathrm{Id}_S$ (j = inclusion of S in X), which is a *deformation* of Id_X, i.e. there exists a homotopy $H : X \times I \longrightarrow X$ relative to S between Id_X and $j \circ r$:

$$H(x, 0) = x, \quad H(x, 1) = r(x) \ \text{ on } \ X, \qquad H(x, t) = x \ \text{ on } \ S \times I.$$

Then X and S are homotopically equivalent. In particular X is said to be *contractible* if X has a deformation retraction onto a point x_0. In this case

$$H^q(X, M) = H^q(\{x_0\}, M) = \begin{cases} M & \text{for } q = 0 \\ 0 & \text{for } q \neq 0. \end{cases}$$

(14.6) Corollary. *If X is a compact differentiable manifold, the cohomology groups $H^q(X, R)$ are finitely generated over R.*

Proof. Lemma 6.9 shows that X has a finite covering $\mathfrak{U}$ such that the intersections $U_{\alpha_0 \ldots \alpha_q}$ are contractible. Hence $\mathfrak{U}$ is acyclic, $H^q(X, R) = H^q\big(C^\bullet(\mathfrak{U}, R)\big)$ and each Čech cochain space is a finitely generated (free) module. □

(14.7) Example: Cohomology Groups of Spheres. Set

$$S^n = \big\{x \in \mathbb{R}^{n+1} \ ; \ x_0^2 + x_1^2 + \ldots + x_n^2 = 1\big\}, \quad n \geq 1.$$

We will prove by induction on n that

$$(14.8) \quad H^q(S^n, M) = \begin{cases} M & \text{for } q = 0 \text{ or } q = n \\ 0 & \text{otherwise.} \end{cases}$$

As S^n is connected, we have $H^0(S^n, M) = M$. In order to compute the higher cohomology groups, we apply the Mayer-Vietoris exact sequence 3.11 to the covering (U_1, U_2) with

$$U_1 = S^n \smallsetminus \{(-1, 0, \ldots, 0)\}, \qquad U_2 = S^n \smallsetminus \{(1, 0, \ldots, 0)\}.$$

Then $U_1, U_2 \approx \mathbb{R}^n$ are contractible, and $U_1 \cap U_2$ can be retracted by deformation on the equator $S^n \cap \{x_0 = 0\} \approx S^{n-1}$. Omitting M in the notations of cohomology groups, we get exact sequences

$$(14.9') \qquad H^0(U_1) \oplus H^0(U_2) \longrightarrow H^0(U_1 \cap U_2) \longrightarrow H^1(S^n) \longrightarrow 0,$$

$$(14.9'') \qquad 0 \longrightarrow H^{q-1}(U_1 \cap U_2) \longrightarrow H^q(S^n) \longrightarrow 0, \quad q \geq 2.$$

For $n = 1$, $U_1 \cap U_2$ consists of two open arcs, so we have

$$H^0(U_1) \oplus H^0(U_2) = H^0(U_1 \cap U_2) = M \times M,$$

and the first arrow in (14.9′) is $(m_1, m_2) \longmapsto (m_2 - m_1, m_2 - m_1)$. We infer easily that $H^1(S^1) = M$ and that

$$H^q(S^1) = H^{q-1}(U_1 \cap U_2) = 0 \quad \text{for} \quad q \geq 2.$$

For $n \geq 2$, $U_1 \cap U_2$ is connected, so the first arrow in (14.9′) is onto and $H^1(S^n) = 0$. The second sequence (14.9″) yields $H^q(S^n) \simeq H^{q-1}(S^{n-1})$. An induction concludes the proof. □

14.B. Relative Cohomology Groups and Excision Theorem

Let X be a topological space and S a subspace. We denote by $M^{[q]}(X, S)$ the subgroup of sections $u \in M^{[q]}(X)$ such that $u(x_0, \ldots, x_q) = 0$ when

$$(x_0, \ldots, x_q) \in S^q, \quad x_1 \in V(x_0), \ \ldots, \ x_q \in V(x_0, \ldots, x_{q-1}).$$

Then $M^{[\bullet]}(X, S)$ is a subcomplex of $M^{[\bullet]}(X)$ and we define the *relative cohomology group* of the pair (X, S) with values in M as

$$(14.10) \quad H^q(X, S\,; M) = H^q\big(M^{[\bullet]}(X, S)\big).$$

By definition of $M^{[q]}(X, S)$, there is an exact sequence

$$(14.11) \quad 0 \longrightarrow M^{[q]}(X, S) \longrightarrow M^{[q]}(X) \longrightarrow (M_{\restriction S})^{[q]}(S) \longrightarrow 0.$$

The reader should take care of the fact that $(M_{\restriction S})^{[q]}(S)$ does not coincide with the module of sections $M^{[q]}(S)$ of the sheaf $M^{[q]}$ on X, except if S is open. The snake lemma shows that there is an "exact sequence of the pair":

$$(14.12) \quad H^q(X, S\,; M) \to H^q(X, M) \to H^q(S, M) \to H^{q+1}(X, S\,; M) \cdots.$$

We have in particular $H^0(X,S\,;\,M)=M^E$, where E is the set of connected components of X which do not meet S. More generally, for a triple (X,S,T) with $X\supset S\supset T$, there is an "exact sequence of the triple":

$$(14.12')\qquad 0\longrightarrow M^{[q]}(X,S)\longrightarrow M^{[q]}(X,T)\longrightarrow M^{[q]}(S,T)\longrightarrow 0,$$
$$H^q(X,S\,;M)\longrightarrow H^q(X,T\,;\,M)\longrightarrow H^q(S,T\,;\,M)\longrightarrow H^{q+1}(X,S\,;\,M).$$

The definition of the cup product in (8.2) shows that $\alpha\smile\beta$ vanishes on $S\cup S'$ if α,β vanish on S, S' respectively. Therefore, we obtain a bilinear map

$$(14.13)\quad H^p(X,S\,;\,M)\times H^q(X,S'\,;\,M')\longrightarrow H^{p+q}(X,S\cup S'\,;\,M\otimes M').$$

If $f:(X,S)\longrightarrow(Y,T)$ is a morphism of pairs, i.e. a continuous map $X\to Y$ such that $f(S)\subset T$, there is an induced pull-back morphism

$$(14.14)\quad f^\star:\quad H^q(Y,T\,;\,M)\longrightarrow H^q(X,S\,;\,M)$$

which is compatible with the cup product. Two morphisms of pairs f,g are said to be homotopic when there is a pair homotopy $H:(X\times I,S\times I)\longrightarrow(Y,T)$. An application of the exact sequence of the pair shows that

$$\pi^\star:H^q(X,S\,;\,M)\longrightarrow H^q(X\times I,S\times I\,;\,M)$$

is an isomorphism. It follows as above that $f^\star=g^\star$ as soon as f,g are homotopic.

(14.15) Excision theorem. *For subspaces $\overline{T}\subset S^\circ$ of X, the restriction morphism $H^q(X,S\,;\,M)\longrightarrow H^q(X\smallsetminus T,S\smallsetminus T\,;\,M)$ is an isomorphism.*

Proof. Under our assumption, it is not difficult to check that the surjective restriction map $M^{[q]}(X,S)\longrightarrow M^{[q]}(X\smallsetminus T,S\smallsetminus T)$ is also injective, because the kernel consists of sections $u\in M^{[q]}(X)$ such that $u(x_0,\ldots,x_q)=0$ on $(X\smallsetminus T)^{q+1}\cup S^{q+1}$, and this set is a neighborhood of the diagonal of X^{q+1}. □

(14.16) Proposition. *If S is open or strongly paracompact in X, the relative cohomology groups can be written in terms of cohomology groups with supports in $X\smallsetminus S$:*

$$H^q(X,S\,;\,M)\simeq H^q_{X\smallsetminus S}(X,M).$$

In particular, if $X\smallsetminus S$ is relatively compact in X, we have

$$H^q(X,S\,;\,M)\simeq H^q_c(X\smallsetminus S,M).$$

Proof. We have an exact sequence

$$(14.17)\quad 0\longrightarrow M^{[\bullet]}_{X\smallsetminus S}(X)\longrightarrow M^{[\bullet]}(X)\longrightarrow M^{[\bullet]}(S)\longrightarrow 0$$

where $M^{[\bullet]}_{X\smallsetminus S}(X)$ denotes sections with support in $X \smallsetminus S$. If S is open, then $M^{[\bullet]}(S) = (M_{\restriction S})^{[\bullet]}(S)$, hence $M^{[\bullet]}_{X\smallsetminus S}(X) = M^{[\bullet]}(X, S)$ and the result follows. If S is strongly paracompact, Prop. 4.7 and Th. 9.10 show that

$$H^q\big(M^{[\bullet]}(S)\big) = H^q\big(\varinjlim_{\Omega\supset S} M^{[\bullet]}(\Omega)\big) = \varinjlim_{\Omega\supset S} H^q(\Omega, M) = H^q(S, M_{\restriction S}).$$

If we consider the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & M^{[\bullet]}_{X\smallsetminus S}(X) & \longrightarrow & M^{[\bullet]}(X) & \longrightarrow & M^{[\bullet]}(S) & \longrightarrow 0 \\
& \big\downarrow & & \big\downarrow \mathrm{Id} & & \big\downarrow \restriction S & \\
0 \longrightarrow & M^{[\bullet]}(X, S) & \longrightarrow & M^{[\bullet]}(X) & \longrightarrow & (M_{\restriction S})^{[\bullet]}(S) & \longrightarrow 0
\end{array}$$

we see that the last two vertical arrows induce isomorphisms in cohomology. Therefore, the first one also does. $\square$

(14.18) Corollary. *Let X, Y be locally compact spaces and $f, g : X \to Y$ proper maps. We say that f, g are properly homotopic if they are homotopic through a proper homotopy $H : X \times I \longrightarrow Y$. Then*

$$f^\star = g^\star \ : \ H^q_c(Y, M) \longrightarrow H^q_c(X, M).$$

Proof. Let $\widehat{X} = X \cup \{\infty\}$, $\widehat{Y} = Y \cup \{\infty\}$ be the Alexandrov compactifications of X, Y. Then f, g, H can be extended as continuous maps

$$\widehat{f}, \widehat{g} \ : \ \widehat{X} \longrightarrow \widehat{Y}, \qquad \widehat{H} \ : \ \widehat{X} \times I \longrightarrow \widehat{Y}$$

with $\widehat{f}(\infty) = \widehat{g}(\infty) = H(\infty, t) = \infty$, so that $\widehat{f}, \widehat{g}$ are homotopic as maps $(\widehat{X}, \infty) \longrightarrow (\widehat{Y}, \infty)$. Proposition 14.16 implies $H^q_c(X, M) = H^q(\widetilde{X}, \infty\, ; M)$ and the result follows. $\square$

15. Künneth Formula

15.A. Flat Modules and Tor Functors

The goal of this section is to investigate homological properties related to tensor products. We work in the category of modules over a commutative ring R with unit. All tensor products appearing here are tensor products over R. The starting point is the observation that tensor product with a given module is a right exact functor: if $0 \to A \to B \to C \to 0$ is an exact sequence and M a R-module, then

$$A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is exact, but the map $A \otimes M \longrightarrow B \otimes M$ need not be injective. A counterexample is given by the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \quad \text{over} \quad R = \mathbb{Z}$$

tensorized by $M = \mathbb{Z}/2\mathbb{Z}$. However, the injectivity holds if M is a free R-module. More generally, one says that M is a *flat R-module* if the tensor product by M preserves injectivity, or equivalently, if $\otimes M$ is a left exact functor.

A *flat resolution* $C_\bullet$ of a R-module A is a homology exact sequence

$$\cdots \longrightarrow C_q \longrightarrow C_{q-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow A \longrightarrow 0$$

where C_q are flat R-modules and $C_q = 0$ for $q < 0$. Such a resolution always exists because every module A is a quotient of a free module C_0. Inductively, we take C_{q+1} to be a free module such that $\ker(C_q \to C_{q-1})$ is a quotient of C_{q+1}. In terms of homology groups, we have $H_0(C_\bullet) = A$ and $H_q(C_\bullet) = 0$ for $q \neq 0$. Given R-modules A, B and free resolutions $d' : C_\bullet \longrightarrow A$, $d'' : D_\bullet \longrightarrow B$, we consider the double homology complex

$$K_{p,q} = C_p \otimes D_q, \qquad d = d' \otimes \mathrm{Id} + (-1)^p \,\mathrm{Id} \otimes d''$$

and the associated first and second spectral sequences. Since C_p is free, we have

$$E^1_{p,q} = H_q(C_p \otimes D_\bullet) = \begin{cases} C_p \otimes B & \text{for} \quad q = 0, \\ 0 & \text{for} \quad q \neq 0. \end{cases}$$

Similarly, the second spectral sequence also collapses and we have

$$H_l(K_\bullet) = H_l(C_\bullet \otimes B) = H_l(A \otimes D_\bullet).$$

This implies in particular that the homology groups $H_l(K_\bullet)$ do not depend on the choice of the resolutions $C_\bullet$ or $D_\bullet$.

(15.1) Definition. *The q-th torsion module of A and B is*

$$\mathrm{Tor}_q(A, B) = H_q(K_\bullet) = H_q(C_\bullet \otimes B) = H_q(A \otimes D_\bullet).$$

Since the definition of $K_\bullet$ is symmetric with respect to A and B, we have $\mathrm{Tor}_q(A, B) \simeq \mathrm{Tor}_q(B, A)$. By the right-exactness of $\otimes B$, we find in particular $\mathrm{Tor}_0(A, B) = A \otimes B$. Moreover, if B is flat, $\otimes B$ is also left exact, thus $\mathrm{Tor}_q(A, B) = 0$ for all $q \geq 1$ and all modules A. If $0 \to A \to A' \to A'' \to 0$ is an exact sequence, there is a corresponding exact sequence of homology complexes

$$0 \longrightarrow A \otimes D_\bullet \longrightarrow A' \otimes D_\bullet \longrightarrow A'' \otimes D_\bullet \longrightarrow 0,$$

thus a long exact sequence

$$(15.2)\qquad \begin{array}{l} \longrightarrow \mathrm{Tor}_q(A,B) \longrightarrow \mathrm{Tor}_q(A',B) \longrightarrow \mathrm{Tor}_q(A'',B) \longrightarrow \mathrm{Tor}_{q-1}(A,B) \\ \cdots \longrightarrow \quad A\otimes B \quad \longrightarrow \quad A'\otimes B \quad \longrightarrow \quad A''\otimes B \quad \longrightarrow \quad 0. \end{array}$$

It follows that B is flat if and only if $\mathrm{Tor}_1(A,B)=0$ for every R-module A.

Suppose now that R is a *principal ring*. Then every module A has a free resolution $0\to C_1\to C_0\to A\to 0$ because the kernel of any surjective map $C_0\to A$ is free (every submodule of a free module is free). It follows that one always has $\mathrm{Tor}_q(A,B)=0$ for $q\ge 2$. In this case, we denote $\mathrm{Tor}_1(A,B)=A\star B$ and call it the *torsion product* of A and B. The above exact sequence (15.2) reduces to

$$(15.3)\qquad 0\to A\star B\to A'\star B\to A''\star B\to A\otimes B\to A'\otimes B\to A''\otimes B\to 0.$$

In order to compute $A\star B$, we may restrict ourselves to finitely generated modules, because every module is a direct limit of such modules and the $\star$ product commutes with direct limits. Over a principal ring R, every finitely generated module is a direct sum of a free module and of cyclic modules R/aR. It is thus sufficient to compute $R/aR\star R/bR$. The obvious free resolution $R\xrightarrow{a\times} R$ of R/aR shows that $R/aR\star R/bR$ is the kernel of the map $R/bR\xrightarrow{a\times} R/bR$. Hence

$$(15.4)\qquad R/aR\star R/bR\simeq R/(a\wedge b)R$$

where $a\wedge b$ denotes the greatest common divisor of a and b. It follows that a module B is flat if and only if it is torsion free. If R is a field, every R-module is free, thus $A\star B=0$ for all A and B.

15.B. Künneth and Universal Coefficient Formulas

The algebraic Künneth formula describes the cohomology groups of the tensor product of two differential complexes.

(15.5) Algebraic Künneth formula. *Let $(K^\bullet,d')$, $(L^\bullet,d'')$ be complexes of R-modules and $(K\otimes L)^\bullet$ the simple complex associated to the double complex $(K\otimes L)^{p,q}=K^p\otimes L^q$. If $K^\bullet$ or $L^\bullet$ is torsion free, there is a split exact sequence*

$$0\to \bigoplus_{p+q=l} H^p(K^\bullet)\otimes H^q(L^\bullet)\xrightarrow{\mu} H^l\big((K\otimes L)^\bullet\big)\to \bigoplus_{p+q=l+1} H^p(K^\bullet)\star H^q(L^\bullet)\to 0$$

where the map μ is defined by $\mu(\{k^p\}\times\{l^q\})=\{k^p\otimes l^q\}$ for all cocycles $\{k^p\}\in H^p(K^\bullet)$, $\{l^q\}\in H^q(L^\bullet)$.

(15.6) Corollary. *If R is a field, or if one of the graded modules $H^\bullet(K^\bullet)$, $H^\bullet(L^\bullet)$ is torsion free, then*

$$H^l\big((K\otimes L)^\bullet\big) \simeq \bigoplus_{p+q=l} H^p(K^\bullet)\otimes H^q(L^\bullet).$$

Proof. Assume for example that $K^\bullet$ is torsion free. There is a short exact sequence of complexes

$$0 \longrightarrow Z^\bullet \longrightarrow K^\bullet \xrightarrow{d'} B^{\bullet+1} \longrightarrow 0$$

where $Z^\bullet, B^\bullet \subset K^\bullet$ are respectively the graded modules of cocycles and coboundaries in $K^\bullet$, considered as subcomplexes with zero differential. As $B^{\bullet+1}$ is torsion free, the tensor product of the above sequence with $L^\bullet$ is still exact. The corresponding long exact sequence for the associated simple complexes yields:

$$\begin{aligned} &H^l\big((B\otimes L)^\bullet\big) \longrightarrow H^l\big((Z\otimes L)^\bullet\big) \longrightarrow H^l\big((K\otimes L)^\bullet\big) \xrightarrow{d'} H^{l+1}\big((B\otimes L)^\bullet\big)\\ &\qquad\qquad\longrightarrow H^{l+1}\big((Z\otimes L)^\bullet\big)\cdots. \end{aligned}\tag{15.7}$$

The first and last arrows are connecting homomorphisms; in this situation, they are easily seen to be induced by the inclusion $B^\bullet \subset Z^\bullet$. Since the differential of $Z^\bullet$ is zero, the simple complex $(Z\otimes L)^\bullet$ is isomorphic to the direct sum $\bigoplus_p Z^p\otimes L^{\bullet-p}$, where Z^p is torsion free. Similar properties hold for $(B\otimes L)^\bullet$, hence

$$H^l\big((Z\otimes L)^\bullet\big) = \bigoplus_{p+q=l} Z^p\otimes H^q(L^\bullet), \qquad H^l\big((B\otimes L)^\bullet\big) = \bigoplus_{p+q=l} B^p\otimes H^q(L^\bullet).$$

The exact sequence

$$0 \longrightarrow B^p \longrightarrow Z^p \longrightarrow H^p(K^\bullet) \longrightarrow 0$$

tensorized by $H^q(L^\bullet)$ yields an exact sequence of the type (15.3):

$$\begin{aligned} 0 \to H^p(K^\bullet)\star H^q(L^\bullet) \to B^p\otimes H^q(L^\bullet) \to Z^p\otimes H^q(L^\bullet)&\\ \to H^p(K^\bullet)\otimes H^q(L^\bullet) \to 0.& \end{aligned}$$

By the above equalities, we get

$$\begin{aligned} 0 \longrightarrow \bigoplus_{p+q=l} H^p(K^\bullet)\star H^q(L^\bullet) \longrightarrow H^l\big((B\otimes L)^\bullet\big) \longrightarrow H^l\big((Z\otimes L)^\bullet\big)&\\ \longrightarrow \bigoplus_{p+q=l} H^p(K^\bullet)\otimes H^q(L^\bullet) \longrightarrow 0.& \end{aligned}$$

In our initial long exact sequence (15.7), the cokernel of the first arrow is thus $\bigoplus_{p+q=l} H^p(K^\bullet)\otimes H^q(L^\bullet)$ and the kernel of the last arrow is the torsion sum $\bigoplus_{p+q=l+1} H^p(K^\bullet)\star H^q(L^\bullet)$. This gives the exact sequence of the lemma. We leave the computation of the map μ as an exercise for the reader. The splitting assertion can be obtained by observing that there always exists a torsion free complex $\widetilde{K}^\bullet$ that splits (i.e. $\widetilde{Z}^\bullet \subset \widetilde{K}^\bullet$ splits), and a morphism $\widetilde{K}^\bullet \longrightarrow K^\bullet$ inducing an isomorphism in cohomology; then the projection $\widetilde{K}^\bullet \longrightarrow \widetilde{Z}^\bullet$ yields a projection

$$H^l\big((\widetilde{K}\otimes L)^\bullet\big) \longrightarrow H^l\big((\widetilde{Z}\otimes L)^\bullet\big) \simeq \bigoplus_{p+q=l} \widetilde{Z}^p \otimes H^q(L^\bullet)$$
$$\longrightarrow \bigoplus_{p+q=l} H^p(\widetilde{K}^\bullet)\otimes H^q(L^\bullet).$$

To construct $\widetilde{K}^\bullet$, let $\widetilde{Z}^\bullet \longrightarrow Z^\bullet$ be a surjective map with $\widetilde{Z}^\bullet$ free, $\widetilde{B}^\bullet$ the inverse image of $B^\bullet$ in $\widetilde{Z}^\bullet$ and $\widetilde{K}^\bullet = \widetilde{Z}^\bullet \oplus \widetilde{B}^{\bullet+1}$, where the differential $\widetilde{K}^\bullet \longrightarrow \widetilde{K}^{\bullet+1}$ is given by $\widetilde{Z}^\bullet \longrightarrow 0$ and $\widetilde{B}^{\bullet+1} \subset \widetilde{Z}^{\bullet+1} \oplus 0$; as $\widetilde{B}^\bullet$ is free, the map $\widetilde{B}^{\bullet+1} \longrightarrow B^{\bullet+1}$ can be lifted to a map $\widetilde{B}^{\bullet+1} \longrightarrow K^\bullet$, and this lifting combined with the composite $\widetilde{Z}^\bullet \to Z^\bullet \subset K^\bullet$ yields the required complex morphism $\widetilde{K}^\bullet = \widetilde{Z}^\bullet \oplus \widetilde{B}^{\bullet+1} \longrightarrow K^\bullet$. □

(15.8) Universal coefficient formula. *Let $K^\bullet$ be a complex of R-modules and M a R-module such that either $K^\bullet$ or M is torsion free. Then there is a split exact sequence*

$$0 \longrightarrow H^p(K^\bullet)\otimes M \longrightarrow H^p(K^\bullet \otimes M) \longrightarrow H^{p+1}(K^\bullet) \star M \longrightarrow 0.$$

Indeed, this is a special case of Formula 15.5 when the complex $L^\bullet$ is reduced to one term $L^0 = M$. In general, it is interesting to observe that the spectral sequence of $K^\bullet \otimes L^\bullet$ collapses in E_2 if $K^\bullet$ is torsion free: $H^k\big((K\otimes L)^\bullet\big)$ is in fact the direct sum of the terms $E_2^{p,q} = H^p\big(K^\bullet \otimes H^q(L^\bullet)\big)$ thanks to (15.8).

15.C. Künneth Formula for Sheaf Cohomology H

ere we apply the general algebraic machinery to compute cohomology groups over a product space $X\times Y$. The main argument is a combination of the Leray spectral sequence with the universal coefficient formula for sheaf cohomology.

(15.9) Theorem. *Let $\mathcal{A}$ be a sheaf of R-modules over a topological space X and M a R-module. Assume that either $\mathcal{A}$ or M is torsion free and that either X is compact or M is finitely generated. Then there is a split exact sequence*

$$0 \longrightarrow H^p(X,\mathcal{A})\otimes M \longrightarrow H^p(X,\mathcal{A}\otimes M) \longrightarrow H^{p+1}(X,\mathcal{A}) \star M \longrightarrow 0.$$

Proof. If M is finitely generated, we get $(\mathcal{A}\otimes M)^{[\bullet]}(X) = \mathcal{A}^{[\bullet]}(X)\otimes M$ easily, so the above exact sequence is a consequence of Formula 15.8. If X is compact, we may consider Čech cochains $C^q(\mathcal{U},\mathcal{A}\otimes M)$ over finite coverings. There is an obvious morphism

$$C^q(\mathcal{U},\mathcal{A})\otimes M\longrightarrow C^q(\mathcal{U},\mathcal{A}\otimes M)$$

but this morphism need not be surjective nor injective. However, since

$$(\mathcal{A}\otimes M)_x=\mathcal{A}_x\otimes M=\varinjlim_{V\ni x}\ \mathcal{A}(V)\otimes M,$$

the following properties are easy to verify:

a) If $c\in C^q(\mathcal{U},\mathcal{A}\otimes M)$, there is a refinement $\mathcal{V}$ of $\mathcal{U}$ and $\rho:\mathcal{V}\longrightarrow\mathcal{U}$ such that $\rho^\star c\in C^q(\mathcal{V},\mathcal{A}\otimes M)$ is in the image of $C^q(\mathcal{V},\mathcal{A})\otimes M$.

b) If a tensor $t\in C^q(\mathcal{U},\mathcal{A})\otimes M$ is mapped to 0 in $C^q(\mathcal{U},\mathcal{A}\otimes M)$, there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\rho^\star t\in C^q(\mathcal{V},\mathcal{A})\otimes M$ equals 0.

From a) and b) it follows that

$$\check{H}^q(X,\mathcal{A}\otimes M)=\varinjlim_{\mathcal{U}}\ H^q\big(C^\bullet(\mathcal{U},\mathcal{A}\otimes M)\big)=\varinjlim_{\mathcal{U}}\ H^q\big(C^\bullet(\mathcal{U},\mathcal{A})\otimes M\big)$$

and the desired exact sequence is the direct limit of the exact sequences of Formula 15.8 obtained for $K^\bullet=C^\bullet(\mathcal{U},\mathcal{A})$. □

(15.10) Theorem (Künneth). *Let $\mathcal{A}$ and $\mathcal{B}$ be sheaves of R-modules over topological spaces X and Y. Assume that $\mathcal{A}$ is torsion free, that Y is compact and that either X is compact or the cohomology groups $H^q(Y,\mathcal{B})$ are finitely generated R-modules. There is a split exact sequence*

$$0\longrightarrow\bigoplus_{p+q=l}H^p(X,\mathcal{A})\otimes H^q(Y,\mathcal{B})\xrightarrow{\ \mu\ }H^l(X\times Y,\mathcal{A}\boxtimes\mathcal{B})$$
$$\longrightarrow\bigoplus_{p+q=l+1}H^p(X,\mathcal{A})\star H^q(Y,\mathcal{B})\longrightarrow 0$$

where μ is the map given by the cartesian product $\bigoplus\alpha_p\otimes\beta_q\longmapsto\sum\alpha_p\times\beta_q$.

Proof. We compute $H^l(X,\mathcal{A}\boxtimes\mathcal{B})$ by means of the Leray spectral sequence of the projection $\pi:X\times Y\longrightarrow X$. This means that we are considering the differential sheaf $\mathcal{L}^q=\pi_\star(\mathcal{A}\boxtimes\mathcal{B})^{[q]}$ and the double complex

$$K^{p,q}=(\mathcal{L}^q)^{[p]}(X).$$

By (12.5′) we have ${}_KE_2^{p,q}=H^p\big(X,\mathcal{H}^q(\mathcal{L}^\bullet)\big)$. As Y is compact, the cohomology sheaves $\mathcal{H}^q(\mathcal{L}^\bullet)=R^q\pi_\star(\mathcal{A}\boxtimes\mathcal{B})$ are given by

$$R^q\pi_\star(\mathcal{A}\boxtimes\mathcal{B})_x=H^q(\{x\}\times Y,\mathcal{A}\boxtimes\mathcal{B}_{\restriction\{x\}\times Y})=H^q(Y,\mathcal{A}_x\otimes\mathcal{B})=\mathcal{A}_x\otimes H^q(Y,\mathcal{B})$$

thanks to the compact case of Th. 15.9 where $M=\mathcal{A}_x$ is torsion free. We obtain therefore

$$R^q\pi_\star(\mathcal{A}\boxtimes\mathcal{B}) = \mathcal{A}\otimes H^q(Y,\mathcal{B}),$$
$$_KE_2^{p,q} = H^p\big(X,\mathcal{A}\otimes H^q(Y,\mathcal{B})\big).$$

Theorem 15.9 shows that the E_2-term is actually given by the desired exact sequence, but it is not a priori clear that the spectral sequence collapses in E_2. In order to check this, we consider the double complex

$$C^{p,q} = \mathcal{A}^{[p]}(X)\otimes\mathcal{B}^{[q]}(Y)$$

and construct a natural morphism $C^{\bullet,\bullet}\longrightarrow K^{\bullet,\bullet}$. We may consider $K^{p,q} = \big(\pi_\star(\mathcal{A}\boxtimes\mathcal{B})^{[q]}\big)^{[p]}(X)$ as the set of equivalence classes of functions

$$h\big(\xi_0,\ldots,\xi_p\big)\in\pi_\star(\mathcal{A}\boxtimes\mathcal{B})^{[q]}_{\xi_p} = \varinjlim\ (\mathcal{A}\boxtimes\mathcal{B})^{[q]}\big(\pi^{-1}\big(V(\xi_p)\big)\big)$$

or more precisely

$$h\big(\xi_0,\ldots,\xi_p\,;(x_0,y_0),\ldots,(x_q,y_q)\big)\in\mathcal{A}_{x_q}\otimes\mathcal{B}_{y_q}\quad\text{with}$$
$$\xi_0\in X,\quad \xi_j\in V(\xi_0,\ldots,\xi_{j-1}),\quad 1\le j\le p,$$
$$(x_0,y_0)\in V(\xi_0,\ldots,\xi_p)\times Y,$$
$$(x_j,y_j)\in V\big(\xi_0,\ldots,\xi_p\,;(x_0,y_0),\ldots,(x_{j-1},y_{j-1})\big),\quad 1\le j\le q.$$

Then $f\otimes g\in C^{p,q}$ is mapped to $h\in K^{p,q}$ by the formula

$$h\big(\xi_0,\ldots,\xi_p\,;(x_0,y_0),\ldots,(x_q,y_q)\big) = f(\xi_0,\ldots,\xi_p)(x_q)\otimes g(y_0,\ldots,y_q).$$

As $\mathcal{A}^{[p]}(X)$ is torsion free, we find

$$_CE_1^{p,q} = \mathcal{A}^{[p]}(X)\otimes H^q(Y,\mathcal{B}).$$

Since either X is compact or $H^q(Y,\mathcal{B})$ finitely generated, Th. 15.9 yields

$$_CE_2^{p,q} = H^p\big(X,\mathcal{A}\otimes H^q(Y,\mathcal{B})\big)\simeq {}_KE_2^{p,q}$$

hence $H^l(K^\bullet)\simeq H^l(C^\bullet)$ and the algebraic Künneth formula 15.5 concludes the proof. □

(15.11) Remark. The exact sequences of Th. 15.9 and of Künneth's theorem also hold for cohomology groups with compact support, provided that X and Y are locally compact and $\mathcal{A}$ (or $\mathcal{B}$) is torsion free. This is an immediate consequence of Prop. 7.12 on direct limits of cohomology groups over compact subsets.

(15.12) Corollary. *When $\mathcal{A}$ and $\mathcal{B}$ are torsion free constant sheaves, e.g. $\mathcal{A}=\mathcal{B}=\mathbb{Z}$ or $\mathbb{R}$, the Künneth formula holds as soon as X or Y has the same homotopy type as a finite cell complex.*

Proof. If Y satisfies the assumption, we may suppose in fact that Y is a finite cell complex by the homotopy invariance. Then Y is compact and $H^\bullet(Y,\mathcal{B})$ is finitely generated, so Th. 15.10 can be applied. □

16. Poincaré duality

16.A. Injective Modules and Ext Functors

The study of duality requires some algebraic preliminaries on the Hom functor and its derived functors Ext^q. Let R be a commutative ring with unit, M a R-module and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

an exact sequence of R-modules. Then we have exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(M, A) &\longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C), \\ \mathrm{Hom}_R(A, M) &\longleftarrow \mathrm{Hom}_R(B, M) \longleftarrow \mathrm{Hom}_R(C, M) \longleftarrow 0, \end{aligned}$$

i.e. $\mathrm{Hom}(M, \bullet)$ is a covariant left exact functor and $\mathrm{Hom}(\bullet, M)$ a contravariant right exact functor. The module M is said to be *projective* if $\mathrm{Hom}(M, \bullet)$ is also right exact, and *injective* if $\mathrm{Hom}(\bullet, M)$ is also left exact. Every free R-module is projective. Conversely, if M is projective, any surjective morphism $F \longrightarrow M$ from a free module F onto M must split $\big(\mathrm{Id}_M$ has a preimage in $\mathrm{Hom}(M, F)\big)$; if R is a principal ring, "projective" is therefore equivalent to "free".

(16.1) Proposition. *Over a principal ring R, a module M is injective if and only if it is divisible, i.e. if for every $x \in M$ and $\lambda \in R \smallsetminus \{0\}$, there exists $y \in M$ such that $\lambda y = x$.*

Proof. If M is injective, the exact sequence $0 \longrightarrow R \xrightarrow{\lambda\times} R \longrightarrow R/\lambda R \longrightarrow 0$ shows that

$$M = \mathrm{Hom}(R, M) \xrightarrow{\lambda\times} \mathrm{Hom}(R, M) = M$$

must be surjective, thus M is divisible.

Conversely, assume that R is divisible. Let $f : A \longrightarrow M$ be a morphism and $B \supset A$. Zorn's lemma implies that there is a maximal extension $\widetilde{f} : \widetilde{A} \longrightarrow M$ of f where $A \subset \widetilde{A} \subset B$. If $\widetilde{A} \neq B$, select $x \in B \smallsetminus \widetilde{A}$ and consider the ideal I of elements $\lambda \in R$ such that $\lambda x \in \widetilde{A}$. As R is principal we have $I = \lambda_0 R$ for some λ_0. If $\lambda_0 \neq 0$, select $y \in M$ such that $\lambda_0 y = \widetilde{f}(\lambda_0 x)$ and if $\lambda_0 = 0$ take y arbitrary. Then $\widetilde{f}$ can be extended to $\widetilde{A} + Rx$ by letting $\widetilde{f}(x) = y$. This is a contradiction, so we must have $\widetilde{A} = B$. □

(16.2) Proposition. *Every module M can be embedded in an injective module $\widetilde{M}$.*

Proof. Assume first $R = \mathbb{Z}$. Then set

$$M' = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), \qquad M'' = \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z}) \subset \mathbb{Q}/\mathbb{Z}^{M'}.$$

Since $\mathbb{Q}/\mathbb{Z}$ is divisible, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}^{M'}$ are injective. It is therefore sufficient to show that the canonical morphism $M \longrightarrow M''$ is injective. In fact, for any $x \in M \smallsetminus \{0\}$, the subgroup $\mathbb{Z}x$ is cyclic (finite or infinite), so there is a non trivial morphism $\mathbb{Z}x \longrightarrow \mathbb{Q}/\mathbb{Z}$, and we can extend this morphism into a morphism $u : M \longrightarrow \mathbb{Q}/\mathbb{Z}$. Then $u \in M'$ and $u(x) \neq 0$, so $M \longrightarrow M''$ is injective.

Now, for an arbitrary ring R, we set $\widetilde{M} = \mathrm{Hom}_{\mathbb{Z}}\big(R, \mathbb{Q}/\mathbb{Z}^{M'}\big)$. There are R-linear embeddings

$$M = \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(R, \mathbb{Q}/\mathbb{Z}^{M'}\big) = \widetilde{M}$$

and since $\mathrm{Hom}_R(\bullet, \widetilde{M}) \simeq \mathrm{Hom}_{\mathbb{Z}}\big(\bullet, \mathbb{Q}/\mathbb{Z}^{M'}\big)$, it is clear that $\widetilde{M}$ is injective over the ring R. □

As a consequence, any module has a (cohomological) resolution by injective modules. Let A, B be given R-modules, let $d' : B \to D^\bullet$ be an injective resolution of B and let $d'' : C_\bullet \to A$ be a free (or projective) resolution of A. We consider the cohomology double complex

$$K^{p,q} = \mathrm{Hom}(C_q, D^p), \qquad d = d' + (-1)^p (d'')^\dagger$$

($\dagger$ means transposition) and the associated first and second spectral sequences. Since $\mathrm{Hom}(\bullet, D^p)$ and $\mathrm{Hom}(C_q, \bullet)$ are exact, we get

$$E_1^{p,0} = \mathrm{Hom}(A, D^p), \qquad \widetilde{E}_1^{p,0} = \mathrm{Hom}(C_p, B),$$
$$E_1^{p,q} = \widetilde{E}_1^{p,q} = 0 \quad \text{for} \quad q \neq 0.$$

Therefore, both spectral sequences collapse in E_1 and we get

$$H^l(K^\bullet) = H^l\big(\mathrm{Hom}(A, D^\bullet)\big) = H^l\big(\mathrm{Hom}(C_\bullet, B)\big) \ ;$$

in particular, the cohomology groups $H^l(K^\bullet)$ do not depend on the choice of the resolutions $C_\bullet$ or $D^\bullet$.

(16.3) Definition. *The q-th extension module of A, B is*

$$\mathrm{Ext}^q_R(A, B) = H^q(K^\bullet) = H^q\big(\mathrm{Hom}(A, D^\bullet)\big) = H^q\big(\mathrm{Hom}(C_\bullet, B)\big).$$

By the left exactness of $\mathrm{Hom}(A, \bullet)$, we get in particular $\mathrm{Ext}^0(A, B) = \mathrm{Hom}(A, B)$. If A is projective or B injective, then clearly $\mathrm{Ext}^q(A, B) = 0$ for all $q \geq 1$. Any exact sequence $0 \to A \to A' \to A'' \to 0$ is converted into an exact sequence by $\mathrm{Hom}(\bullet, D^\bullet)$, thus we get a long exact sequence

$$0 \longrightarrow \mathrm{Hom}(A'', B) \longrightarrow \mathrm{Hom}(A', B) \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Ext}^1(A'', B) \cdots$$
$$\longrightarrow \mathrm{Ext}^q(A'', B) \longrightarrow \mathrm{Ext}^q(A', B) \longrightarrow \mathrm{Ext}^q(A, B) \longrightarrow \mathrm{Ext}^{q+1}(A'', B) \cdots$$

Similarly, any exact sequence $0 \to B \to B' \to B'' \to 0$ yields

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B') \longrightarrow \mathrm{Hom}(A,B'') \longrightarrow \mathrm{Ext}^1(A,B) \cdots\\ \longrightarrow \mathrm{Ext}^q(A,B) \longrightarrow \mathrm{Ext}^q(A,B') \longrightarrow \mathrm{Ext}^q(A,B'') \longrightarrow \mathrm{Ext}^{q+1}(A,B) \cdots\end{aligned}$$

Suppose now that R is a principal ring. Then the resolutions $C_\bullet$ or $D^\bullet$ can be taken of length 1 (any quotient of a divisible module is divisible), thus $\mathrm{Ext}^q(A,B)$ is always 0 for $q \ge 2$. In this case, we simply denote $\mathrm{Ext}^1(A,B) = \mathrm{Ext}(A,B)$. When A is finitely generated, the computation of $\mathrm{Ext}(A,B)$ can be reduced to the cyclic case, since $\mathrm{Ext}(A,B) = 0$ when A is free. For $A = R/aR$, the obvious free resolution $R \xrightarrow{a\times} R$ gives

(16.4) $\quad \mathrm{Ext}_R(R/aR, B) = B/aB.$

(16.5) Lemma. *Let $K_\bullet$ be a homology complex and let $M \to M^\bullet$ be an injective resolution of a R-module M. Let $L^\bullet$ be the simple complex associated to $L^{p,q} = \mathrm{Hom}_R(K_q, M^p)$. There is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}\big(H_{q-1}(K_\bullet), M\big) \longrightarrow H^q(L^\bullet) \longrightarrow \mathrm{Hom}\,\big(H_q(K_\bullet), M\big) \longrightarrow 0.$$

Proof. As the functor $\mathrm{Hom}_R(\bullet, M^p)$ is exact, we get

$$\begin{aligned}{}_LE_1^{p,q} &= \mathrm{Hom}\,\big(H_q(K_\bullet), M^p\big),\\ {}_LE_2^{p,q} &= \begin{cases}\mathrm{Hom}\,\big(H_q(K_\bullet), M\big) & \text{for} \quad p = 0,\\ \mathrm{Ext}\big(H_q(K_\bullet), M\big) & \text{for} \quad p = 1,\\ 0 & \text{for} \quad p \ge 2.\end{cases}\end{aligned}$$

The spectral sequence collapses in E_2, therefore we get

$$\begin{aligned}G_0\big(H^q(L^\bullet)\big) &= \mathrm{Hom}\,\big(H_q(K_\bullet), M\big),\\ G_1\big(H^q(L^\bullet)\big) &= \mathrm{Ext}\big(H_{q-1}(K_\bullet), M\big)\end{aligned}$$

and the expected exact sequence follows. By the same arguments as at the end of the proof of Formula 15.5, we may assume that $K_\bullet$ is split, so that there is a projection $K_q \longrightarrow Z_q$. Then the composite morphism

$$\begin{aligned}\mathrm{Hom}\,\big(H_q(K_\bullet), M\big) = \mathrm{Hom}(Z_q/B_q, M) \longrightarrow \mathrm{Hom}(K_q/B_q, M)&\\ \subset Z^q(L^\bullet) \longrightarrow H^q(L^\bullet)&\end{aligned}$$

defines a splitting of the exact sequence. □

16.B. Poincaré Duality for Sheaves

Let $\mathcal{A}$ be a sheaf of abelian groups on a locally compact topological space X of finite topological dimension $n = \operatorname{topdim} X$. By 13.12 c), $\mathcal{A}$ admits a soft resolution $\mathcal{L}^\bullet$ of length n. If $M \to M^0 \to M^1 \to 0$ is an injective resolution of a R-module M, we introduce the double complex of presheaves $\mathcal{F}^{p,q}_{\mathcal{A},M}$ defined by

$$\mathcal{F}^{p,q}_{\mathcal{A},M}(U) = \operatorname{Hom}_R\big(\mathcal{L}^{n-q}_c(U), M^p\big), \tag{16.6}$$

where the restriction map $\mathcal{F}^{p,q}_{\mathcal{A},M}(U) \longrightarrow \mathcal{F}^{p,q}_{\mathcal{A},M}(V)$ is the adjoint of the inclusion $\mathcal{L}^{n-q}_c(V) \longrightarrow \mathcal{L}^{n-q}_c(U)$ when $V \subset U$. As $\mathcal{L}^{n-q}$ is soft, any $f \in \mathcal{L}^{n-q}_c(U)$ can be written as $f = \sum f_\alpha$ with (f_α) subordinate to any open covering (U_α) of U ; it follows easily that $\mathcal{F}^{p,q}_{\mathcal{A},M}$ satisfy axioms (II-2.4) of sheaves. The injectivity of M^p implies that $\mathcal{F}^{p,q}_{\mathcal{A},M}$ is a flabby sheaf. By Lemma 16.5, we get a split exact sequence

$$\begin{aligned} &0 \longrightarrow \operatorname{Ext}\big(H^{n-q+1}_c(X,\mathcal{A}), M\big) \longrightarrow H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(X)\big) \\ &\qquad\qquad\longrightarrow \operatorname{Hom}\big(H^{n-q}_c(X,\mathcal{A}), M\big) \longrightarrow 0. \end{aligned} \tag{16.7}$$

This can be seen as an abstract Poincaré duality formula, relating the cohomology groups of a differential sheaf $\mathcal{F}^\bullet_{\mathcal{A},M}$ "dual" of $\mathcal{A}$ to the dual of the cohomology with compact support of $\mathcal{A}$. In concrete applications, it still remains to compute $H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(X)\big)$. This can be done easily when X is a manifold and $\mathcal{A}$ is a constant or locally constant sheaf.

16.C. Poincaré Duality on Topological Manifolds

Here, X denotes a paracompact topological manifold of dimension n.

(16.8) Definition. *Let L be a R-module. A locally constant sheaf of stalk L on X is a sheaf $\mathcal{A}$ such that every point has a neighborhood Ω on which $\mathcal{A}_{\restriction\Omega}$ is R-isomorphic to the constant sheaf L.*

Thus, a locally constant sheaf $\mathcal{A}$ can be seen as a discrete fiber bundle over X whose fibers are R-modules and whose transition automorphisms are R-linear. If X is locally contractible, a locally constant sheaf of stalk L is given, up to isomorphism, by a representation $\rho : \pi_1(X) \longrightarrow \operatorname{Aut}_R(L)$ of the fundamental group of X, up to conjugation; denoting by $\widetilde{X}$ the universal covering of X, the sheaf $\mathcal{A}$ associated to ρ can be viewed as the quotient of $\widetilde{X} \times L$ by the diagonal action of $\pi_1(X)$. We leave the reader check himself the details of these assertions: in fact similar arguments will be explained in full details in §V-6 when properties of flat vector bundles are discussed.

Let $\mathcal{A}$ be a locally constant sheaf of stalk L, let $\mathcal{L}^\bullet$ be a soft resolution of $\mathcal{A}$ and $\mathcal{F}^{p,q}_{\mathcal{A},M}$ the associated flabby sheaves. For an arbitrary open set $U \subset X$, Formula (16.7) gives a (non canonical) isomorphism

$$H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(U)\big) \simeq \mathrm{Hom}\,\big(H_c^{n-q}(U,\mathcal{A}),M\big) \oplus \mathrm{Ext}\big(H_c^{n-q+1}(U,\mathcal{A}),M\big)$$

and in the special case $q=0$ a canonical isomorphism

$$(16.9)\quad H^0\big(\mathcal{F}^\bullet_{\mathcal{A},M}(U)\big) = \mathrm{Hom}\,\big(H_c^n(U,\mathcal{A}),M\big).$$

For an open subset Ω homeomorphic to $\mathbb{R}^n$, we have $\mathcal{A}_{\restriction\Omega} \simeq L$. Proposition 14.16 and the exact sequence of the pair yield

$$H_c^q(\Omega,L) \simeq H^q(S^n,\{\infty\}\,;\,L) = \begin{cases} L & \text{for}\ \ q=n,\\ 0 & \text{for}\ \ q\neq n.\end{cases}$$

If $\Omega \simeq \mathbb{R}^n$, we find

$$H^0\big(\mathcal{F}^\bullet_{\mathcal{A},M}(\Omega)\big) \simeq \mathrm{Hom}(L,M), \qquad H^1\big(\mathcal{F}^\bullet_{\mathcal{A},M}(\Omega)\big) \simeq \mathrm{Ext}(L,M)$$

and $H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(\Omega)\big) = 0$ for $q\neq 0,1$. Consider open sets $V\subset\Omega$ where V is a deformation retract of Ω. Then the restriction map $H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(\Omega)\big) \longrightarrow H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(V)\big)$ is an isomorphism. Taking the direct limit over all such neighborhoods V of a given point $x\in\Omega$, we see that $\mathcal{H}^0(\mathcal{F}^\bullet_{\mathcal{A},M})$ and $\mathcal{H}^1(\mathcal{F}^\bullet_{\mathcal{A},M})$ are locally constant sheaves of stalks $\mathrm{Hom}(L,M)$ and $\mathrm{Ext}(L,M)$, and that $\mathcal{H}^q(\mathcal{F}^\bullet_{\mathcal{A},M})=0$ for $q\neq 0,1$. When $\mathrm{Ext}(L,M)=0$, the complex $\mathcal{F}^\bullet_{\mathcal{A},M}$ is thus a flabby resolution of $\mathcal{H}^0 = \mathcal{H}^0(\mathcal{F}^\bullet_{\mathcal{A},M})$ and we get isomorphisms

$$(16.10)\ H^q\big(\mathcal{F}^\bullet_{\mathcal{A},M}(X)\big) = H^q(X,\mathcal{H}^0),$$
$$(16.11)\ \mathcal{H}^0(U) = H^0\big(\mathcal{F}^\bullet_{\mathcal{A},M}(U)\big) = \mathrm{Hom}\,\big(H_c^n(U,\mathcal{A}),M\big).$$

(16.12) Definition. *The locally constant sheaf* $\tau_X = \mathcal{H}^0(\mathcal{F}^\bullet_{\mathbb{Z},\mathbb{Z}})$ *of stalk* $\mathbb{Z}$ *defined by*

$$\tau_X(U) = \mathrm{Hom}_{\mathbb{Z}}\,\big(H_c^n(U,\mathbb{Z}),\mathbb{Z}\big)$$

is called the orientation sheaf (or dualizing sheaf) of X*.*

This sheaf is given by a homomorphism $\pi_1(X)\longrightarrow\{1,-1\}$; it is not difficult to check that τ_X coincides with the trivial sheaf $\mathbb{Z}$ if and only if X is orientable (cf. exercice 18.?). In general, $H_c^n(U,\mathcal{A}) = H_c^n(U,\mathbb{Z})\otimes_{\mathbb{Z}}\mathcal{A}(U)$ for any small open set U on which $\mathcal{A}$ is trivial, thus

$$\mathcal{H}^0(\mathcal{F}^\bullet_{\mathcal{A},M}) = \tau_X \otimes_{\mathbb{Z}} \mathrm{Hom}(\mathcal{A},M).$$

A combination of (16.7) and (16.10) then gives:

(16.13) Poincaré duality theorem. *Let* X *be a topological manifold, let* $\mathcal{A}$ *be a locally constant sheaf over* X *of stalk* L *and let* M *be a* R*-module such that* $\mathrm{Ext}(L,M)=0$*. There is a split exact sequence*

$$0 \longrightarrow \operatorname{Ext}\big(H_c^{n-q+1}(X, \mathcal{A}), M\big) \longrightarrow H^q\big(X, \tau_X \otimes \operatorname{Hom}(\mathcal{A}, M)\big) \\ \longrightarrow \operatorname{Hom}\big(H_c^{n-q}(X, \mathcal{A}), M\big) \longrightarrow 0.$$

In particular, if either X is orientable or R has characteristic 2, then

$$0 \longrightarrow \operatorname{Ext}\big(H_c^{n-q+1}(X, R), R\big) \longrightarrow H^q(X, R) \longrightarrow \operatorname{Hom}\big(H_c^{n-q}(X, R), R\big) \\ \longrightarrow 0. \qquad \square$$

(16.14) Corollary. *Let X be a connected topological manifold, $n = \dim X$. Then for any R-module L*

a) $H_c^n(X, \tau_X \otimes L) \simeq L$;

b) $H_c^n(X, L) \simeq L/2L$ *if X is not orientable.*

Proof. First assume that L is free. For $q = 0$ and $\mathcal{A} = \tau_X \otimes L$, the Poincaré duality formula gives an isomorphism

$$\operatorname{Hom}\big(H_c^n(X, \tau_X \otimes L), M\big) \simeq \operatorname{Hom}(L, M)$$

and the isomorphism is functorial with respect to morphisms $M \longrightarrow M'$. Taking $M = L$ or $M = H_c^n(X, \tau_X \otimes L)$, we easily obtain the existence of inverse morphisms $H_c^n(X, \tau_X \otimes L) \longrightarrow L$ and $L \longrightarrow H_c^n(X, \tau_X \otimes L)$, hence equality a). Similarly, for $\mathcal{A} = L$ we get

$$\operatorname{Hom}\big(H_c^n(X, L), M\big) \simeq H^0\big(X, \tau_X \otimes \operatorname{Hom}(L, M)\big).$$

If X is non orientable, then τ_X is non trivial and the global sections of the sheaf $\tau_X \otimes \operatorname{Hom}(L, M)$ consist of 2-torsion elements of $\operatorname{Hom}(L, M)$, that is

$$\operatorname{Hom}\big(H_c^n(X, L), M\big) \simeq \operatorname{Hom}(L/2L, M).$$

Formula b) follows. If L is not free, the result can be extended by using a free resolution $0 \to L_1 \to L_0 \to L \to 0$ and the associated long exact sequence. $\square$

(16.15) Remark. If X is a connected non compact n-dimensional manifold, it can be proved (exercise 18.?) that $H^n(X, \mathcal{A}) = 0$ for every locally constant sheaf $\mathcal{A}$ on X. $\square$

Assume from now on that X is oriented. Replacing M by $L \otimes M$ and using the obvious morphism $M \longrightarrow \operatorname{Hom}(L, L \otimes M)$, the Poincaré duality theorem yields a morphism

$$(16.16) \quad H^q(X, M) \longrightarrow \operatorname{Hom}\big(H_c^{n-q}(X, L), L \otimes M\big),$$

in other words, a bilinear pairing

$$(16.16') \quad H_c^{n-q}(X, L) \times H^q(X, M) \longrightarrow L \otimes M.$$

(16.17) Proposition. *Up to the sign, the above pairing is given by the cup product, modulo the identification $H_c^n(X, L \otimes M) \simeq L \otimes M$.*

Proof. By functoriality in L, we may assume $L = R$. Then we make the following special choices of resolutions:

$$\mathcal{L}^q = R^{[q]} \quad \text{for} \quad q < n, \qquad \mathcal{L}^n = \ker(R^{[q]} \longrightarrow R^{[q+1]}),$$
$$M^0 = \text{an injective module containing } \; M_c^{[n]}(X)/d^{n-1}M_c^{[n-1]}(X).$$

We embed M in M^0 by $\lambda \mapsto u \otimes_{\mathbb{Z}} \lambda$ where $u \in \mathbb{Z}^{[n]}(X)$ is a representative of a generator of $H_c^n(X, \mathbb{Z})$, and we set $M^1 = M^0/M$. The projection map $M^0 \longrightarrow M^1$ can be seen as an extension of

$$\widetilde{d^n} \;:\; M_c^{[n]}(X)/d^{n-1}M_c^{[n-1]}(X) \longrightarrow d^n M_c^{[n]}(X),$$

since $\operatorname{Ker} \widetilde{d^n} \simeq H_c^n(X, M) = M$. The inclusion $d^n M_c^{[n]}(X) \subset M^1$ can be extended into a map $\pi : M_c^{[n+1]}(X) \longrightarrow M^1$. The cup product bilinear map

$$M^{[q]}(U) \times R_c^{[n-q]}(U) \longrightarrow M_c^{[n]}(X) \longrightarrow M^0$$

gives rise to a morphism $M^{[q]}(U) \longrightarrow \mathcal{F}^q_{R,M}(U)$ defined by

$$(16.18) \qquad \begin{array}{rccc} M^{[q]}(U) \longrightarrow & \operatorname{Hom}\big(\mathcal{L}_c^{n-q}(U), M^0\big) & \oplus & \operatorname{Hom}\big(\mathcal{L}_c^{n-q+1}(U), M^1\big) \\ f \longmapsto & (g \longmapsto f \smile g) & \oplus & \big(h \longmapsto \pi(f \smile h)\big). \end{array}$$

This morphism is easily seen to give a morphism of differential sheaves $M^{[\bullet]} \longrightarrow \mathcal{F}^\bullet_{R,M}$, when $M^{[\bullet]}$ is truncated in degree n, i.e. when $M^{[n]}$ is replaced by $\operatorname{Ker} d^n$. The induced morphism

$$M = \mathcal{H}^0(M^{[\bullet]}) \longrightarrow \mathcal{H}^0(\mathcal{F}^\bullet_{R,M})$$

is then the identity map, hence the cup product morphism (16.18) actually induces the Poincaré duality map (16.16). $\square$

(16.19) Remark. If X is an oriented differentiable manifold, the natural isomorphism $H_c^n(X, \mathbb{R}) \simeq \mathbb{R}$ given by 16.14 a) corresponds in De Rham cohomology to the integration morphism $f \longmapsto \int_X f$, $f \in \mathcal{D}_n(X)$. Indeed, by a partition of unity, we may assume that $\operatorname{Supp} f \subset \Omega \simeq \mathbb{R}^n$. The proof is thus reduced to the case $X = \mathbb{R}^n$, which itself reduces to $X = \mathbb{R}$ since the cup product is compatible with the wedge product of forms. Let us consider the covering $\mathcal{U} = (]k-1, k+1[)_{k \in \mathbb{Z}}$ and a partition of unity (ψ_k) subordinate to $\mathcal{U}$. The Čech differential

$$AC^0(\mathcal{U}, \mathbb{Z}) \longrightarrow AC^1(\mathcal{U}, \mathbb{Z})$$
$$(c_k) \longmapsto (c_{k\,k+1}) = (c_{k+1} - c_k)$$

shows immediately that the generators of $H^1_c(\mathbb{R},\mathbb{Z})$ are the 1-cocycles c such that $c_{01}=\pm 1$ and $c_{k\,k+1}=0$ for $k\neq 0$. By Formula (6.12), the associated closed differential form is

$$f = c_{01}\psi_1 d\psi_0 + c_{10}\psi_0 d\psi_1,$$

hence $f=\pm\mathbf{1}_{[0,1]}d\psi_0$ and f does satisfy $\int_{\mathbb{R}} f = \pm 1$.

(16.20) Corollary. *If X is an oriented C^∞ manifold, the bilinear map*

$$H^{n-q}_c(X,\mathbb{R})\times H^q(X,\mathbb{R})\longrightarrow\mathbb{R},\qquad (\{f\},\{g\})\longmapsto\int_X f\wedge g$$

is well defined and identifies $H^q(X,\mathbb{R})$ to the dual of $H^{n-q}_c(X,\mathbb{R})$.

Chapter V
Hermitian Vector Bundles

This chapter introduces the basic definitions concerning vector bundles and connections. In the first sections, the concepts of connection, curvature form, first Chern class are considered in the framework of differentiable manifolds. Although we are mainly interested in complex manifolds, the ideas which will be developed in the next chapters also involve real analysis and real geometry as essential tools. In the second part, the special features of connections over complex manifolds are investigated in detail: Chern connections, first Chern class of type $(1,1)$, induced curvature forms on sub- and quotient bundles, ... These general concepts are then illustrated by the example of universal vector bundles over $\mathbb{P}^n$ and over Grassmannians.

1. Definition of Vector Bundles

Let M be a C^∞ differentiable manifold of dimension m and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ be the scalar field. A (real, complex) *vector bundle* of rank r over M is a C^∞ manifold E together with

i) a C^∞ map $\pi : E \longrightarrow M$ called the projection,

ii) a $\mathbb{K}$-vector space structure of dimension r on each fiber $E_x = \pi^{-1}(x)$

such that the vector space structure is *locally trivial.* This means that there exists an open covering $(V_\alpha)_{\alpha\in I}$ of M and C^∞ diffeomorphisms called *trivializations*

$$\theta_\alpha : E_{\restriction V_\alpha} \longrightarrow V_\alpha \times \mathbb{K}^r, \qquad \text{where } \ E_{\restriction V_\alpha} = \pi^{-1}(V_\alpha),$$

such that for every $x \in V_\alpha$ the map

$$E_x \xrightarrow{\theta_\alpha} \{x\} \times \mathbb{K}^r \longrightarrow \mathbb{K}^r$$

is a linear isomorphism. For each $\alpha, \beta \in I$, the map

$$\theta_{\alpha\beta} = \theta_\alpha \circ \theta_\beta^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{K}^r \longrightarrow (V_\alpha \cap V_\beta) \times \mathbb{K}^r$$

acts as a linear automorphism on each fiber $\{x\} \times \mathbb{K}^r$. It can thus be written

$$\theta_{\alpha\beta}(x,\xi) = (x, g_{\alpha\beta}(x)\cdot\xi), \qquad (x,\xi) \in (V_\alpha \cap V_\beta) \times \mathbb{K}^r$$

where $(g_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$ is a collection of invertible matrices with coefficients in $C^\infty(V_\alpha\cap V_\beta,\mathbb{K})$, satisfying the cocycle relation

$$(1.1)\qquad g_{\alpha\beta}\,g_{\beta\gamma}=g_{\alpha\gamma}\quad\text{on}\quad V_\alpha\cap V_\beta\cap V_\gamma.$$

The collection $(g_{\alpha\beta})$ is called a *system of transition matrices*. Conversely, any collection of invertible matrices satisfying (1.1) defines a vector bundle E, obtained by gluing the charts $V_\alpha\times\mathbb{K}^r$ via the identifications $\theta_{\alpha\beta}$.

(1.2) Example. The product manifold $E=M\times\mathbb{K}^r$ is a vector bundle over M, and is called the *trivial vector bundle* of rank r over M. We shall often simply denote it $\mathbb{K}^r$ for brevity.

(1.3) Example. A much more interesting example of real vector bundle is the *tangent bundle* TM ; if $\tau_\alpha:V_\alpha\longrightarrow\mathbb{R}^n$ is a collection of coordinate charts on M, then $\theta_\alpha=\pi\times d\tau_\alpha:TM_{\restriction V_\alpha}\longrightarrow V_\alpha\times\mathbb{R}^m$ define trivializations of TM and the transition matrices are given by $g_{\alpha\beta}(x)=d\tau_{\alpha\beta}(x^\beta)$ where $\tau_{\alpha\beta}=\tau_\alpha\circ\tau_\beta^{-1}$ and $x^\beta=\tau_\beta(x)$. The dual $T^\star M$ of TM is called the *cotangent bundle* and the p-th exterior power $\Lambda^pT^\star M$ is called the bundle of differential forms of degree p on M.

(1.4) Definition. *If $\Omega\subset M$ is an open subset and k a positive integer or $+\infty$, we let $C^k(\Omega,E)$ denote the space of C^k sections of $E_{\restriction\Omega}$, i.e. the space of C^k maps $s:\Omega\longrightarrow E$ such that $s(x)\in E_x$ for all $x\in\Omega$ (that is $\pi\circ s=\mathrm{Id}_\Omega$).*

Let $\theta:E_{\restriction V}\longrightarrow V\times\mathbb{K}^r$ be a trivialization of E. To θ, we associate the C^∞ *frame* $(e_1,\ldots,e_r)$ of $E_{\restriction V}$ defined by

$$e_\lambda(x)=\theta^{-1}(x,\varepsilon_\lambda),\qquad x\in V,$$

where (ε_λ) is the standard basis of $\mathbb{K}^r$. A section $s\in C^k(V,E)$ can then be represented in terms of its components $\theta(s)=\sigma=(\sigma_1,\ldots,\sigma_r)$ by

$$s=\sum_{1\le\lambda\le r}\sigma_\lambda e_\lambda\quad\text{on}\quad V,\qquad\sigma_\lambda\in C^k(V,\mathbb{K}).$$

Let (θ_α) be a family of trivializations relative to a covering (V_α) of M. Given a global section $s\in C^k(M,E)$, the components $\theta_\alpha(s)=\sigma^\alpha=(\sigma_1^\alpha,\ldots,\sigma_r^\alpha)$ satisfy the *transition relations*

$$(1.5)\qquad\sigma^\alpha=g_{\alpha\beta}\,\sigma^\beta\quad\text{on}\quad V_\alpha\cap V_\beta.$$

Conversely, any collection of vector valued functions $\sigma^\alpha:V_\alpha\longrightarrow\mathbb{K}^r$ satisfying the transition relations defines a global section s of E.

More generally, we shall also consider differential forms on M with values in E. Such forms are nothing else than sections of the tensor product bundle $\Lambda^pT^\star M\otimes_{\mathbb{R}}E$. We shall write

(1.6) $\quad C_p^k(\Omega, E) = C^k(\Omega, \Lambda^p T^\star M \otimes_{\mathbb{R}} E)$

(1.7) $\quad C_\bullet^k(\Omega, E) = \bigoplus_{0\le p\le m} C_p^k(\Omega, E).$

2. Linear Connections

A (linear) connection D on the bundle E is a linear differential operator of order 1 acting on $C_\bullet^\infty(M, E)$ and satisfying the following properties:

(2.1) $\quad D : C_q^\infty(M, E) \longrightarrow C_{q+1}^\infty(M, E),$

(2.1′) $\quad D(f \wedge s) = df \wedge s + (-1)^p f \wedge Ds$

for any $f \in C_p^\infty(M, \mathbb{K})$ and $s \in C_q^\infty(M, E)$, where df stands for the usual exterior derivative of f.

Assume that $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{K}^r$ is a trivialization of $E_{\restriction\Omega}$, and let $(e_1, \ldots, e_r)$ be the corresponding frame of $E_{\restriction\Omega}$. Then any $s \in C_q^\infty(\Omega, E)$ can be written in a unique way

$$s = \sum_{1\le\lambda\le r} \sigma_\lambda \otimes e_\lambda, \qquad \sigma_\lambda \in C_q^\infty(\Omega, \mathbb{K}).$$

By axiom (2.1′) we get

$$Ds = \sum_{1\le\lambda\le r} \big(d\sigma_\lambda \otimes e_\lambda + (-1)^p \sigma_\lambda \wedge De_\lambda\big).$$

If we write $De_\mu = \sum_{1\le\lambda\le r} a_{\lambda\mu} \otimes e_\lambda$ where $a_{\lambda\mu} \in C_1^\infty(\Omega, \mathbb{K})$, we thus have

$$Ds = \sum_\lambda \big(d\sigma_\lambda + \sum_\mu a_{\lambda\mu} \wedge \sigma_\mu\big) \otimes e_\lambda.$$

Identify $E_{\restriction\Omega}$ with $\Omega \times \mathbb{K}^r$ via θ and denote by d the trivial connection $d\sigma = (d\sigma_\lambda)$ on $\Omega \times \mathbb{K}^r$. Then the operator D can be written

(2.2) $\quad Ds \simeq_\theta d\sigma + A \wedge \sigma$

where $A = (a_{\lambda\mu}) \in C_1^\infty(\Omega, \mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^r))$. Conversely, it is clear that any operator D defined in such a way is a connection on $E_{\restriction\Omega}$. The matrix 1-form A will be called the *connection form* of D associated to the trivialization θ. If $\widetilde\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{K}^r$ is another trivialization and if we set

$$g = \widetilde\theta \circ \theta^{-1} \in C^\infty(\Omega, \mathrm{Gl}(\mathbb{K}^r))$$

then the new components $\widetilde\sigma = (\widetilde\sigma_\lambda)$ are related to the old ones by $\widetilde\sigma = g\sigma$. Let $\widetilde A$ be the connection form of D with respect to $\widetilde\theta$. Then

$$\begin{aligned}
Ds &\simeq_{\widetilde{\theta}} \ d\widetilde{\sigma} + \widetilde{A} \wedge \widetilde{\sigma} \\
Ds &\simeq_{\theta} \ g^{-1}(d\widetilde{\sigma} + \widetilde{A} \wedge \widetilde{\sigma}) = g^{-1}(d(g\sigma) + \widetilde{A} \wedge g\sigma) \\
&= \ d\sigma + (g^{-1}\widetilde{A}g + g^{-1}dg) \wedge \sigma.
\end{aligned}$$

Therefore we obtain the *gauge transformation law*:

$$(2.3) \quad A = g^{-1}\widetilde{A}g + g^{-1}dg.$$

3. Curvature Tensor

Let us compute $D^2 : C^\infty_q(M,E) \to C^\infty_{q+2}(M,E)$ with respect to the trivialization $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{K}^r$. We obtain

$$\begin{aligned}
D^2 s &\simeq_\theta \ d(d\sigma + A \wedge \sigma) + A \wedge (d\sigma + A \wedge \sigma) \\
&= d^2\sigma + (dA \wedge \sigma - A \wedge d\sigma) + (A \wedge d\sigma + A \wedge A \wedge \sigma) \\
&= (dA + A \wedge A) \wedge \sigma.
\end{aligned}$$

It follows that there exists a global 2-form $\Theta(D) \in C^\infty_2(M, \mathrm{Hom}(E,E))$ called *the curvature tensor* of D, such that

$$D^2 s = \Theta(D) \wedge s,$$

given with respect to any trivialization θ by

$$(3.1) \quad \Theta(D) \simeq_\theta \ dA + A \wedge A.$$

(3.2) Remark. If E is of rank $r = 1$, then $A \in C^\infty_1(M, \mathbb{K})$ and $\mathrm{Hom}(E,E)$ is canonically isomorphic to the trivial bundle $M \times \mathbb{K}$, because the endomorphisms of each fiber E_x are homotheties. With the identification $\mathrm{Hom}(E,E) = \mathbb{K}$, the curvature tensor $\Theta(D)$ can be considered as a closed 2-form with values in $\mathbb{K}$:

$$(3.3) \quad \Theta(D) = dA.$$

In this case, the gauge transformation law can be written

$$(3.4) \quad A = \widetilde{A} + g^{-1}dg, \qquad g = \widetilde{\theta} \circ \theta^{-1} \in C^\infty(\Omega, \mathbb{K}^\star).$$

It is then immediately clear that $dA = d\widetilde{A}$, and this equality shows again that dA does not depend on θ. □

Now, we show that the curvature tensor is closely related to commutation properties of covariant derivatives.

(3.5) Definition. *If ξ is a C^∞ vector field with values in TM, the covariant derivative of a section $s \in C^\infty(M,E)$ in the direction ξ is the section $\xi_D \cdot s \in C^\infty(M,E)$ defined by $\xi_D \cdot s = Ds \cdot \xi$.*

(3.6) Proposition. *For all sections $s \in C^\infty(M,E)$ and all vector fields $\xi, \eta \in C^\infty(M,TM)$, we have*

$$\xi_D \cdot (\eta_D \cdot s) - \eta_D \cdot (\xi_D \cdot s) = [\xi,\eta]_D \cdot s + \Theta(D)(\xi,\eta) \cdot s$$

where $[\xi,\eta] \in C^\infty(M,TM)$ is the Lie bracket of ξ, η.

Proof. Let $(x_1, \ldots, x_m)$ be local coordinates on an open set $\Omega \subset M$. Let $\theta : E_{\restriction\Omega} \longrightarrow \Omega \times \mathbb{K}^r$ be a trivialization of E and let A be the corresponding connection form. If $\xi = \sum \xi_j\, \partial/\partial x_j$ and $A = \sum A_j\, dx_j$, we find

$$(3.7) \quad \xi_D s \simeq_\theta (d\sigma + A\sigma) \cdot \xi = \sum_j \xi_j \Big(\frac{\partial\sigma}{\partial x_j} + A_j \cdot \sigma\Big).$$

Now, we compute the above commutator $[\xi_D, \eta_D]$ at a given point $z_0 \in \Omega$. Without loss of generality, we may assume $A(z_0) = 0$; in fact, one can always find a gauge transformation g near z_0 such that $g(z_0) = \mathrm{Id}$ and $dg(z_0) = A(z_0)$; then (2.3) yields $\widetilde{A}(z_0) = 0$. If $\eta = \sum \eta_k\, \partial/\partial x_k$, we find $\eta_D \cdot s \simeq_\theta \sum \eta_k\, \partial\sigma/\partial x_k$ at z_0, hence

$$\begin{aligned}
&\eta_D \cdot (\xi_D \cdot s) \simeq_\theta \sum_k \eta_k \frac{\partial}{\partial x_k} \sum_j \xi_j \Big(\frac{\partial\sigma}{\partial x_j} + A_j \cdot \sigma\Big),\\
&\xi_D \cdot (\eta_D \cdot s) - \eta_D \cdot (\xi_D \cdot s) \simeq_\theta\\
&\qquad\qquad \simeq_\theta \sum_{j,k} \Big(\xi_k \frac{\partial\eta_j}{\partial x_k} - \eta_k \frac{\partial\xi_j}{\partial x_k}\Big)\frac{\partial\sigma}{\partial x_j} + \sum_{j,k} \frac{\partial A_j}{\partial x_k}(\xi_j\eta_k - \eta_j\xi_k) \cdot \sigma\\
&\qquad\qquad = d\sigma([\xi,\eta]) + dA(\xi,\eta) \cdot \sigma,
\end{aligned}$$

whereas $\Theta(D) \simeq_\theta dA$ and $[\xi,\eta]_D s \simeq_\theta d\sigma([\xi,\eta])$ at point z_0. □

4. Operations on Vector Bundles

Let E, F be vector bundles of rank r_1, r_2 over M. Given any functorial operation on vector spaces, a corresponding operation can be defined on bundles by applying the operation on each fiber. For example $E^\star$, $E \oplus F$, $\mathrm{Hom}(E,F)$ are defined by

$$(E^\star)_x = (E_x)^\star, \qquad (E \oplus F)_x = E_x \oplus F_x, \qquad \mathrm{Hom}(E,F)_x = \mathrm{Hom}(E_x, F_x).$$

The bundles E and F can be trivialized over the same covering V_α of M (otherwise take a common refinement). If $(g_{\alpha\beta})$ and $(\gamma_{\alpha\beta})$ are the transition matrices of E and F, then for example $E \otimes F$, $\Lambda^k E$, $E^\star$ are the bundles defined by the transition matrices $g_{\alpha\beta} \otimes \gamma_{\alpha\beta}$, $\Lambda^k g_{\alpha\beta}$, $(g^\dagger_{\alpha\beta})^{-1}$ where $\dagger$ denotes transposition.

Suppose now that E, F are equipped with connections D_E, D_F. Then natural connections can be associated to all derived bundles. Let us mention a few cases. First, we let

$$(4.1) \quad D_{E\oplus F} = D_E \oplus D_F.$$

It follows immediately that

$$(4.1') \quad \Theta(D_{E\oplus F}) = \Theta(D_E) \oplus \Theta(D_F).$$

$D_{E\otimes F}$ will be defined in such a way that the usual formula for the differentiation of a product remains valid. For every $s \in C^\infty_\bullet(M, E)$, $t \in C^\infty_\bullet(M, F)$, the wedge product $s \wedge t$ can be combined with the bilinear map $E \times F \longrightarrow E \otimes F$ in order to obtain a section $s \wedge t \in C^\infty(M, E \otimes F)$ of degree $\deg s + \deg t$. Then there exists a unique connection $D_{E\otimes F}$ such that

$$(4.2) \quad D_{E\otimes F}(s \wedge t) = D_E s \wedge t + (-1)^{\deg\, s} s \wedge D_F t.$$

As the products $s \wedge t$ generate $C^\infty_\bullet(M, E \otimes F)$, the uniqueness is clear. If E, F are trivial on an open set $\Omega \subset M$ and if A_E, A_F, are their connection 1-forms, the induced connection $D_{E\otimes F}$ is given by the connection form $A_E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes A_F$. The existence follows. An easy computation shows that $D^2_{E\otimes F}(s \wedge t) = D^2_E s \wedge t + s \wedge D^2_F t$, thus

$$(4.2') \quad \Theta(D_{E\otimes F}) = \Theta(D_E) \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \Theta(D_F).$$

Similarly, there are unique connections $D_{E^\star}$ and $D_{\mathrm{Hom}(E,F)}$ such that

$$(4.3) \qquad (D_{E^\star} u) \cdot s = d(u \cdot s) - (-1)^{\deg\, u} u \cdot D_E s,$$

$$(4.4) \qquad (D_{\mathrm{Hom}(E,F)} v) \cdot s = D_F(v \cdot s) - (-1)^{\deg\, v} v \cdot D_E s$$

whenever $s \in C^\infty_\bullet(M, E)$, $u \in C^\infty_\bullet(M, E^\star)$, $v \in C^\infty_\bullet\big(\mathrm{Hom}(E, F)\big)$. It follows that

$$0 = d^2(u \cdot s) = \big(\Theta(D_{E^\star}) \cdot u\big) \cdot s + u \cdot \big(\Theta(D_E) \cdot s\big).$$

If $\dagger$ denotes the transposition operator $\mathrm{Hom}(E, E) \to \mathrm{Hom}(E^\star, E^\star)$, we thus get

$$(4.3') \quad \Theta(D_{E^\star}) = -\Theta(D_E)^\dagger.$$

With the identification $\mathrm{Hom}(E, F) = E^\star \otimes F$, Formula (4.2′) implies

$$(4.4') \quad \Theta(D_{\mathrm{Hom}(E,F)}) = \mathrm{Id}_{E^\star} \otimes \Theta(D_F) - \Theta(D_E)^\dagger \otimes \mathrm{Id}_F\,.$$

Finally, $\Lambda^k E$ carries a natural connection $D_{\Lambda^k E}$. For every $s_1, \ldots, s_k$ in $C^\infty_\bullet(M, E)$ of respective degrees $p_1, \ldots, p_k$, this connection satisfies

$$(4.5)\qquad D_{\Lambda^k E}(s_1 \wedge \ldots \wedge s_k) = \sum_{1\le j\le k} (-1)^{p_1+\ldots+p_{j-1}} s_1 \wedge \ldots D_E s_j \ldots \wedge s_k,$$

$$(4.5')\qquad \Theta(D_{\Lambda^k E}) \cdot (s_1 \wedge \ldots \wedge s_k) = \sum_{1\le j\le k} s_1 \wedge \ldots \wedge \Theta(D_E) \cdot s_j \wedge \ldots \wedge s_k.$$

In particular, the *determinant bundle*, defined by $\det E = \Lambda^r E$ where r is the rank of E, has a curvature form given by

$$(4.6)\qquad \Theta(D_{\det E}) = \mathcal{T}_E\big(\Theta(D_E)\big)$$

where $\mathcal{T}_E : \mathrm{Hom}(E,E) \longrightarrow \mathbb{K}$ is the trace operator. As a conclusion of this paragraph, we mention the following simple identity.

(4.7) Bianchi identity. $D_{\mathrm{Hom}(E,E)}\big(\Theta(D_E)\big) = 0.$

Proof. By definition of $D_{\mathrm{Hom}(E,E)}$, we find for any $s \in C^\infty(M,E)$

$$\begin{aligned} D_{\mathrm{Hom}(E,E)}\big(\Theta(D_E)\big) \cdot s &= D_E\big(\Theta(D_E) \cdot s\big) - \Theta(D_E) \cdot (D_E s) \\ &= D_E^3 s - D_E^3 s = 0. \end{aligned}$$ □

5. Pull-Back of a Vector Bundle

Let $\widetilde{M}$, M be C^∞ manifolds and $\psi : \widetilde{M} \to M$ a smooth map. If E is a vector bundle on M, one can define in a natural way a C^∞ vector bundle $\widetilde{\pi} : \widetilde{E} \to \widetilde{M}$ and a C^∞ linear morphism $\Psi : \widetilde{E} \to E$ such that the diagram

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\Psi} & E \\ \downarrow{\scriptstyle\widetilde{\pi}} & & \downarrow{\scriptstyle\pi} \\ \widetilde{M} & \xrightarrow{\psi} & M \end{array}$$

commutes and such that $\Psi : \widetilde{E}_x \longrightarrow E_{\psi(x)}$ is an isomorphism for every $x \in M$. The bundle $\widetilde{E}$ can be defined by

$$(5.1)\qquad \widetilde{E} = \{(\widetilde{x},\xi) \in \widetilde{M} \times E \ ;\ \psi(\widetilde{x}) = \pi(\xi)\}$$

and the maps $\widetilde{\pi}$ and Ψ are then the restrictions to $\widetilde{E}$ of the projections of $\widetilde{M} \times E$ on $\widetilde{M}$ and E respectively.

If $\theta_\alpha : E_{\restriction V_\alpha} \longrightarrow V_\alpha \times \mathbb{K}^r$ are trivializations of E, the maps

$$\widetilde{\theta}_\alpha = \theta_\alpha \circ \Psi : \widetilde{E}_{\restriction \psi^{-1}(V_\alpha)} \longrightarrow \psi^{-1}(V_\alpha) \times \mathbb{K}^r$$

define trivializations of $\widetilde{E}$ with respect to the covering $\widetilde{V}_\alpha = \psi^{-1}(V_\alpha)$ of $\widetilde{M}$. The corresponding system of transition matrices is given by

(5.2) $\quad \widetilde{g}_{\alpha\beta} = g_{\alpha\beta} \circ \psi \quad$ on $\widetilde{V}_\alpha \cap \widetilde{V}_\beta$.

(5.3) Definition. *$\widetilde{E}$ is termed the pull-back of E under the map ψ and is denoted $\widetilde{E} = \psi^\star E$.*

Let D be a connection on E. If (A_α) is the collection of connection forms of D with respect to the θ_α's, one can define a connection $\widetilde{D}$ on $\widetilde{E}$ by the collection of connection forms $\widetilde{A}_\alpha = \psi^\star A_\alpha \in C_1^\infty\big(\widetilde{V}_\alpha, \mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^r)\big)$, i.e. for every $\widetilde{s} \in C_p^\infty(\widetilde{V}_\alpha, \widetilde{E})$

$$\widetilde{D}\widetilde{s} \simeq_{\widetilde{\theta}_\alpha} d\widetilde{\sigma} + \psi^\star A_\alpha \wedge \widetilde{\sigma}.$$

Given any section $s \in C_p^\infty(M, E)$, one defines a pull back $\psi^\star s$ which is a section in $C_p^\infty(\widetilde{M}, \widetilde{E})$: for $s = f \otimes u$, $f \in C_p^\infty(M, \mathbb{K})$, $u \in C^\infty(M, E)$, set $\psi^\star s = \psi^\star f \otimes (u \circ \psi)$. Then we have the formula

(5.4) $\quad \widetilde{D}(\psi^\star s) = \psi^\star(Ds)$.

Using (5.4), a simple computation yields

(5.5) $\quad \Theta(\widetilde{D}) = \psi^\star(\Theta(D))$.

6. Parallel Translation and Flat Vector Bundles

Let $\gamma : [0,1] \longrightarrow M$ be a smooth curve and $s : [0,1] \to E$ a C^∞ section of E along γ, i.e. a C^∞ map s such that $s(t) \in E_{\gamma(t)}$ for all $t \in [0,1]$. Then s can be viewed as a section of $\widetilde{E} = \gamma^\star E$ over $[0,1]$. The *covariant derivative* of s is the section of E along γ defined by

(6.1) $\quad \dfrac{Ds}{dt} = \widetilde{D}s(t) \cdot \dfrac{d}{dt} \in E_{\gamma(t)},$

where $\widetilde{D}$ is the induced connection on $\widetilde{E}$. If A is a connection form of D with respect to a trivialization $\theta : E_{\restriction\Omega} \longrightarrow \Omega \times \mathbb{K}^r$, we have $\widetilde{D}s \simeq_\theta d\sigma + \gamma^\star A \cdot \sigma$, i.e.

(6.2) $\quad \dfrac{Ds}{dt} \simeq_\theta \dfrac{d\sigma}{dt} + \big(A(\gamma(t)) \cdot \gamma'(t)\big) \cdot \sigma(t) \quad$ for $\gamma(t) \in \Omega$.

For $v \in E_{\gamma(0)}$ given, the Cauchy uniqueness and existence theorem for ordinary linear differential equations implies that there exists a unique section s of $\widetilde{E}$ such that $s(0) = v$ and $Ds/dt = 0$.

(6.3) Definition. *The linear map*

$$T_\gamma : E_{\gamma(0)} \longrightarrow E_{\gamma(1)}, \qquad v = s(0) \longmapsto s(1)$$

is called parallel translation along γ.

If $\gamma = \gamma_2\gamma_1$ is the composite of two paths γ_1, γ_2 such that $\gamma_2(0) = \gamma_1(1)$, it is clear that $T_\gamma = T_{\gamma_2} \circ T_{\gamma_1}$, and the inverse path $\gamma^{-1} : t \mapsto \gamma(1-t)$ is such that $T_{\gamma^{-1}} = T_\gamma^{-1}$. It follows that T_γ is a linear isomorphism from $E_{\gamma(0)}$ onto $E_{\gamma(1)}$.

More generally, if $h : W \longrightarrow M$ is a C^∞ map from a domain $W \subset \mathbb{R}^p$ into M and if s is a section of $h^\star E$, we define covariant derivatives $Ds/\partial t_j$, $1 \le j \le p$, by $\widetilde{D} = h^\star D$ and

$$(6.4) \quad \frac{Ds}{\partial t_j} = \widetilde{D}s \cdot \frac{\partial}{\partial t_j}.$$

Since $\partial/\partial t_j$, $\partial/\partial t_k$ commute and since $\Theta(\widetilde{D}) = h^\star\,\Theta(D)$, Prop. 3.6 implies

$$(6.5) \quad \frac{D}{\partial t_j}\frac{Ds}{\partial t_k} - \frac{D}{\partial t_k}\frac{Ds}{\partial t_j} = \Theta(\widetilde{D})\Big(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}\Big) \cdot s = \Theta(D)_{h(t)}\Big(\frac{\partial h}{\partial t_j}, \frac{\partial h}{\partial t_k}\Big) \cdot s(t).$$

(6.6) Definition. *The connection* D *is said to be flat if* $\Theta(D) = 0$.

Assume from now on that D is flat. We then show that T_γ only depends on the homotopy class of γ. Let $h : [0,1] \times [0,1] \longrightarrow M$ be a smooth homotopy $h(t,u) = \gamma_u(t)$ from γ_0 to γ_1 with fixed end points $a = \gamma_u(0)$, $b = \gamma_u(1)$. Let $v \in E_a$ be given and let $s(t,u)$ be such that $s(0,u) = v$ and $Ds/\partial t = 0$ for all $u \in [0,1]$. Then s is C^∞ in both variables (t,u) by standard theorems on the dependence of parameters. Moreover (6.5) implies that the covariant derivatives $D/\partial t$, $D/\partial u$ commute. Therefore, if we set $s' = Ds/\partial u$, we find $Ds'/\partial t = 0$ with initial condition $s'(0,u) = 0$ (recall that $s(0,u)$ is a constant). The uniqueness of solutions of differential equations implies that s' is identically zero on $[0,1] \times [0,1]$, in particular $T_{\gamma_u}(v) = s(1,u)$ must be constant, as desired.

(6.7) Proposition. *Assume that* D *is flat. If* Ω *is a simply connected open subset of* M, *then* $E_{\restriction\Omega}$ *admits a* C^∞ *parallel frame* $(e_1, \ldots, e_r)$, *in the sense that* $De_\lambda = 0$ *on* Ω, $1 \le \lambda \le r$. *For any two simply connected open subsets* Ω, Ω' *the transition automorphism between the corresponding parallel frames* (e_λ) *and* (e'_λ) *is locally constant.*

The converse statement "*E has parallel frames near every point implies that* $\Theta(D) = 0$ " can be immediately verified from the equality $\Theta(D) = D^2$.

Proof. Choose a base point $a \in \Omega$ and define a linear isomorphism $\Phi : \Omega \times E_a \longrightarrow E_{\restriction\Omega}$ by sending (x,v) on $T_\gamma(v) \in E_x$, where γ is any path from a to x in Ω (two such paths are always homotopic by hypothesis). Now, for any path γ from a to x, we have by construction $(D/dt)\Phi(\gamma(t),v) = 0$. Set $e_v(x) = \Phi(x,v)$. As γ may reach any point $x \in \Omega$ with an arbitrary tangent vector $\xi = \gamma'(1) \in T_xM$, we get $De_v(x) \cdot \xi = (D/dt)\Phi(\gamma(t), v)_{\restriction t=1} = 0$. Hence De_v is parallel for any fixed vector $v \in E_a$; Prop. 6.7 follows. □

Assume that M is connected. Let a be a base point and $\widetilde{M} \longrightarrow M$ the universal covering of M. The manifold $\widetilde{M}$ can be considered as the set of pairs $(x, [\gamma])$, where $[\gamma]$ is a homotopy class of paths from a to x. Let $\pi_1(M)$ be the fundamental group of M with base point a, acting on $\widetilde{M}$ on the left by $[\kappa] \cdot (x, [\gamma]) = (x, [\gamma\kappa^{-1}])$. If D is flat, $\pi_1(M)$ acts also on E_a by $([\kappa], v) \mapsto T_\kappa(v)$, $[\kappa] \in \pi_1(M)$, $v \in E_a$, and we have a well defined map

$$\Psi : \widetilde{M} \times E_a \longrightarrow E, \qquad \Psi(x, [\gamma]) = T_\gamma(v).$$

Then Ψ is invariant under the left action of $\pi_1(M)$ on $\widetilde{M} \times E_a$ defined by

$$[\kappa] \cdot \big((x, [\gamma]), v\big) = \big((x, [\gamma\kappa^{-1}]), T_\kappa(v)\big),$$

therefore we have an isomorphism $E \simeq (\widetilde{M} \times E_a)/\pi_1(M)$.

Conversely, let S be a $\mathbb{K}$-vector space of dimension r together with a left action of $\pi_1(M)$. The quotient $E = (\widetilde{M} \times S)/\pi_1(M)$ is a vector bundle over M with locally constant transition automorphisms $(g_{\alpha\beta})$ relatively to any covering (V_α) of M by simply connected open sets. The relation $\sigma^\alpha = g_{\alpha\beta}\,\sigma^\beta$ implies $d\sigma^\alpha = g_{\alpha\beta}\,d\sigma^\beta$ on $V_\alpha \cap V_\beta$. We may therefore define a connection D on E by letting $Ds \simeq_{\theta_\alpha} d\sigma^\alpha$ on each V_α. Then clearly $\Theta(D) = 0$.

7. Hermitian Vector Bundles and Connections

A complex vector bundle E is said to be *hermitian* if a positive definite hermitian form $|\ \ |^2$ is given on each fiber E_x in such a way that the map $E \to \mathbb{R}_+,\ \xi \mapsto |\xi|^2$ is smooth. The notion of a euclidean (real) vector bundle is similar, so we leave the reader adapt our notations to that case.

Let $\theta : E_{\restriction\Omega} \longrightarrow \Omega \times \mathbb{C}^r$ be a trivialization and let $(e_1, \ldots, e_r)$ be the corresponding frame of $E_{\restriction\Omega}$. The associated inner product of E is given by a positive definite hermitian matrix $(h_{\lambda\mu})$ with C^∞ coefficients on Ω, such that

$$\langle e_\lambda(x), e_\mu(x)\rangle = h_{\lambda\mu}(x), \qquad \forall x \in \Omega.$$

When E is hermitian, one can define a natural sesquilinear map

$$\begin{aligned} C^\infty_p(M,E) \times C^\infty_q(M,E) &\longrightarrow C^\infty_{p+q}(M,\mathbb{C}) \\ (s,t) &\longmapsto \{s,t\} \end{aligned} \tag{7.1}$$

combining the wedge product of forms with the hermitian metric on E ; if $s = \sum \sigma_\lambda \otimes e_\lambda$, $t = \sum \tau_\mu \otimes e_\mu$, we let

$$\{s,t\} = \sum_{1\le\lambda,\mu\le r} \sigma_\lambda \wedge \overline{\tau}_\mu \, \langle e_\lambda, e_\mu\rangle.$$

A connection D is said to be compatible with the hermitian structure of E, or briefly *hermitian*, if for every $s \in C^\infty_p(M,E)$, $t \in C^\infty_q(M,E)$ we have

$$(7.2) \quad d\{s,t\} = \{Ds,t\} + (-1)^p\{s,Dt\}.$$

Let $(e_1,\ldots,e_r)$ be an *orthonormal frame* of $E_{\restriction\Omega}$. Denote $\theta(s) = \sigma = (\sigma_\lambda)$ and $\theta(t) = \tau = (\tau_\lambda)$. Then

$$\{s,t\} = \{\sigma,\tau\} = \sum_{1\le\lambda\le r} \sigma_\lambda \wedge \overline{\tau}_\lambda,$$
$$d\{s,t\} = \{d\sigma,\tau\} + (-1)^p\{\sigma,d\tau\}.$$

Therefore $D_{\restriction\Omega}$ is hermitian if and only if its connection form A satisfies

$$\{A\sigma,\tau\} + (-1)^p\{\sigma,A\tau\} = \{(A+A^\star)\wedge\sigma,\tau\} = 0$$

for all σ,τ, i.e.

$$(7.3) \quad A^\star = -A \quad \text{or} \quad (\overline{a_{\mu\lambda}}) = -(a_{\lambda\mu}).$$

This means that $\mathrm{i}A$ is a 1-form with values in the space $\mathrm{Herm}(\mathbb{C}^r,\mathbb{C}^r)$ of hermitian matrices. The identity $d^2\{s,t\} = 0$ implies $\{D^2s,t\} + \{s,D^2t\} = 0$, i.e. $\{\Theta(D)\wedge s,t\} + \{s,\Theta(D)\wedge t\} = 0$. Therefore $\Theta(D)^\star = -\Theta(D)$ and the curvature tensor $\Theta(D)$ is such that

$$\mathrm{i}\,\Theta(D) \in C^\infty_2(M,\mathrm{Herm}(E,E)).$$

(7.4) Special case. If E is a hermitian line bundle ($r=1$), $D_{\restriction\Omega}$ is a hermitian connection if and only if its connection form A associated to any given orthonormal frame of $E_{\restriction\Omega}$ is a 1-form with purely imaginary values.

If $\theta,\widetilde\theta : E_{\restriction\Omega} \to \Omega$ are two such trivializations on a simply connected open subset $\Omega \subset M$, then $g = \widetilde\theta\circ\theta^{-1} = e^{\mathrm{i}\varphi}$ for some real phase function $\varphi \in C^\infty(\Omega,\mathbb{R})$. The gauge transformation law can be written

$$A = \widetilde{A} + \mathrm{i}\,d\varphi.$$

In this case, we see that $\mathrm{i}\,\Theta(D) \in C^\infty_2(M,\mathbb{R})$.

(7.5) Remark. If $s,s' \in C^\infty(M,E)$ are two sections of E along a smooth curve $\gamma : [0,1] \longrightarrow M$, one can easily verify the formula

$$\frac{d}{dt}\langle s(t),s'(t)\rangle = \langle\frac{Ds}{dt},s'\rangle + \langle s,\frac{Ds'}{dt}\rangle.$$

In particular, if $(e_1, \ldots, e_r)$ is a parallel frame of E along γ such that $\big(e_1(0), \ldots, e_r(0)\big)$ is orthonormal, then $\big(e_1(t), \ldots, e_r(t)\big)$ is orthonormal for all t. All parallel translation operators T_γ defined in §6 are thus isometries of the fibers. It follows that E has a flat hermitian connection D if and only if E can be defined by means of locally constant unitary transition automorphisms $g_{\alpha\beta}$, or equivalently if E is isomorphic to the hermitian bundle $(\widetilde{M}\times S)/\pi_1(M)$ defined by a unitary representation of $\pi_1(M)$ in a hermitian vector space S. Such a bundle E is said to be *hermitian flat.*

8. Vector Bundles and Locally Free Sheaves

We denote here by $\mathcal{E}$ the sheaf of germs of C^∞ complex functions on M. Let $F \longrightarrow M$ be a C^∞ complex vector bundle of rank r. We let $\mathcal{F}$ be the sheaf of germs of C^∞ sections of F, i.e. the sheaf whose space of sections on an open subset $U \subset M$ is $\mathcal{F}(U) = C^\infty(U, F)$. It is clear that $\mathcal{F}$ is a $\mathcal{E}$-module. Furthermore, if $F_{\restriction\Omega} \simeq \Omega \times \mathbb{C}^r$ is trivial, the sheaf $\mathcal{F}_{\restriction\Omega}$ is isomorphic to $\mathcal{E}^r_{\restriction\Omega}$ as a $\mathcal{E}_{\restriction\Omega}$-module.

(8.1) Definition. *A sheaf $\mathcal{S}$ of modules over a sheaf of rings $\mathcal{R}$ is said to be locally free of rank k if every point in the base has a neighborhood Ω such that $\mathcal{S}_{\restriction\Omega}$ is $\mathcal{R}$-isomorphic to $\mathcal{R}^k_{\restriction\Omega}$.*

Suppose that $\mathcal{S}$ is a locally free $\mathcal{E}$-module of rank r. There exists a covering $(V_\alpha)_{\alpha\in I}$ of M and sheaf isomorphisms

$$\theta_\alpha : \mathcal{S}_{\restriction V_\alpha} \longrightarrow \mathcal{E}^r_{\restriction V_\alpha}.$$

Then we have transition isomorphisms $g_{\alpha\beta} = \theta_\alpha \circ \theta_\beta^{-1} : \mathcal{E}^r \to \mathcal{E}^r$ defined on $V_\alpha \cap V_\beta$, and such an isomorphism is the multiplication by an invertible matrix with C^∞ coefficients on $V_\alpha \cap V_\beta$. The concepts of vector bundle and of locally free $\mathcal{E}$-module are thus completely equivalent.

Assume now that $F \longrightarrow M$ is a line bundle ($r = 1$). Then every collection of transition automorphisms $g = (g_{\alpha\beta})$ defines a Čech 1-cocycle with values in the multiplicative sheaf $\mathcal{E}^\star$ of invertible C^∞ functions on M. In fact the definition of the Čech differential (cf. (IV-5.1)) gives $(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta}$, and we have $\delta g = 1$ in view of (1.1). Let θ'_α be another family of trivializations and $(g'_{\alpha\beta})$ the associated cocycle (it is no loss of generality to assume that both are defined on the same covering since we may otherwise take a refinement). Then we have

$$\theta'_\alpha \circ \theta_\alpha^{-1} : V_\alpha \times \mathbb{C} \longrightarrow V_\alpha \times \mathbb{C}, \qquad (x, \xi) \longmapsto (x, u_\alpha(x)\xi), \qquad u_\alpha \in \mathcal{E}^\star(V_\alpha).$$

It follows that $g_{\alpha\beta} = g'_{\alpha\beta} u_\alpha^{-1} u_\beta$, i.e. that the Čech 1-cocycles g, g' differ only by the Čech 1-coboundary δu. Therefore, there is a well defined map which associates to every line bundle F over M the Čech cohomology class $\{g\} \in H^1(M, \mathcal{E}^\star)$ of its cocycle of transition automorphisms. It is easy to verify that the cohomology classes associated to two line bundles F, F' are equal if and only if these bundles are isomorphic: if $g = g' \cdot \delta u$, then the collection of maps

$$\begin{aligned} F_{\restriction V_\alpha} \xrightarrow{\theta_\alpha} V_\alpha \times \mathbb{C} &\longrightarrow V_\alpha \times \mathbb{C} \xrightarrow{\theta_\alpha'^{-1}} F'_{\restriction V_\alpha} \\ (x, \xi) &\longmapsto (x, u_\alpha(x)\xi) \end{aligned}$$

defines a global isomorphism $F \to F'$. It is clear that the multiplicative group structure on $H^1(M, \mathcal{E}^\star)$ corresponds to the tensor product of line bundles (the inverse of a line bundle being given by its dual). We may summarize this discussion by the following:

(8.2) Theorem. *The group of isomorphism classes of complex C^∞ line bundles is in one-to-one correspondence with the Čech cohomology group $H^1(M, \mathcal{E}^\star)$.*

9. First Chern Class

Throughout this section, we assume that E is a complex line bundle (that is, $\operatorname{rk} E = r = 1$). Let D be a connection on E. By (3.3), $\Theta(D)$ is a closed 2-form on M. Moreover, if D' is another connection on E, then (2.2) shows that $D' = D + \Gamma \wedge \bullet$ where $\Gamma \in C_1^\infty(M, \mathbb{C})$. By (3.3), we get

$$(9.1) \quad \Theta(D') = \Theta(D) + d\Gamma.$$

This formula shows that the De Rham class $\{\Theta(D)\} \in H^2_{DR}(M, \mathbb{C})$ does not depend on the particular choice of D. If D is chosen to be hermitian with respect to a given hermitian metric on E (such a connection can always be constructed by means of a partition of unity) then $\mathrm{i}\,\Theta(D)$ is a real 2-form, thus $\{\mathrm{i}\,\Theta(D)\} \in H^2_{DR}(M, \mathbb{R})$. Consider now the one-to-one correspondence given by Th. 8.2:

$$\begin{aligned} \{\text{isomorphism classes of line bundles}\} &\longrightarrow H^1(M, \mathcal{E}^\star) \\ \text{class } \{E\} \text{ defined by the cocycle } (g_{\alpha\beta}) &\longmapsto \text{class of } (g_{\alpha\beta}). \end{aligned}$$

Using the exponential exact sequence of sheaves

$$\begin{aligned} 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E} &\longrightarrow \mathcal{E}^\star \longrightarrow 1 \\ f &\longmapsto e^{2\pi \mathrm{i} f} \end{aligned}$$

and the fact that $H^1(M, \mathcal{E}) = H^2(M, \mathcal{E}) = 0$, we obtain:

(9.2) Theorem and Definition. *The coboundary morphism*

$$H^1(M,\mathcal{E}^\star) \xrightarrow{\partial} H^2(M,\mathbb{Z})$$

is an isomorphism. The first Chern class of a line bundle E is the image $c_1(E)$ in $H^2(M,\mathbb{Z})$ of the Čech cohomology class of the 1-cocycle $(g_{\alpha\beta})$ associated to E :

$$c_1(E) = \partial\{(g_{\alpha\beta})\}. \tag{9.3}$$

Consider the natural morphism

$$H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{R}) \simeq H^2_{DR}(M,\mathbb{R}) \tag{9.4}$$

where the isomorphism $\simeq$ is that given by the De Rham-Weil isomorphism theorem and the sign convention of Formula (IV-6.11).

(9.5) Theorem. *The image of $c_1(E)$ in $H^2_{DR}(M,\mathbb{R})$ under* (9.4) *coincides with the De Rham cohomology class $\{\frac{\mathrm{i}}{2\pi}\Theta(D)\}$ associated to any (hermitian) connection D on E.*

Proof. Choose an open covering $(V_\alpha)_{\alpha\in I}$ of M such that E is trivial on each V_α, and such that all intersections $V_\alpha\cap V_\beta$ are simply connected (as in §IV-6, choose the V_α to be small balls relative to a given locally finite covering of M by coordinate patches). Denote by A_α the connection forms of D with respect to a family of isometric trivializations

$$\theta_\alpha :\ E_{\restriction V_\alpha} \longrightarrow V_\alpha \times \mathbb{C}^r.$$

Let $g_{\alpha\beta}\in\mathcal{E}^\star(V_\alpha\cap V_\beta)$ be the corresponding transition automorphisms. Then $|g_{\alpha\beta}|=1$, and as $V_\alpha\cap V_\beta$ is simply connected, we may choose real functions $u_{\alpha\beta}\in\mathcal{E}(V_\alpha\cap V_\beta)$ such that

$$g_{\alpha\beta} = \exp(2\pi\mathrm{i}\, u_{\alpha\beta}).$$

By definition, the first Chern class $c_1(E)$ is the Čech 2-cocycle

$$\begin{aligned} c_1(E) =&\partial\{(g_{\alpha\beta})\} = \{((\delta u)_{\alpha\beta\gamma})\} \in H^2(M,\mathbb{Z}) \quad \text{where}\\ (\delta u)_{\alpha\beta\gamma} :=& u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}. \end{aligned}$$

Now, if $\mathcal{E}^q$ (resp. $\mathcal{Z}^q$) denotes the sheaf of real (resp. real d-closed) q-forms on M, the short exact sequences

$$\begin{aligned} &0\longrightarrow \mathcal{Z}^1 \longrightarrow \mathcal{E}^1 \xrightarrow{d} \mathcal{Z}^2 \longrightarrow 0\\ &0\longrightarrow \mathbb{R} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{Z}^1 \longrightarrow 0 \end{aligned}$$

yield isomorphisms (with the sign convention of (IV-6.11))

$$H^2_{DR}(M,\mathbb{R}) := H^0(M,\mathcal{Z}^2)/dH^0(M,\mathcal{E}^1) \xrightarrow{-\partial} H^1(M,\mathcal{Z}^1), \tag{9.6}$$

$$H^1(M,\mathcal{Z}^1) \xrightarrow{\partial} H^2(M,\mathbb{R}). \tag{9.7}$$

Formula 3.4 gives $A_\beta = A_\alpha + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$. Since $\Theta(D) = dA_\alpha$ on V_α, the image of $\{\frac{\mathrm{i}}{2\pi}\Theta(D)\}$ under (9.6) is the Čech 1-cocycle with values in $\mathcal{Z}^1$

$$\Big\{-\frac{\mathrm{i}}{2\pi}(A_\beta - A_\alpha)\Big\} = \Big\{\frac{1}{2\pi\mathrm{i}} g_{\alpha\beta}^{-1} dg_{\alpha\beta}\Big\} = \{du_{\alpha\beta}\}$$

and the image of this cocycle under (9.7) is the Čech 2-cocycle $\{\delta u\}$ in $H^2(M,\mathbb{R})$. But $\{\delta u\}$ is precisely the image of $c_1(E) \in H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{R})$. $\square$

Let us assume now that M is oriented and that $s \in C^\infty(M,E)$ is transverse to the zero section of E, i.e. that $Ds \in \mathrm{Hom}(TM,E)$ is surjective at every point of the zero set $Z := s^{-1}(0)$. Then Z is an oriented 2-codimensional submanifold of M (the orientation of Z is uniquely defined by those of M and E). We denote by $[Z]$ the current of integration over Z and by $\{[Z]\} \in H^2_{DR}(M,\mathbb{R})$ its cohomology class.

(9.8) Theorem. *We have* $\{[Z]\} = c_1(E)_{\mathbb{R}}$.

Proof. Consider the differential 1-form

$$u = s^{-1} \otimes Ds \in C_1^\infty(M \smallsetminus Z, \mathbb{C}).$$

Relatively to any trivialization θ of $E_{\restriction\Omega}$, one has $D_{\restriction\Omega} \simeq_\theta d + A \wedge \bullet$, thus

$$u_{\restriction\Omega} = \frac{d\sigma}{\sigma} + A \quad \text{where} \quad \sigma = \theta(s).$$

It follows that u has locally integrable coefficients on M. If $d\sigma/\sigma$ is considered as a current on Ω, then

$$d\Big(\frac{d\sigma}{\sigma}\Big) = d\Big(\sigma^\star \frac{dz}{z}\Big) = \sigma^\star d\Big(\frac{dz}{z}\Big) = \sigma^\star(2\pi\mathrm{i}\delta_0) = 2\pi\mathrm{i}[Z]$$

because of the Cauchy residue formula (cf. Lemma I-2.10) and because σ is a submersion in a neighborhood of Z (cf. (I-1.19)). Now, we have $dA = \Theta(D)$ and Th. 9.8 follows from the resulting equality:

$$du = 2\pi\mathrm{i}\,[Z] + \Theta(E). \tag{9.9}$$

$\square$

10. Connections of Type (1,0) and (0,1) over Complex Manifolds

Let X be a complex manifold, $\dim_{\mathbb{C}} X = n$ and E a C^∞ vector bundle of rank r over X ; here, E is not assumed to be holomorphic. We denote by $C^\infty_{p,q}(X,E)$ the space of C^∞ sections of the bundle $\Lambda^{p,q}T^\star X \otimes E$. We have therefore a direct sum decomposition

$$C_l^\infty(X,E) = \bigoplus_{p+q=l} C_{p,q}^\infty(X,E).$$

Connections of type $(1,0)$ or $(0,1)$ are operators acting on vector valued forms, which imitate the usual operators d', d'' acting on $C_{p,q}^\infty(X,\mathbb{C})$. More precisely, a connection of type (1,0) on E is a differential operator D' of order 1 acting on $C_{\bullet,\bullet}^\infty(X,E)$ and satisfying the following two properties:

$$(10.1)\quad D' : C_{p,q}^\infty(X,E) \longrightarrow C_{p+1,q}^\infty(X,E),$$

$$(10.1')\quad D'(f\wedge s) = d'f\wedge s + (-1)^{\deg f} f\wedge D's$$

for any $f \in C_{p_1,q_1}^\infty(X,\mathbb{C})$, $s \in C_{p_2,q_2}^\infty(X,E)$. The definition of a connection D'' of type (0,1) is similar. If $\theta : E_{\restriction\Omega} \to \Omega\times\mathbb{C}^r$ is a C^∞ trivialization of $E_{\restriction\Omega}$ and if $\sigma = (\sigma_\lambda) = \theta(s)$, then all such connections D' and D'' can be written

$$(10.2')\quad D's \simeq_\theta d'\sigma + A'\wedge\sigma,$$

$$(10.2'')\quad D''s \simeq_\theta d''\sigma + A''\wedge\sigma$$

where $A' \in C_{1,0}^\infty\big(\Omega, \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r)\big)$, $A'' \in C_{0,1}^\infty\big(\Omega, \mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^r)\big)$ are arbitrary forms with matrix coefficients.

It is clear that $D = D' + D''$ is then a connection in the sense of §2 ; conversely any connection D admits a unique decomposition $D = D' + D''$ in terms of a (1,0)-connection and a (0,1)-connection.

Assume now that E has a hermitian structure and that θ is an *isometry*. The connection D is hermitian if and only if the connection form $A = A' + A''$ satisfies $A^\star = -A$, and this condition is equivalent to $A' = -(A'')^\star$. From this observation, we get immediately:

(10.3) Proposition. *Let D_0'' be a given $(0,1)$-connection on a hermitian bundle $\pi : E \to X$. Then there exists a unique hermitian connection $D = D' + D''$ such that $D'' = D_0''$.*

11. Holomorphic Vector Bundles

From now on, the vector bundles E in which we are interested are supposed to have a *holomorphic structure*:

(11.1) Definition. *A vector bundle $\pi : E \to X$ is said to be holomorphic if E is a complex manifold, if the projection map π is holomorphic and if there exists a covering $(V_\alpha)_{\alpha\in I}$ of X and a family of holomorphic trivializations $\theta_\alpha : E_{\restriction V_\alpha} \to V_\alpha \times \mathbb{C}^r$.*

It follows that the transition matrices $g_{\alpha\beta}$ are holomorphic on $V_\alpha\cap V_\beta$. In complete analogy with the discussion of §8, we see that the concept of holomorphic vector bundle is equivalent to the concept of locally free sheaf of modules over the ring $\mathcal{O}$ of germs of holomorphic functions on X. We shall denote by $\mathcal{O}(E)$ the associated sheaf of germs of holomorphic sections of E. In the case $r = 1$, there is a one-to-one correspondence between the isomorphism classes of holomorphic line bundles and the Čech cohomology group $H^1(X, \mathcal{O}^\star)$.

(11.2) Definition. *The group $H^1(X, \mathcal{O}^\star)$ of isomorphism classes of holomorphic line bundles is called the Picard group of X.*

If $s \in C^\infty_{p,q}(X, E)$, the components $\sigma^\alpha = (\sigma^\alpha_\lambda)_{1\le\lambda\le r} = \theta_\alpha(s)$ of s under θ_α are related by

$$\sigma^\alpha = g_{\alpha\beta} \cdot \sigma^\beta \quad \text{on} \quad V_\alpha \cap V_\beta.$$

Since $d''g_{\alpha\beta} = 0$, it follows that

$$d''\sigma^\alpha = g_{\alpha\beta} \cdot d''\sigma^\beta \quad \text{on} \quad V_\alpha \cap V_\beta.$$

The collection of forms $(d''\sigma^\alpha)$ therefore corresponds to a unique global $(p, q+1)$-form $d''s$ such that $\theta_\alpha(d''s) = d''\sigma^\alpha$, and the operator d'' defined in this way is a $(0, 1)$-connection on E.

(11.3) Definition. *The operator d'' is called the canonical $(0, 1)$-connection of the holomorphic bundle E.*

It is clear that $d''^2 = 0$. Therefore, for any integer $p = 0, 1, \ldots, n$, we get a complex

$$C^\infty_{p,0}(X, E) \xrightarrow{d''} \cdots \longrightarrow C^\infty_{p,q}(X, E) \xrightarrow{d''} C^\infty_{p,q+1}(X, E) \longrightarrow \cdots$$

known as the *Dolbeault complex* of $(p, \bullet)$-forms with values in E.

(11.4) Notation. *The q-th cohomology group of the Dolbeault complex is denoted $H^{p,q}(X, E)$ and is called the (p, q) Dolbeault cohomology group with values in E.*

The Dolbeault-Grothendieck lemma I-2.11 shows that the complex of sheaves $d'' : \mathcal{C}^\infty_{0,\bullet}(X, E)$ is a soft resolution of the sheaf $\mathcal{O}(E)$. By the De Rham-Weil isomorphism theorem IV-6.4, we get:

(11.5) Proposition. $H^{0,q}(X, E) \simeq H^q\big(X, \mathcal{O}(E)\big)$.

Most often, we will identify the locally free sheaf $\mathcal{O}(E)$ and the bundle E itself ; the above sheaf cohomology group will therefore be simply denoted $H^q(X, E)$. Another standard notation in analytic or algebraic geometry is:

(11.6) Notation. *If X is a complex manifold, Ω_X^p denotes the vector bundle $\Lambda^p T^\star X$ or its sheaf of sections.*

It is clear that the complex $C^\infty_{p,\bullet}(X,E)$ is identical to the complex $C^\infty_{0,\bullet}(X,\Omega_X^p\otimes E)$, therefore we obtain a canonical isomorphism:

(11.7) Dolbeault isomorphism. $H^{p,q}(X,E)\simeq H^q(X,\Omega_X^p\otimes E)$.

In particular, $H^{p,0}(X,E)$ is the space of global holomorphic sections of the bundle $\Omega_X^p\otimes E$.

12. Chern Connection

Let $\pi : E\to X$ be a *hermitian holomorphic* vector bundle of rank r. By Prop. 10.3, there exists a unique hermitian connection D such that $D''=d''$.

(12.1) Definition. *The unique hermitian connection D such that $D''=d''$ is called the Chern connection of E. The curvature tensor of this connection will be denoted by $\Theta(E)$ and is called the Chern curvature tensor of E.*

Let us compute D with respect to an arbitrary *holomorphic trivialization* $\theta : E_{\restriction\Omega}\to\Omega\times\mathbb{C}^r$. Let $H=(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ denote the hermitian matrix with C^∞ coefficients representing the metric along the fibers of $E_{\restriction\Omega}$. For any $s,t\in C^\infty_{\bullet,\bullet}(X,E)$ and $\sigma=\theta(s)$, $\tau=\theta(t)$ one can write

$$\{s,t\}=\sum_{\lambda,\mu}h_{\lambda\mu}\sigma_\lambda\wedge\overline{\tau}_\mu=\sigma^\dagger\wedge H\overline{\tau},$$

where $\sigma^\dagger$ is the transposed matrix of σ. It follows that

$$\begin{aligned}\{Ds,t\}+(-1)^{\deg s}\{s,Dt\}&=d\{s,t\}\\&=(d\sigma)^\dagger\wedge H\overline{\tau}+(-1)^{\deg\sigma}\sigma^\dagger\wedge(dH\wedge\overline{\tau}+H\overline{d\tau})\\&=\big(d\sigma+\overline{H}^{-1}d'\overline{H}\wedge\sigma\big)^\dagger\wedge H\overline{\tau}+(-1)^{\deg\sigma}\sigma^\dagger\wedge\overline{(d\tau+\overline{H}^{-1}d'\overline{H}\wedge\tau)}\end{aligned}$$

using the fact that $dH=d'H+\overline{d'\overline{H}}$ and $\overline{H}^\dagger=H$. Therefore the Chern connection D coincides with the hermitian connection defined by

$$Ds\simeq_\theta d\sigma+\overline{H}^{-1}d'\overline{H}\wedge\sigma, \tag{12.2}$$

$$D'\simeq_\theta d'+\overline{H}^{-1}d'\overline{H}\wedge\bullet=\overline{H}^{-1}d'(\overline{H}\bullet),\qquad D''=d''. \tag{12.3}$$

It is clear from this relations that $D'^2=D''^2=0$. Consequently D^2 is given by to $D^2=D'D''+D''D'$, and the curvature tensor $\Theta(E)$ is of type $(1,1)$. Since $d'd''+d''d'=0$, we get

$$(D'D'' + D''D')s \simeq_\theta \overline{H}^{-1} d'\overline{H} \wedge d''\sigma + d''(\overline{H}^{-1} d'\overline{H} \wedge \sigma) = d''(\overline{H}^{-1} d'\overline{H}) \wedge \sigma.$$

(12.4) Theorem. *The Chern curvature tensor is such that*

$$\mathrm{i}\,\Theta(E) \in C^\infty_{1,1}(X, \mathrm{Herm}(E,E)).$$

If $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{C}^r$ is a holomorphic trivialization and if H is the hermitian matrix representing the metric along the fibers of $E_{\restriction\Omega}$, then

$$\mathrm{i}\,\Theta(E) = \mathrm{i}\,d''(\overline{H}^{-1} d'\overline{H}) \quad \text{on} \quad \Omega.$$

Let $(e_1, \ldots, e_r)$ be a C^∞ orthonormal frame of E over a coordinate patch $\Omega \subset X$ with complex coordinates $(z_1, \ldots, z_n)$. On Ω the Chern curvature tensor can be written

$$(12.5) \quad \mathrm{i}\Theta(E) = \mathrm{i} \sum_{1\le j,k\le n,\ 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \otimes e_\lambda^\star \otimes e_\mu$$

for some coefficients $c_{jk\lambda\mu} \in \mathbb{C}$. The hermitian property of $\mathrm{i}\Theta(E)$ means that $\overline{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$.

(12.6) Special case. When $r = \mathrm{rank}\ E = 1$, the hermitian matrix H is a positive function which we write $H = e^{-\varphi}$, $\varphi \in C^\infty(\Omega, \mathbb{R})$. By the above formulas we get

$$(12.7) \quad D' \simeq_\theta d' - d'\varphi \wedge \bullet = e^\varphi d'(e^{-\varphi}\bullet),$$

$$(12.8) \quad \mathrm{i}\Theta(E) = \mathrm{i}d'd''\varphi \quad \text{on} \quad \Omega.$$

Especially, we see that $\mathrm{i}\,\Theta(E)$ is a *closed* real (1,1)-form on X.

(12.9) Remark. In general, it is not possible to find local frames $(e_1, \ldots, e_r)$ of $E_{\restriction\Omega}$ that are simultaneously *holomorphic* and *orthonormal*. In fact, we have in this case $H = (\delta_{\lambda\mu})$, so a necessary condition for the existence of such a frame is that $\Theta(E) = 0$ on Ω. Conversely, if $\Theta(E) = 0$, Prop. 6.7 and Rem. 7.5 show that E possesses local orthonormal parallel frames (e_λ) ; we have in particular $D''e_\lambda = 0$, so (e_λ) is holomorphic; such a bundle E arising from a unitary representation of $\pi_1(X)$ is said to be *hermitian flat*. The next proposition shows in a more local way that the Chern curvature tensor is the obstruction to the existence of orthonormal holomorphic frames: a holomorphic frame can be made "almost orthonormal" only up to curvature terms of order 2 in a neighborhood of any point.

(12.10) Proposition. *For every point $x_0 \in X$ and every coordinate system $(z_j)_{1\le j\le n}$ at x_0, there exists a holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$ in a neighborhood of x_0 such that*

$$\langle e_\lambda(z), e_\mu(z)\rangle = \delta_{\lambda\mu} - \sum_{1\le j,k\le n} c_{jk\lambda\mu}\, z_j \overline{z}_k + O(|z|^3)$$

where $(c_{jk\lambda\mu})$ *are the coefficients of the Chern curvature tensor* $\Theta(E)_{x_0}$. *Such a frame* (e_λ) *is called a normal coordinate frame at* x_0.

Proof. Let (h_λ) be a holomorphic frame of E. After replacing (h_λ) by suitable linear combinations with constant coefficients, we may assume that $\big(h_\lambda(x_0)\big)$ is an orthonormal basis of E_{x_0}. Then the inner products $\langle h_\lambda, h_\mu\rangle$ have an expansion

$$\langle h_\lambda(z), h_\mu(z)\rangle = \delta_{\lambda\mu} + \sum_j (a_{j\lambda\mu}\, z_j + a'_{j\lambda\mu}\, \overline{z}_j) + O(|z|^2)$$

for some complex coefficients $a_{j\lambda\mu}$, $a'_{j\lambda\mu}$ such that $a'_{j\lambda\mu} = \overline{a}_{j\mu\lambda}$. Set first

$$g_\lambda(z) = h_\lambda(z) - \sum_{j,\mu} a_{j\lambda\mu}\, z_j\, h_\mu(z).$$

Then there are coefficients $a_{jk\lambda\mu}$, $a'_{jk\lambda\mu}$, $a''_{jk\lambda\mu}$ such that

$$\begin{aligned}\langle g_\lambda(z), g_\mu(z)\rangle &= \delta_{\lambda\mu} + O(|z|^2)\\ &= \delta_{\lambda\mu} + \sum_{j,k}\big(a_{jk\lambda\mu}\, z_j\overline{z}_k + a'_{jk\lambda\mu}\, z_j z_k + a''_{jk\lambda\mu}\overline{z}_j\overline{z}_k\big) + O(|z|^3).\end{aligned}$$

The holomorphic frame (e_λ) we are looking for is

$$e_\lambda(z) = g_\lambda(z) - \sum_{j,k,\mu} a'_{jk\lambda\mu}\, z_j z_k\, g_\mu(z).$$

Since $a''_{jk\lambda\mu} = \overline{a}'_{jk\mu\lambda}$, we easily find

$$\begin{aligned}\langle e_\lambda(z), e_\mu(z)\rangle &= \delta_{\lambda\mu} + \sum_{j,k} a_{jk\lambda\mu}\, z_j\overline{z}_k + O(|z|^3),\\ d'\langle e_\lambda, e_\mu\rangle = \{D'e_\lambda, e_\mu\} &= \sum_{j,k} a_{jk\lambda\mu}\,\overline{z}_k\, dz_j + O(|z|^2),\\ \Theta(E)\cdot e_\lambda = D''(D'e_\lambda) &= \sum_{j,k,\mu} a_{jk\lambda\mu}\, d\overline{z}_k\wedge dz_j\otimes e_\mu + O(|z|),\end{aligned}$$

therefore $c_{jk\lambda\mu} = -a_{jk\lambda\mu}$. □

13. Lelong-Poincaré Equation and First Chern Class

Our goal here is to extend the Lelong-Poincaré equation III-2.15 to any meromorphic section of a holomorphic line bundle.

(13.1) Definition. *A meromorphic section of a bundle $E \to X$ is a section s defined on an open dense subset of X, such that for every trivialization $\theta_\alpha : E_{\restriction V_\alpha} \to V_\alpha \times \mathbb{C}^r$ the components of $\sigma^\alpha = \theta_\alpha(s)$ are meromorphic functions on V_α.*

Let E be a hermitian line bundle, s a meromorphic section which does not vanish on any component of X and $\sigma = \theta(s)$ the corresponding meromorphic function in a trivialization $\theta : E_{\restriction \Omega} \to \Omega \times \mathbb{C}$. The divisor of s is the current on X defined by $\operatorname{div} s_{\restriction \Omega} = \operatorname{div} \sigma$ for all trivializing open sets Ω. One can write $\operatorname{div} s = \sum m_j Z_j$, where the sets Z_j are the irreducible components of the sets of zeroes and poles of s (cf. § II-5). The Lelong-Poincaré equation (II-5.32) gives

$$\frac{\mathrm{i}}{\pi} d'd'' \log |\sigma| = \sum m_j [Z_j],$$

and from the equalities $|s|^2 = |\sigma|^2 e^{-\varphi}$ and $d'd''\varphi = \Theta(E)$ we get

$$(13.2) \quad id'd'' \log |s|^2 = 2\pi \sum m_j [Z_j] - \mathrm{i}\, \Theta(E).$$

This equality can be viewed as a complex analogue of (9.9) (except that here the hypersurfaces Z_j are not necessarily smooth). In particular, if s is a *non vanishing holomorphic* section of $E_{\restriction \Omega}$, we have

$$(13.3) \quad \mathrm{i}\, \Theta(E) = -\mathrm{i}d'd'' \log |s|^2 \quad \text{on} \quad \Omega.$$

(13.4) Theorem. *Let $E \to X$ be a line bundle and let s be a meromorphic section of E which does not vanish identically on any component of X. If $\sum m_j Z_j$ is the divisor of s, then*

$$c_1(E)_{\mathbb{R}} = \Big\{ \sum m_j [Z_j] \Big\} \in H^2(X, \mathbb{R}).$$

Proof. Apply Formula (13.2) and Th. 9.5, and observe that the bidimension $(1,1)$-current $\mathrm{i}d'd'' \log |s|^2 = d\big(\mathrm{i}d'' \log |s|^2\big)$ has zero cohomology class. □

(13.5) Example. If $\Delta = \sum m_j Z_j$ is an arbitrary divisor on X, we associate to Δ the sheaf $\mathcal{O}(\Delta)$ of germs of meromorphic functions f such that $\operatorname{div}(f) + \Delta \geq 0$. Let (V_α) be a covering of X and u_α a meromorphic function on V_α such that $\operatorname{div}(u_\alpha) = \Delta$ on V_α. Then $\mathcal{O}(\Delta)_{\restriction V_\alpha} = u_\alpha^{-1}\mathcal{O}$, thus $\mathcal{O}(\Delta)$ is a locally free $\mathcal{O}$-module of rank 1. This sheaf can be identified to the line bundle E over X defined by the cocycle $g_{\alpha\beta} := u_\alpha / u_\beta \in \mathcal{O}^\star(V_\alpha \cap V_\beta)$. In fact, there is a sheaf isomorphism $\mathcal{O}(\Delta) \longrightarrow \mathcal{O}(E)$ defined by

$$\mathcal{O}(\Delta)(\Omega) \ni f \longmapsto s \in \mathcal{O}(E)(\Omega) \quad \text{with} \quad \theta_\alpha(s) = f u_\alpha \quad \text{on } \Omega \cap V_\alpha.$$

The constant meromorphic function $f = 1$ induces a meromorphic section s of E such that $\operatorname{div} s = \operatorname{div} u_\alpha = \Delta$; in the special case when $\Delta \geq 0$, the section s is holomorphic and its zero set $s^{-1}(0)$ is the support of Δ. By Th. 13.4, we have

$$(13.6) \quad c_1\big(\mathcal{O}(\Delta)\big)_{\mathbb{R}} = \{[\Delta]\}.$$

Let us consider the exact sequence $1 \to \mathcal{O}^\star \to \mathcal{M}^\star \to \mathrm{Div} \to 0$ already described in (II-5.36). There is a corresponding cohomology exact sequence

$$(13.7) \quad \mathcal{M}^\star(X) \longrightarrow \mathrm{Div}(X) \xrightarrow{\partial^0} H^1(X, \mathcal{O}^\star).$$

The connecting homomorphism ∂^0 is equal to the map

$$\Delta \longmapsto \text{isomorphism class of } \mathcal{O}(\Delta)$$

defined above. The kernel of this map consists of divisors which are divisors of global meromorphic functions in $\mathcal{M}^\star(X)$. In particular, two divisors Δ_1 and Δ_2 give rise to isomorphic line bundles $\mathcal{O}(\Delta_1) \simeq \mathcal{O}(\Delta_2)$ if and only if $\Delta_2 - \Delta_1 = \operatorname{div}(f)$ for some global meromorphic function $f \in \mathcal{M}^\star(X)$; such divisors are called *linearly equivalent*. The image of ∂^0 consists of classes of line bundles E such that E has a global meromorphic section which does not vanish on any component of X. Indeed, if s is such a section and $\Delta = \operatorname{div} s$, there is an isomorphism

$$(13.8) \qquad \mathcal{O}(\Delta) \xrightarrow{\simeq} \mathcal{O}(E), \qquad f \longmapsto fs. \qquad \square$$

The last result of this section is a characterization of 2-forms on X which can be written as the curvature form of a hermitian holomorphic line bundle.

(13.9) Theorem. *Let X be an arbitrary complex manifold.*

a) *For any hermitian line bundle E over M, the Chern curvature form $\frac{\mathrm{i}}{2\pi}\Theta(E)$ is a closed real $(1,1)$-form whose De Rham cohomology class is the image of an integral class.*

b) *Conversely, let ω be a C^∞ closed real $(1,1)$-form such that the class $\{\omega\} \in H^2_{DR}(X, \mathbb{R})$ is the image of an integral class. Then there exists a hermitian line bundle $E \to X$ such that $\frac{\mathrm{i}}{2\pi}\Theta(E) = \omega$.*

Proof. a) is an immediate consequence of Formula (12.9) and Th. 9.5, so we have only to prove the converse part b). By Prop. III-1.20, there exist an open covering (V_α) of X and functions $\varphi_\alpha \in C^\infty(V_\alpha, \mathbb{R})$ such that $\frac{\mathrm{i}}{2\pi} d'd''\varphi_\alpha = \omega$

on V_α. It follows that the function $\varphi_\beta - \varphi_\alpha$ is pluriharmonic on $V_\alpha \cap V_\beta$. If (V_α) is chosen such that the intersections $V_\alpha \cap V_\beta$ are simply connected, then Th. I-3.35 yields holomorphic functions $f_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ such that

$$2\,\mathrm{Re}\, f_{\alpha\beta} = \varphi_\beta - \varphi_\alpha \quad \text{on} \quad V_\alpha \cap V_\beta.$$

Now, our aim is to prove (roughly speaking) that $\big(\exp(-f_{\alpha\beta})\big)$ is a cocycle in $\mathcal{O}^\star$ that defines the line bundle E we are looking for. The Čech differential $(\delta f)_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}$ takes values in the constant sheaf $\mathrm{i}\mathbb{R}$ because

$$2\,\mathrm{Re}\,(\delta f)_{\alpha\beta\gamma} = (\varphi_\gamma - \varphi_\beta) - (\varphi_\gamma - \varphi_\alpha) + (\varphi_\beta - \varphi_\alpha) = 0.$$

Consider the real 1-forms $A_\alpha = \frac{\mathrm{i}}{4\pi}(d''\varphi_\alpha - d'\varphi_\alpha)$. As $d'(\varphi_\beta - \varphi_\alpha)$ is equal to $d'(f_{\alpha\beta} + \overline{f}_{\alpha\beta}) = df_{\alpha\beta}$, we get

$$(\delta A)_{\alpha\beta} = A_\beta - A_\alpha = \frac{\mathrm{i}}{4\pi} d(\overline{f}_{\alpha\beta} - f_{\alpha\beta}) = \frac{1}{2\pi} d\,\mathrm{Im}\, f_{\alpha\beta}.$$

Since $\omega = dA_\alpha$, it follows by (9.6) and (9.7) that the Čech cohomology class $\{\delta(\frac{1}{2\pi}\,\mathrm{Im}\, f_{\alpha\beta})\}$ is equal to $\{\omega\} \in H^2(X,\mathbb{R})$, which is by hypothesis the image of a 2-cocycle $(n_{\alpha\beta\gamma}) \in H^2(X,\mathbb{Z})$. Thus we can write

$$\delta\Big(\frac{1}{2\pi}\,\mathrm{Im}\, f_{\alpha\beta}\Big) = (n_{\alpha\beta\gamma}) + \delta(c_{\alpha\beta})$$

for some 1-chain $(c_{\alpha\beta})$ with values in $\mathbb{R}$. If we replace $f_{\alpha\beta}$ by $f_{\alpha\beta} - 2\pi\mathrm{i} c_{\alpha\beta}$, then we can achieve $c_{\alpha\beta} = 0$, so $\delta(f_{\alpha\beta}) \in 2\pi\mathrm{i}\mathbb{Z}$ and $g_{\alpha\beta} := \exp(-f_{\alpha\beta})$ will be a cocycle with values in $\mathcal{O}^\star$. Since

$$\varphi_\beta - \varphi_\alpha = 2\,\mathrm{Re}\, f_{\alpha\beta} = -\log |g_{\alpha\beta}|^2,$$

the line bundle E associated to this cocycle admits a global hermitian metric defined in every trivialization by the matrix $H_\alpha = (\exp(-\varphi_\alpha))$ and therefore

$$\frac{\mathrm{i}}{2\pi}\Theta(E) = \frac{\mathrm{i}}{2\pi} d'd''\varphi_\alpha = \omega \quad \text{on} \quad V_\alpha. \qquad \square$$

14. Exact Sequences of Hermitian Vector Bundles

Let us consider an exact sequence of holomorphic vector bundles over X :

$$(14.1) \quad 0 \longrightarrow S \xrightarrow{\ j\ } E \xrightarrow{\ g\ } Q \longrightarrow 0.$$

Then E is said to be an *extension of S by Q*. A (holomorphic, resp. C^∞) splitting of the exact sequence is a (holomorphic, resp. C^∞) homomorphism $h : Q \longrightarrow E$ which is a right inverse of the projection $E \longrightarrow Q$, i.e. such that $g \circ h = \mathrm{Id}_Q$.

Assume that a C^∞ hermitian metric on E is given. Then S and Q can be endowed with the induced and quotient metrics respectively. Let us denote by D_E, D_S, D_Q the corresponding Chern connections. The adjoint homomorphisms

$$j^\star : E \longrightarrow S, \qquad g^\star : Q \longrightarrow E$$

are C^∞ and can be described respectively as the orthogonal projection of E onto S and as the orthogonal splitting of the exact sequence (14.1). They yield a C^∞ (in general non analytic) isomorphism

$$(14.2) \qquad j^\star \oplus g : E \xrightarrow{\simeq} S \oplus Q.$$

(14.3) Theorem. *According to the C^∞ isomorphism $j^\star \oplus g$, D_E can be written*

$$D_E = \begin{pmatrix} D_S & -\beta^\star \\ \beta & D_Q \end{pmatrix}$$

where $\beta \in C^\infty_{1,0}\big(X, \mathrm{Hom}(S,Q)\big)$ is called the second fundamental of S in E and where $\beta^\star \in C^\infty_{0,1}\big(X, \mathrm{Hom}(Q,S)\big)$ is the adjoint of β. Furthermore, the following identities hold:

a) $D'_{\mathrm{Hom}(S,E)} j = g^\star \circ \beta$, $\quad d''j = 0$;
b) $D'_{\mathrm{Hom}(E,Q)} g = -\beta \circ j^\star$, $\quad d''g = 0$;
c) $D'_{\mathrm{Hom}(E,S)} j^\star = 0$, $\quad d''j^\star = \beta^\star \circ g$;
d) $D'_{\mathrm{Hom}(Q,E)} g^\star = 0$, $\quad d''g^\star = -j \circ \beta^\star$.

Proof. If we define $\nabla_E \simeq D_S \oplus D_Q$ via (14.2), then ∇_E is a hermitian connection on E. By (7.3), we have therefore $D_E = \nabla_E + \Gamma \wedge \bullet$, where $\Gamma \in C^\infty_1(X, \mathrm{Hom}(E,E))$ and $\Gamma^\star = -\Gamma$. Let us write

$$\Gamma = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \qquad \alpha^\star = -\alpha,\ \delta^\star = -\delta,\ \gamma = -\beta^\star,$$

$$(14.4) \qquad D_E = \begin{pmatrix} D_S + \alpha & \gamma \\ \beta & D_Q + \delta \end{pmatrix}.$$

For any section $u \in C^\infty_{\bullet,\bullet}(X,E)$ we have

$$\begin{aligned} D_E u &= D_E(jj^\star u + g^\star g u) \\ &= jD_S(j^\star u) + g^\star D_Q(gu) + (D_{\mathrm{Hom}(S,E)} j) \wedge j^\star u + (D_{\mathrm{Hom}(E,Q)} g^\star) \wedge gu. \end{aligned}$$

A comparison with (14.4) yields

$$\begin{aligned} D_{\mathrm{Hom}(S,E)} j &= j \circ \alpha + g^\star \circ \beta, \\ D_{\mathrm{Hom}(E,Q)} g^\star &= j \circ \gamma + g^\star \circ \delta, \end{aligned}$$

Since j is holomorphic, we have $d''j = j \circ \alpha^{0,1} + g^\star \circ \beta^{0,1} = 0$, thus $\alpha^{0,1} = \beta^{0,1} = 0$. But $\alpha^\star = -\alpha$, hence $\alpha = 0$ and $\beta \in C^\infty_{1,0}(\mathrm{Hom}(S,Q))$; identity a) follows. Similarly, we get

$$\begin{aligned} D_S(j^\star u) &= j^\star D_E u + (D_{\mathrm{Hom}(E,S)} j^\star) \wedge u, \\ D_Q(gu) &= g D_E u + (D_{\mathrm{Hom}(E,Q)} g) \wedge u, \end{aligned}$$

and comparison with (14.4) yields

$$\begin{aligned} D_{\mathrm{Hom}(E,S)} j^\star &= -\alpha \circ j^\star - \gamma \circ g = \beta^\star \circ g, \\ D_{\mathrm{Hom}(E,Q)} g &= -\beta \circ j^\star - \delta \circ g. \end{aligned}$$

Since $d''g = 0$, we get $\delta^{0,1} = 0$, hence $\delta = 0$. Identities b), c), d) follow from the above computations. □

(14.5) Theorem. *We have $d''(\beta^\star) = 0$, and the Chern curvature of E is*

$$\Theta(E) = \begin{pmatrix} \Theta(S) - \beta^\star \wedge \beta & D'_{\mathrm{Hom}(Q,S)} \beta^\star \\ d''\beta & \Theta(Q) - \beta \wedge \beta^\star \end{pmatrix}.$$

Proof. A computation of D_E^2 yields

$$D_E^2 = \begin{pmatrix} D_S^2 - \beta^\star \wedge \beta & -(D_S \circ \beta^\star + \beta^\star \circ D_Q) \\ \beta \circ D_S + D_Q \circ \beta & D_Q^2 - \beta \wedge \beta^\star \end{pmatrix}.$$

Formula (13.4) implies

$$\begin{aligned} D_{\mathrm{Hom}(S,Q)} \beta &= \beta \circ D_S + D_Q \circ \beta, \\ D_{\mathrm{Hom}(Q,S)} \beta^\star &= D_S \circ \beta^\star + \beta^\star \circ D_Q. \end{aligned}$$

Since D_E^2 is of type (1,1), it follows that $d''\beta^\star = D''_{\mathrm{Hom}(Q,S)} \beta^\star = 0$. The proof is achieved. □

A consequence of Th. 14.5 is that $\Theta(S)$ and $\Theta(Q)$ are given in terms of $\Theta(E)$ by the following formulas, where $\Theta(E)_{\restriction S}$, $\Theta(E)_{\restriction Q}$ denote the blocks in the matrix of $\Theta(E)$ corresponding to $\mathrm{Hom}(S,S)$ and $\mathrm{Hom}(Q,Q)$:

$$\Theta(S) = \Theta(E)_{\restriction S} + \beta^\star \wedge \beta, \tag{14.6}$$

$$\Theta(Q) = \Theta(E)_{\restriction Q} + \beta \wedge \beta^\star. \tag{14.7}$$

By 14.3 c) the second fundamental form β vanishes identically if and only if the orthogonal splitting $E \simeq S \oplus Q$ is holomorphic ; then we have $\Theta(E) = \Theta(S) \oplus \Theta(Q)$.

Next, we show that the d''-cohomology class $\{\beta^\star\} \in H^{0,1}\big(X, \mathrm{Hom}(Q,S)\big)$ characterizes the isomorphism class of E among all extensions of S by Q.

Two extensions E and F are said to be isomorphic if there is a commutative diagram of holomorphic maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & S & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \end{array} \tag{14.8}$$

in which the rows are exact sequences. The central vertical arrow is then necessarily an isomorphism. It is easily seen that $0 \to S \to E \to Q \to 0$ has a holomorphic splitting if and only if E is isomorphic to the trivial extension $S \oplus Q$.

(14.9) Proposition. *The correspondence*

$$\{E\} \longmapsto \{\beta^\star\}$$

induces a bijection from the set of isomorphism classes of extensions of S by Q onto the cohomology group $H^1\big(X, \mathrm{Hom}(Q,S)\big)$. In particular $\{\beta^\star\}$ vanishes if and only if the exact sequence

$$0 \longrightarrow S \xrightarrow{\ j\ } E \xrightarrow{\ g\ } Q \longrightarrow 0$$

splits holomorphically.

Proof. a) The map is well defined, i.e. $\{\beta^\star\}$ does not depend on the choice of the hermitian metric on E. Indeed, a new hermitian metric produces a new C^∞ splitting $\widehat{g}^\star$ and a new form $\widehat{\beta}^\star$ such that $d''\widehat{g}^\star = -j \circ \widehat{\beta}^\star$. Then $gg^\star = g\widehat{g}^\star = \mathrm{Id}_Q$, thus $\widehat{g} - g = j \circ v$ for some section $v \in C^\infty\big(X, \mathrm{Hom}(Q,S)\big)$. It follows that $\widehat{\beta}^\star - \beta^\star = -d''v$. Moreover, it is clear that an isomorphic extension F has the same associated form $\beta^\star$ if F is endowed with the image of the hermitian metric of E.

b) The map is injective. Let E and F be extensions of S by Q. Select C^∞ splittings $E, F \simeq S \oplus Q$. We endow S, Q with arbitrary hermitian metrics and E, F with the direct sum metric. Then we have corresponding $(0,1)$-connections

$$D''_E = \begin{pmatrix} D''_S & -\beta^\star \\ 0 & D''_Q \end{pmatrix}, \qquad D''_F = \begin{pmatrix} D''_S & -\widetilde{\beta}^\star \\ 0 & D''_Q \end{pmatrix}.$$

Assume that $\widetilde{\beta}^\star = \beta^\star + d''v$ for some $v \in C^\infty\big(X, \mathrm{Hom}(Q,S)\big)$. The isomorphism $\Psi : E \longrightarrow F$ of class C^∞ defined by the matrix

$$\begin{pmatrix} \mathrm{Id}_S & v \\ 0 & \mathrm{Id}_Q \end{pmatrix}.$$

is then holomorphic, because the relation $D''_S \circ v - v \circ D''_Q = d''v = \widetilde{\beta}^\star - \beta^\star$ implies

$$\begin{aligned} D''_{\mathrm{Hom}(E,F)}\Psi &= D''_F \circ \Psi - \Psi \circ D''_E \\ &= \begin{pmatrix} D''_S & -\widetilde{\beta}^\star \\ 0 & D''_Q \end{pmatrix} \begin{pmatrix} \mathrm{Id}_S & v \\ 0 & \mathrm{Id}_Q \end{pmatrix} - \begin{pmatrix} \mathrm{Id}_S & v \\ 0 & \mathrm{Id}_Q \end{pmatrix} \begin{pmatrix} D''_S & -\beta^\star \\ 0 & D''_Q \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\widetilde{\beta}^\star + \beta^\star + (D''_S \circ v - v \circ D''_Q) \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

Hence the extensions E and F are isomorphic.

c) The map is surjective. Let γ be an arbitrary d''-closed $(0,1)$-form on X with values in $\mathrm{Hom}(Q,S)$. We define E as the C^∞ hermitian vector bundle $S \oplus Q$ endowed with the $(0,1)$-connection

$$D''_E = \begin{pmatrix} D''_S & \gamma \\ 0 & D''_Q \end{pmatrix}.$$

We only have to show that this connection is induced by a holomorphic structure on E ; then we will have $\beta^\star = -\gamma$. However, the Dolbeault-Grothendieck lemma implies that there is a covering of X by open sets U_α on which $\gamma = d''v_\alpha$ for some $v_\alpha \in C^\infty\big(U_\alpha, \mathrm{Hom}(Q,S)\big)$. Part b) above shows that the matrix

$$\begin{pmatrix} \mathrm{Id}_S & v_\alpha \\ 0 & \mathrm{Id}_Q \end{pmatrix}$$

defines an isomorphism ψ_α from $E_{\restriction U_\alpha}$ onto the trivial extension $(S \oplus Q)_{\restriction U_\alpha}$ such that $D''_{\mathrm{Hom}(E,S\oplus Q)}\psi_\alpha = 0$. The required holomorphic structure on $E_{\restriction U_\alpha}$ is the inverse image of the holomorphic structure of $(S \oplus Q)_{\restriction U_\alpha}$ by ψ_α ; it is independent of α because $v_\alpha - v_\beta$ and $\psi_\alpha \circ \psi_\beta^{-1}$ are holomorphic on $U_\alpha \cap U_\beta$. □

(14.10) Remark. If E and F are extensions of S by Q such that the corresponding forms $\beta^\star$ and $\widetilde{\beta}^\star = u \circ \beta^\star \circ v^{-1}$ differ by $u \in H^0\big(X, \mathrm{Aut}(S)\big)$, $v \in H^0\big(X, \mathrm{Aut}(Q)\big)$, it is easy to see that the bundles E and F are isomorphic. To see this, we need only replace the vertical arrows representing the identity maps of S and Q in (14.8) by u and v respectively. Thus, if we want to classify isomorphism classes of bundles E which are extensions of S by Q rather than the extensions themselves, the set of classes is the quotient of $H^1\big(X, \mathrm{Hom}(Q,S)\big)$ by the action of $H^0\big(X, \mathrm{Aut}(S)\big) \times H^0\big(X, \mathrm{Aut}(Q)\big)$. In particular, if S, Q are line bundles and if X is compact connected, then $H^0\big(X, \mathrm{Aut}(S)\big)$, $H^0\big(X, \mathrm{Aut}(Q)\big)$ are equal to $\mathbb{C}^\star$ and the set of classes is the projective space $P\big(H^1(X, \mathrm{Hom}(Q,S))\big)$.

15. Line Bundles $\mathcal{O}(k)$ over $\mathbb{P}^n$

15.A. Algebraic properties of $\mathcal{O}(k)$

Let V be a complex vector space of dimension $n+1$, $n \geq 1$. The quotient topological space $P(V) = (V \smallsetminus \{0\})/\mathbb{C}^\star$ is called the *projective space of* V, and can be considered as the set of lines in V if $\{0\}$ is added to each class $\mathbb{C}^\star \cdot x$. Let

$$\begin{aligned} \pi :\ V \smallsetminus \{0\} &\longrightarrow P(V) \\ x &\longmapsto [x] = \mathbb{C}^\star \cdot x \end{aligned}$$

be the canonical projection. When $V = \mathbb{C}^{n+1}$, we simply denote $P(V) = \mathbb{P}^n$. The space $\mathbb{P}^n$ is the quotient S^{2n+1}/S^1 of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the multiplicative action of the unit circle $S^1 \subset \mathbb{C}$, so $\mathbb{P}^n$ is compact. Let $(e_0, \ldots, e_n)$ be a basis of V, and let $(x_0, \ldots, x_n)$ be the coordinates of a vector $x \in V \smallsetminus \{0\}$. Then $(x_0, \ldots, x_n)$ are called the *homogeneous coordinates* of $[x] \in P(V)$. The space $P(V)$ can be covered by the open sets Ω_j defined by $\Omega_j = \{[x] \in P(V)\,;\ x_j \neq 0\}$ and there are homeomorphisms

$$\begin{aligned} \tau_j \ :\ \ \Omega_j &\longrightarrow \mathbb{C}^n \\ [x] &\longmapsto (z_0, \ldots, \widehat{z_j}, \ldots, z_n), \qquad z_l = x_l/x_j \ \text{ for } \ l \neq j. \end{aligned}$$

The collection (τ_j) defines a holomorphic atlas on $P(V)$, thus $P(V) = \mathbb{P}^n$ is a compact n-dimensional complex analytic manifold.

Let $\underline{V}$ be the trivial bundle $P(V) \times V$. We denote by $\mathcal{O}(-1) \subset \underline{V}$ the *tautological line subbundle*

$$(15.1) \quad \mathcal{O}(-1) = \big\{([x], \xi) \in P(V) \times V\ ;\ \xi \in \mathbb{C} \cdot x\big\}$$

such that $\mathcal{O}(-1)_{[x]} = \mathbb{C} \cdot x \subset V$, $x \in V \smallsetminus \{0\}$. Then $\mathcal{O}(-1)_{\restriction \Omega_j}$ admits a non vanishing holomorphic section

$$[x] \longrightarrow \varepsilon_j([x]) = x/x_j = z_0 e_0 + \ldots + e_j + z_{j+1} e_{j+1} + \ldots + z_n e_n,$$

and this shows in particular that $\mathcal{O}(-1)$ is a holomorphic line bundle.

(15.2) Definition. *For every $k \in \mathbb{Z}$, the line bundle $\mathcal{O}(k)$ is defined by*

$$\begin{aligned} \mathcal{O}(1) &= \mathcal{O}(-1)^\star, \qquad \mathcal{O}(0) = P(V) \times \mathbb{C}, \\ \mathcal{O}(k) &= \mathcal{O}(1)^{\otimes k} = \mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1) \quad \text{for} \quad k \geq 1, \\ \mathcal{O}(-k) &= \mathcal{O}(-1)^{\otimes k} \quad \text{for} \quad k \geq 1 \end{aligned}$$

We also introduce the quotient vector bundle $H = \underline{V}/\mathcal{O}(-1)$ of rank n. Therefore we have canonical exact sequences of vector bundles over $P(V)$:

$$(15.3)\quad 0 \to \mathcal{O}(-1) \to \underline{V} \to H \to 0, \qquad 0 \to H^\star \to \underline{V}^\star \to \mathcal{O}(1) \to 0.$$

The total manifold of the line bundle $\mathcal{O}(-1)$ gives rise to the so called *monoidal transformation*, or *Hopf σ-process*:

(15.4) Lemma. *The holomorphic map $\mu : \mathcal{O}(-1) \to V$ defined by*

$$\mu : \mathcal{O}(-1) \hookrightarrow \underline{V} = P(V) \times V \xrightarrow{pr_2} V$$

sends the zero section $P(V) \times \{0\}$ of $\mathcal{O}(-1)$ to the point $\{0\}$ and induces a biholomorphism of $\mathcal{O}(-1) \smallsetminus \big(P(V) \times \{0\}\big)$ onto $V \smallsetminus \{0\}$.

Proof. The inverse map $\mu^{-1} : V \smallsetminus \{0\} \longrightarrow \mathcal{O}(-1)$ is clearly defined by

$$\mu^{-1} : x \longmapsto \big([x], x\big). \qquad \square$$

The space $H^0(\mathbb{P}^n, \mathcal{O}(k))$ of global holomorphic sections of $\mathcal{O}(k)$ can be easily computed by means of the above map μ.

(15.5) Theorem. $H^0\big(P(V), \mathcal{O}(k)\big) = 0$ *for $k < 0$, and there is a canonical isomorphism*

$$H^0\big(P(V), \mathcal{O}(k)\big) \simeq S^k V^\star, \qquad k \geq 0,$$

where $S^k V^\star$ denotes the k-th symmetric power of $V^\star$.

(15.6) Corollary. *We have* $\dim H^0\big(\mathbb{P}^n, \mathcal{O}(k)\big) = \binom{n+k}{n}$ *for $k \geq 0$, and this group is* 0 *for $k < 0$.*

Proof. Assume first that $k \geq 0$. There exists a canonical morphism

$$\Phi : S^k V^\star \longrightarrow H^0\big(P(V), \mathcal{O}(k)\big)\,;$$

indeed, any element $a \in S^k V^\star$ defines a homogeneous polynomial of degree k on V and thus by restriction to $\mathcal{O}(-1) \subset \underline{V}$ a section $\Phi(a) = \widetilde{a}$ of $(\mathcal{O}(-1)^\star)^{\otimes k} = \mathcal{O}(k)$; in other words Φ is induced by the k-th symmetric power $S^k \underline{V}^\star \to \mathcal{O}(k)$ of the canonical morphism $\underline{V}^\star \to \mathcal{O}(1)$ in (15.3).

Assume now that $k \in \mathbb{Z}$ is arbitrary and that s is a holomorphic section of $\mathcal{O}(k)$. For every $x \in V \smallsetminus \{0\}$ we have $s([x]) \in \mathcal{O}(k)_{[x]}$ and $\mu^{-1}(x) \in \mathcal{O}(-1)_{[x]}$. We can therefore associate to s a holomorphic function on $V \smallsetminus \{0\}$ defined by

$$f(x) = s([x]) \cdot \mu^{-1}(x)^k, \qquad x \in V \smallsetminus \{0\}.$$

Since $\dim V = n + 1 \geq 2$, f can be extended to a holomorphic function on V and f is clearly homogeneous of degree k (μ and μ^{-1} are homogeneous of degree 1). It follows that $f = 0$, $s = 0$ if $k < 0$ and that f is a homogeneous polynomial of degree k on V if $k \geq 0$. Thus, there exists a unique element $a \in S^k V^\star$ such that

$$f(x) = a \cdot x^k = \widetilde{a}([x]) \cdot \mu^{-1}(x)^k.$$

Therefore Φ is an isomorphism. □

The tangent bundle on $\mathbb{P}^n$ is closely related to the bundles H and $\mathcal{O}(1)$ as shown by the following proposition.

(15.7) Proposition. *There is a canonical isomorphism of bundles*

$$TP(V) \simeq H \otimes \mathcal{O}(1).$$

Proof. The differential $d\pi_x$ of the projection $\pi : V \smallsetminus \{0\} \to P(V)$ may be considered as a map

$$d\pi_x : V \to T_{[x]}P(V).$$

As $d\pi_x(x) = 0$, $d\pi_x$ can be factorized through $V/\mathbb{C} \cdot x = V/\mathcal{O}(-1)_{[x]} = H_{[x]}$. Hence we get an isomorphism

$$d\widetilde{\pi}_x : H_{[x]} \longrightarrow T_{[x]}P(V),$$

but this isomorphism depends on x and not only on the base point $[x]$ in $P(V)$. The formula $\pi(\lambda x + \xi) = \pi(x + \lambda^{-1}\xi)$, $\lambda \in \mathbb{C}^\star$, $\xi \in V$, shows that $d\pi_{\lambda x} = \lambda^{-1} d\pi_x$, hence the map

$$d\widetilde{\pi}_x \otimes \mu^{-1}(x) \ : \quad H_{[x]} \longrightarrow \big(TP(V) \otimes \mathcal{O}(-1)\big)_{[x]}$$

depends only on $[x]$. Therefore $H \simeq TP(V) \otimes \mathcal{O}(-1)$. □

15.B. Curvature of the Tautological Line Bundle

Assume now that V is a *hermitian vector space*. Then (15.3) yields exact sequences of hermitian vector bundles. We shall compute the curvature of $\mathcal{O}(1)$ and H.

Let $a \in P(V)$ be fixed. Choose an orthonormal basis $(e_0, e_1, \ldots, e_n)$ of V such that $a = [e_0]$. Consider the embedding

$$\mathbb{C}^n \hookrightarrow P(V), \qquad 0 \longmapsto a$$

which sends $z = (z_1, \ldots, z_n)$ to $[e_0 + z_1 e_1 + \cdots + z_n e_n]$. Then

$$\varepsilon(z) = e_0 + z_1 e_1 + \cdots + z_n e_n$$

defines a non-zero holomorphic section of $\mathcal{O}(-1)_{\restriction\mathbb{C}^n}$ and Formula (13.3) for $\Theta\big(\mathcal{O}(1)\big) = -\Theta\big(\mathcal{O}(-1)\big)$ implies

$$\Theta\big(\mathcal{O}(1)\big) = d'd'' \log|\varepsilon(z)|^2 = d'd'' \log(1+|z|^2) \quad \text{on } \mathbb{C}^n, \tag{15.8}$$

$$\Theta\big(\mathcal{O}(1)\big)_a = \sum_{1\le j\le n} dz_j \wedge d\overline{z}_j. \tag{15.8'}$$

On the other hand, Th. 14.3 and (14.7) imply

$$d''g^\star = -j\circ\beta^\star, \qquad \Theta(H) = \beta\wedge\beta^\star,$$

where $j : \mathcal{O}(-1) \longrightarrow \underline{V}$ is the inclusion, $g^\star : H \longrightarrow \underline{V}$ the orthogonal splitting and $\beta^\star \in C^\infty_{0,1}\big(P(V), \mathrm{Hom}(H, \mathcal{O}(-1))\big)$. The images $(\widetilde{e}_1, \ldots, \widetilde{e}_n)$ of $e_1, \ldots, e_n$ in $H = \underline{V}/\mathcal{O}(-1)$ define a holomorphic frame of $H_{\restriction\mathbb{C}^n}$ and we have

$$g^\star\cdot\widetilde{e}_j = e_j - \frac{\langle e_j, \varepsilon\rangle}{|\varepsilon|^2} = e_j - \frac{\overline{z}_j}{1+|z|^2}\varepsilon, \qquad d''g^\star_a\cdot\widetilde{e}_j = -d\overline{z}_j\otimes\varepsilon,$$

$$\beta^\star_a = \sum_{1\le j\le n} d\overline{z}_j\otimes\widetilde{e}^\star_j\otimes\varepsilon, \qquad \beta_a = \sum_{1\le j\le n} dz_j\otimes\varepsilon^\star\otimes\widetilde{e}_j,$$

$$\Theta(H)_a = \sum_{1\le j,k\le n} dz_j\wedge d\overline{z}_k\otimes\widetilde{e}^\star_k\otimes\widetilde{e}_j. \tag{15.9}$$

(15.10) Theorem. *The cohomology algebra $H^\bullet(\mathbb{P}^n, \mathbb{Z})$ is isomorphic to the quotient ring $\mathbb{Z}[h]/(h^{n+1})$ where the generator h is given by $h = c_1(\mathcal{O}(1))$ in $H^2(\mathbb{P}^n, \mathbb{Z})$.*

Proof. Consider the inclusion $\mathbb{P}^{n-1} = P(\mathbb{C}^n\times\{0\}) \subset \mathbb{P}^n$. Topologically, $\mathbb{P}^n$ is obtained from $\mathbb{P}^{n-1}$ by attaching a $2n$-cell B_{2n} to $\mathbb{P}^{n-1}$, via the map

$$\begin{aligned} f\ :\ & B_{2n} \longrightarrow \mathbb{P}^n \\ & z \longmapsto [z, 1-|z|^2], \qquad z\in\mathbb{C}^n, \quad |z|\le 1 \end{aligned}$$

which sends $S^{2n-1} = \{|z| = 1\}$ onto $\mathbb{P}^{n-1}$. That is, $\mathbb{P}^n$ is homeomorphic to the quotient space of $B_{2n} \amalg \mathbb{P}^{n-1}$, where every point $z\in S^{2n-1}$ is identified with its image $f(z)\in\mathbb{P}^{n-1}$. We shall prove by induction on n that

$$H^{2k}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}, \quad 0\le k\le n, \quad \text{otherwise} \quad H^l(\mathbb{P}^n, \mathbb{Z}) = 0. \tag{15.11}$$

The result is clear for $\mathbb{P}^0$, which is reduced to a single point. For $n\ge 1$, consider the covering (U_1, U_2) of $\mathbb{P}^n$ such that U_1 is the image by f of the open ball B°_{2n} and $U_2 = \mathbb{P}^n \smallsetminus \{f(0)\}$. Then $U_1 \approx B^\circ_{2n}$ is contractible, whereas $U_2 = (B_{2n}\smallsetminus\{0\}) \amalg_{S^{2n-1}} \mathbb{P}^{n-1}$. Moreover $U_1\cap U_2 \approx B^\circ_{2n}\smallsetminus\{0\}$ can be retracted on the $(2n-1)$-sphere of radius $1/2$. For $q\ge 2$, the Mayer-Vietoris exact sequence IV-3.11 yields

$$\begin{aligned}\cdots\ \ H^{q-1}(\mathbb{P}^{n-1},\mathbb{Z}) &\longrightarrow H^{q-1}(S^{2n-1},\mathbb{Z})\\ \longrightarrow H^q(\mathbb{P}^n,\mathbb{Z}) \longrightarrow H^q(\mathbb{P}^{n-1},\mathbb{Z}) &\longrightarrow H^q(S^{2n-1},\mathbb{Z})\ \ \cdots .\end{aligned}$$

For $q=1$, the first term has to be replaced by $H^0(\mathbb{P}^{n-1},\mathbb{Z})\oplus\mathbb{Z}$, so that the first arrow is onto. Formula (15.11) follows easily by induction, thanks to our computation of the cohomology groups of spheres in IV-14.6.

We know that $h=c_1(\mathcal{O}(1))\in H^2(\mathbb{P}^n,\mathbb{Z})$. It will follow necessarily that h^k is a generator of $H^{2k}(\mathbb{P}^n,\mathbb{Z})$ if we can prove that h^n is the fundamental class in $H^{2n}(\mathbb{P},\mathbb{Z})$, or equivalently that

$$(15.12)\quad c_1\big(\mathcal{O}(1)\big)^n_{\mathbb{R}}=\int_{\mathbb{P}^n}\Big(\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(1))\Big)^n=1.$$

This equality can be verified directly by means of (15.8), but we will avoid this computation. Observe that the element $e_n^\star\in\big(\mathbb{C}^{n+1}\big)^\star$ defines a section $\widetilde{e}_n^\star$ of $H^0(\mathbb{P}^n,\mathcal{O}(1))$ transverse to 0, whose zero set is the hyperplane $\mathbb{P}^{n-1}$. As $\{\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(1))\}=\{[\mathbb{P}^{n-1}]\}$ by Th. 13.4, we get

$$\begin{aligned}c_1(\mathcal{O}(1))&=\int_{\mathbb{P}^1}[\mathbb{P}^0]=1\qquad\text{for}\ \ n=1\ \ \text{and}\\ c_1(\mathcal{O}(1))^n&=\int_{\mathbb{P}^n}[\mathbb{P}^{n-1}]\wedge\Big(\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(1))\Big)^{n-1}=\int_{\mathbb{P}^{n-1}}\Big(\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(1))\Big)^{n-1}\end{aligned}$$

in general. Since $\mathcal{O}(-1)_{\restriction\mathbb{P}^{n-1}}$ can be identified with the tautological line subbundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ over $\mathbb{P}^{n-1}$, we have $\Theta(\mathcal{O}(1))_{\restriction\mathbb{P}^{n-1}}=\Theta(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ and the proof is achieved by induction on n. □

15.C. Tautological Line Bundle Associated to a Vector Bundle

Let E be a holomorphic vector bundle of rank r over a complex manifold X. The projectivized bundle $P(E)$ is the bundle with $\mathbb{P}^{r-1}$ fibers over X defined by $P(E)_x=P(E_x)$ for all $x\in X$. The points of $P(E)$ can thus be identified with the lines in the fibers of E. For any trivialization $\theta_\alpha:E_{\restriction U_\alpha}\to U_\alpha\times\mathbb{C}^r$ of E we have a corresponding trivialization $\widetilde{\theta}_\alpha:P(E)_{\restriction U_\alpha}\to U_\alpha\times\mathbb{P}^{r-1}$, and it is clear that the transition automorphisms are the projectivizations $\widetilde{g}_{\alpha\beta}\in H^0\big(U_\alpha\cap U_\beta,PGL(r,\mathbb{C})\big)$ of the transition automorphisms $g_{\alpha\beta}$ of E.

Similarly, we have a dual projectivized bundle $P(E^\star)$ whose points can be identified with the hyperplanes of E (every hyperplane F in E_x corresponds bijectively to the line of linear forms in $E_x^\star$ which vanish on F); note that $P(E)$ and $P(E^\star)$ coincide only when $r=\operatorname{rk}E=2$. If $\pi:P(E^\star)\to X$ is the natural projection, there is a tautological hyperplane subbundle S of $\pi^\star E$ over $P(E^\star)$ such that $S_{[\xi]}=\xi^{-1}(0)\subset E_x$ for all $\xi\in E_x^\star\smallsetminus\{0\}$.
$\big[$exercise: check that S is actually locally trivial over $P(E^\star)\big]$.

(15.13) Definition. *The quotient line bundle $\pi^\star E/S$ is denoted $\mathcal{O}_E(1)$ and is called the tautological line bundle associated to E. Hence there is an exact sequence*

$$0 \longrightarrow S \longrightarrow \pi^\star E \longrightarrow \mathcal{O}_E(1) \longrightarrow 0$$

of vector bundles over $P(E^\star)$.

Note that (13.3) applied with $V = E_x^\star$ implies that the restriction of $\mathcal{O}_E(1)$ to each fiber $P(E_x^\star) \simeq \mathbb{P}^{r-1}$ coincides with the line bundle $\mathcal{O}(1)$ introduced in Def. 15.2. Theorem 15.5 can then be extended to the present situation and yields:

(15.14) Theorem. *For every $k \in \mathbb{Z}$, the direct image sheaf $\pi_\star \mathcal{O}_E(k)$ on X vanishes for $k < 0$ and is isomorphic to $\mathcal{O}(S^k E)$ for $k \geq 0$.*

Proof. For $k \geq 0$, the k-th symmetric power of the morphism $\pi^\star E \to \mathcal{O}_E(1)$ gives a morphism $\pi^\star S^k E \to \mathcal{O}_E(k)$. This morphism together with the pull-back morphism yield canonical arrows

$$\Phi_U : H^0(U, S^k E) \xrightarrow{\pi^\star} H^0\big(\pi^{-1}(U), \pi^\star S^k E\big) \longrightarrow H^0\big(\pi^{-1}(U), \mathcal{O}_E(k)\big)$$

for any open set $U \subset X$. The right hand side is by definition the space of sections of $\pi_\star \mathcal{O}_E(k)$ over U, hence we get a canonical sheaf morphism

$$\Phi : \mathcal{O}(S^k E) \longrightarrow \pi_\star \mathcal{O}_E(k).$$

It is easy to check that this Φ coincides with the map Φ introduced in the proof of Cor. 15.6 when X is reduced to a point. In order to check that Φ is an isomorphism, we may suppose that U is chosen so small that $E_{\restriction U}$ is trivial, say $E_{\restriction U} = U \times V$ with $\dim V = r$. Then $P(E^\star) = U \times P(V^\star)$ and $\mathcal{O}_E(1) = p^\star \mathcal{O}(1)$ where $\mathcal{O}(1)$ is the tautological line bundle over $P(V^\star)$ and $p : P(E^\star) \to P(V^\star)$ is the second projection. Hence we get

$$\begin{aligned} H^0\big(\pi^{-1}(U), \mathcal{O}_E(k)\big) &= H^0\big(U \times P(V^\star), p^\star \mathcal{O}(1)\big) \\ &= \mathcal{O}_X(U) \otimes H^0\big(P(V^\star), \mathcal{O}(1)\big) \\ &= \mathcal{O}_X(U) \otimes S^k V = H^0(U, S^k E), \end{aligned}$$

as desired; the reason for the second equality is that $p^\star \mathcal{O}(1)$ coincides with $\mathcal{O}(1)$ on each fiber $\{x\} \times P(V^\star)$ of p, thus any section of $p^\star \mathcal{O}(1)$ over $U \times P(V^\star)$ yields a family of sections $H^0\big(\{x\} \times P(V^\star), \mathcal{O}(k)\big)$ depending holomorphically in x. When $k < 0$ there are no non zero such sections, thus $\pi_\star \mathcal{O}_E(k) = 0$. $\square$

Finally, suppose that E is equipped with a hermitian metric. Then the morphism $\pi^\star E \to \mathcal{O}_E(1)$ endows $\mathcal{O}_E(1)$ with a quotient metric. We are going to compute the associated curvature form $\Theta\big(\mathcal{O}_E(1)\big)$.

Fix a point $x_0 \in X$ and $a \in P(E^\star_{x_0})$. Then Prop. 12.10 implies the existence of a normal coordinate frame $(e_\lambda)_{1\le\lambda\le r})$ of E at x_0 such that a is the hyperplane $\langle e_2,\ldots,e_r\rangle = (e_1^\star)^{-1}(0)$ at x_0. Let $(z_1,\ldots,z_n)$ be local coordinates on X near x_0 and let $(\xi_1,\ldots,\xi_r)$ be coordinates on $E^\star$ with respect to the dual frame $(e_1^\star,\ldots,e_r^\star)$. If we assign $\xi_1 = 1$, then $(z_1,\ldots,z_n,\xi_2,\ldots,\xi_r)$ define local coordinates on $P(E^\star)$ near a, and we have a local section of $\mathcal{O}_E(-1) := \mathcal{O}_E(1)^\star \subset \pi^\star E^\star$ defined by

$$\varepsilon(z,\xi) = e_1^\star(z) + \sum_{2\le\lambda\le r} \xi_\lambda\, e_\lambda^\star(z).$$

The hermitian matrix $(\langle e_\lambda^\star, e_\mu^\star\rangle)$ is just the congugate inverse of $(\langle e_\lambda, e_\mu\rangle) = \mathrm{Id} - \big(\sum c_{jk\lambda\mu}\, z_j\overline{z}_k\big) + O(|z|^3)$, hence we get

$$\langle e_\lambda^\star(z), e_\mu^\star(z)\rangle = \delta_{\lambda\mu} + \sum_{1\le j,k\le n} c_{jk\mu\lambda}\, z_j\overline{z}_k + O(|z|^3),$$

where $(c_{jk\lambda\mu})$ are the curvature coefficients of $\Theta(E)$; accordingly we have $\Theta(E^\star) = -\Theta(E)^\dagger$. We infer from this

$$|\varepsilon(z,\xi)|^2 = 1 + \sum_{1\le j,k\le n} c_{jk11}\, z_j\overline{z}_k + \sum_{2\le\lambda\le r} |\xi_\lambda|^2 + O(|z|^3).$$

Since $\Theta\big(\mathcal{O}_E(1)\big) = d'd''\log|\varepsilon(z,\xi)|^2$, we get

$$\Theta\big(\mathcal{O}_E(1)\big)_a = \sum_{1\le j,k\le n} c_{jk11}\, dz_j\wedge d\overline{z}_k + \sum_{2\le\lambda\le r} d\xi_\lambda\wedge d\overline{\xi}_\lambda.$$

Note that the first summation is simply $-\langle\Theta(E^\star)a,a\rangle/|a|^2 = -$ curvature of $E^\star$ in the direction a. A unitary change of variables then gives the slightly more general formula:

(15.15) Formula. *Let (e_λ) be a normal coordinate frame of E at $x_0 \in X$ and let $\Theta(E)_{x_0} = \sum c_{jk\lambda\mu}\, dz_j\wedge d\overline{z}_k\otimes e_\lambda^\star\otimes e_\mu$. At any point $a \in P(E^\star)$ represented by a vector $\sum a_\lambda e_\lambda^\star \in E^\star_{x_0}$ of norm 1, the curvature of $\mathcal{O}_E(1)$ is*

$$\Theta\big(\mathcal{O}_E(1)\big)_a = \sum_{1\le j,k\le n,\ 1\le\lambda,\mu\le r} c_{jk\mu\lambda}\, a_\lambda\overline{a}_\mu\, dz_j\wedge d\overline{z}_k + \sum_{1\le\lambda\le r-1} d\zeta_\lambda\wedge d\overline{\zeta}_\lambda,$$

where (ζ_λ) are coordinates near a on $P(E^\star)$, induced by unitary coordinates on the hyperplane $a^\perp \subset E^\star_{x_0}$. □

16. Grassmannians and Universal Vector Bundles

16.A. Universal Subbundles and Quotient Vector Bundles

If V is a complex vector space of dimension d, we denote by $G_r(V)$ the set of all r-codimensional vector subspaces of V. Let $a \in G_r(V)$ and $W \subset V$ be fixed such that

$$V = a \oplus W, \qquad \dim_{\mathbb{C}} W = r.$$

Then any subspace $x \in G_r(V)$ in the open subset

$$\Omega_W = \{x \in G_r(V)\ ;\ x \oplus W = V\}$$

can be represented in a unique way as the graph of a linear map u in $\mathrm{Hom}(a, W)$. This gives rise to a covering of $G_r(V)$ by affine coordinate charts $\Omega_W \simeq \mathrm{Hom}(a, W) \simeq \mathbb{C}^{r(d-r)}$. Indeed, let $(e_1, \ldots, e_r)$ and $(e_{r+1}, \ldots, e_n)$ be respective bases of W and a. Every point $x \in \Omega_W$ is the graph of a linear map

$$(16.1) \quad u:\ a \longrightarrow W, \qquad u(e_k) = \sum_{1 \le j \le r} z_{jk} e_j, \qquad r+1 \le k \le d,$$

i.e. $x = \mathrm{Vect}\big(e_k + \sum_{1 \le j \le r} z_{jk} e_j\big)_{r+1 \le k \le d}$. We choose (z_{jk}) as complex coordinates on Ω_W. These coordinates are centered at $a = \mathrm{Vect}(e_{r+1}, \ldots, e_d)$.

(16.2) Proposition. *$G_r(V)$ is a compact complex analytic manifold of dimension $n = r(d-r)$.*

Proof. It is immediate to verify that the coordinate change between two affine charts of $G_r(V)$ is holomorphic. Fix an arbitrary hermitian metric on V. Then the unitary group $U(V)$ is compact and acts transitively on $G_r(V)$. The isotropy subgroup of a point $a \in G_r(V)$ is $U(a) \times U(a^\perp)$, hence $G_r(V)$ is diffeomorphic to the compact quotient space $U(V)/U(a) \times U(a^\perp)$. □

Next, we consider the tautological subbundle $S \subset \underline{V} := G_r(V) \times V$ defined by $S_x = x$ for all $x \in G_r(V)$, and the quotient bundle $Q = \underline{V}/S$ of rank r:

$$(16.3) \quad 0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow Q \longrightarrow 0.$$

An interesting special case is $r = d-1$, $G_{d-1}(V) = P(V)$, $S = \mathcal{O}(-1)$, $Q = H$. The case $r = 1$ is dual, we have the identification $G_1(V) = P(V^\star)$ because every hyperplane $x \subset V$ corresponds bijectively to the line in $V^\star$ of linear forms $\xi \in V^\star$ that vanish on x. Then the bundles $\mathcal{O}(-1) \subset \underline{V}^\star$ and H on $P(V^\star)$ are given by

$$\begin{aligned} \mathcal{O}(-1)_{[\xi]} &= \mathbb{C}.\xi \simeq (V/x)^\star = Q_x^\star, \\ H_{[\xi]} &= V^\star/\mathbb{C}.\xi \simeq x^\star = S_x^\star, \end{aligned}$$

therefore $S = H^\star$, $Q = \mathcal{O}(1)$. This special case will allow us to compute $H^0(G_r(V), Q)$ in general.

(16.4) Proposition. *There is an isomorphism*

$$V = H^0\big(G_r(V), \underline{V}\big) \xrightarrow{\sim} H^0\big(G_r(V), Q\big).$$

Proof. Let $V = W \oplus W'$ be an arbitrary direct sum decomposition of V with $\operatorname{codim} W = r - 1$. Consider the projective space

$$P(W^\star) = G_1(W) \subset G_r(V),$$

its tautological hyperplane subbundle $H^\star \subset \underline{W} = P(W^\star) \times W$ and the exact sequence $0 \to H^\star \to \underline{W} \to \mathcal{O}(1) \to 0$. Then $S_{\restriction P(W^\star)}$ coincides with $H^\star$ and

$$Q_{\restriction P(W^\star)} = (\underline{W} \oplus \underline{W}')/H^\star = (\underline{W}/H^\star) \oplus \underline{W}' = \mathcal{O}(1) \oplus \underline{W}'.$$

Theorem 15.5 implies $H^0(P(W^\star), \mathcal{O}(1)) = W$, therefore the space

$$H^0(P(W^\star), Q_{\restriction P(W^\star)}) = W \oplus W'$$

is generated by the images of the constant sections of $\underline{V}$. Since W is arbitrary, Prop. 16.4 follows immediately. □

Let us compute the tangent space $TG_r(V)$. The linear group $\mathrm{Gl}(V)$ acts transitively on $G_r(V)$, and the tangent space to the isotropy subgroup of a point $x \in G_r(V)$ is the set of elements $u \in \mathrm{Hom}(V, V)$ in the Lie algebra such that $u(x) \subset x$. We get therefore

$$\begin{aligned} T_x G_r(V) &\simeq \mathrm{Hom}(V, V)/\{u \,;\, u(x) \subset x\} \\ &\simeq \mathrm{Hom}(V, V/x)/\big\{\widetilde{u} \,;\, \widetilde{u}(x) = \{0\}\big\} \\ &\simeq \mathrm{Hom}(x, V/x) = \mathrm{Hom}(S_x, Q_x). \end{aligned}$$

(16.5) Corollary. $TG_r(V) = \mathrm{Hom}(S, Q) = S^\star \otimes Q$. □

16.B. Plücker Embedding

There is a natural map, called the *Plücker embedding*,

(16.6) $\quad j_r : G_r(V) \hookrightarrow P(\Lambda^r V^\star)$

constructed as follows. If $x \in G_r(V)$ is defined by r independent linear forms $\xi_1, \ldots, \xi_r \in V^\star$, we set

$$j_r(x) = [\xi_1 \wedge \cdots \wedge \xi_r].$$

Then x is the subspace of vectors $v \in V$ such that $v \lrcorner (\xi_1 \wedge \cdots \wedge \xi_r) = 0$, so j_r is injective. Since the linear group $\mathrm{Gl}(V)$ acts transitively on $G_r(V)$, the rank of the differential dj_r is a constant. As j_r is injective, the constant rank theorem implies:

(16.7) Proposition. *The map j_r is a holomorphic embedding.* □

Now, we define a commutative diagram

$$\begin{array}{ccc} \Lambda^r Q & \xrightarrow{J_r} & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ G_r(V) & \xhookrightarrow{j_r} & P(\Lambda^r V^\star) \end{array} \tag{16.8}$$

as follows: for $x = \xi_1^{-1}(0) \cap \cdots \cap \xi_r^{-1}(0) \in G_r(V)$ and $\widetilde{v} = \widetilde{v}_1 \wedge \cdots \wedge \widetilde{v}_r \in \Lambda^r Q_x$ where $\widetilde{v}_k \in Q_x = V/x$ is the image of $v_k \in V$ in the quotient, we let $J_r(\widetilde{v}) \in \mathcal{O}(1)_{j_r(x)}$ be the linear form on $\mathcal{O}(-1)_{j_r(x)} = \mathbb{C}.\xi_1 \wedge \ldots \wedge \xi_r$ such that

$$\langle J_r(\widetilde{v}), \lambda \xi_1 \wedge \ldots \wedge \xi_r \rangle = \lambda \det \big(\xi_j(v_k)\big), \qquad \lambda \in \mathbb{C}.$$

Then J_r is an isomorphism on the fibers, so $\Lambda^r Q$ can be identified with the pull-back of $\mathcal{O}(1)$ by j_r.

16.C. Curvature of the Universal Vector Bundles

Assume now that V is a hermitian vector space. We shall generalize our curvature computations of §15.C to the present situation. Let $a \in G_r(V)$ be a given point. We take W to be the orthogonal complement of a in V and select an orthonormal basis $(e_1, \ldots, e_d)$ of V such that $W = \mathrm{Vect}(e_1, \ldots, e_r)$, $a = \mathrm{Vect}(e_{r+1}, \ldots, e_d)$. For any point $x \in G_r(V)$ in Ω_W with coordinates (z_{jk}), we set

$$\varepsilon_k(x) = e_k + \sum_{1 \le j \le r} z_{jk} e_j, \qquad r+1 \le k \le d,$$

$$\widetilde{e}_j(x) = \text{image of } e_j \text{ in } Q_x = V/x, \qquad 1 \le j \le r.$$

Then $(\widetilde{e}_1, \ldots, \widetilde{e}_r)$ and $(\varepsilon_{r+1}, \ldots, \varepsilon_d)$ are holomorphic frames of Q and S respectively. If $g^\star : Q \longrightarrow \underline{V}$ is the orthogonal splitting of $g : \underline{V} \longrightarrow Q$, then

$$g^\star \cdot \widetilde{e}_j = e_j + \sum_{r+1 \le k \le d} \zeta_{jk} \varepsilon_k$$

for some $\zeta_{jk} \in \mathbb{C}$. After an easy computation we find

$$0=\langle\widetilde{e}_j,g\varepsilon_k\rangle=\langle g^\star\widetilde{e}_j,\varepsilon_k\rangle=\zeta_{jk}+\overline{z}_{jk}+\sum_{l,m}\zeta_{jm}z_{lm}\overline{z}_{lk},$$

so that $\zeta_{jk}=-\overline{z}_{jk}+O(|z|^2)$. Formula (13.3) yields

$$d''g_a^\star\cdot\widetilde{e}_j=-\sum_{r+1\le k\le d}d\overline{z}_{jk}\otimes\varepsilon_k,$$

$$\beta_a^\star=\sum_{j,k}d\overline{z}_{jk}\otimes\widetilde{e}_j^\star\otimes\varepsilon_k,\qquad \beta_a=\sum_{j,k}dz_{jk}\otimes\varepsilon_k^\star\otimes\widetilde{e}_j,$$

$$\Theta(Q)_a=(\beta\wedge\beta^\star)_a=\sum_{j,k,l}dz_{jk}\wedge d\overline{z}_{lk}\otimes\widetilde{e}_l^\star\otimes\widetilde{e}_j,\tag{16.9}$$

$$\Theta(S)_a=(\beta^\star\wedge\beta)_a=-\sum_{j,k,l}dz_{jk}\wedge d\overline{z}_{jl}\otimes\varepsilon_k^\star\otimes\varepsilon_l.\tag{16.10}$$

Chapter VI
Hodge Theory

The goal of this chapter is to prove a number of basic facts in the Hodge theory of real or complex manifolds. The theory rests essentially on the fact that the De Rham (or Dolbeault) cohomology groups of a compact manifold can be represented by means of spaces of harmonic forms, once a Riemannian metric has been chosen. At this point, some knowledge of basic results about elliptic differential operators is required. The special properties of compact Kähler manifolds are then investigated in detail: Hodge decomposition theorem, hard Lefschetz theorem, Jacobian and Albanese variety, $\ldots$; the example of curves is treated in detail. Finally, the Hodge-Frölicher spectral sequence is applied to get some results on general compact complex manifolds, and it is shown that Hodge decomposition still holds for manifolds in the Fujiki class $(\mathcal{C})$.

§1. Differential Operators on Vector Bundles

We first describe some basic concepts concerning differential operators (symbol, composition, adjunction, ellipticity), in the general setting of vector bundles. Let M be a C^∞ differentiable manifold, $\dim_{\mathbb{R}} M = m$, and let E, F be $\mathbb{K}$-vector bundles over M, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $\operatorname{rank} E = r$, $\operatorname{rank} F = r'$.

(1.1) Definition. *A (linear) differential operator of degree δ from E to F is a $\mathbb{K}$-linear operator $P : C^\infty(M, E) \to C^\infty(M, F)$, $u \mapsto Pu$ of the form*

$$Pu(x) = \sum_{|\alpha| \le \delta} a_\alpha(x) D^\alpha u(x),$$

where $E_{\restriction\Omega} \simeq \Omega \times \mathbb{K}^r$, $F_{\restriction\Omega} \simeq \Omega \times \mathbb{K}^{r'}$ are trivialized locally on some open chart $\Omega \subset M$ equipped with local coordinates $(x_1, \ldots, x_m)$, and where $a_\alpha(x) = \big(a_{\alpha\lambda\mu}(x)\big)_{1\le\lambda\le r',\,1\le\mu\le r}$ are $r' \times r$-matrices with C^∞ coefficients on Ω. Here $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_m)^{\alpha_m}$ as usual, and $u = (u_\mu)_{1\le\mu\le r}$, $D^\alpha u = (D^\alpha u_\mu)_{1\le\mu\le r}$ are viewed as column matrices.

If $t \in \mathbb{K}$ is a parameter, a simple calculation shows that $e^{-tu(x)} P(e^{tu(x)})$ is a polynomial of degree δ in t, of the form

$$e^{-tu(x)} P(e^{tu(x)}) = t^\delta \sigma_P(x, du(x)) + \text{lower order terms } c_j(x) t^j,\ j < \delta,$$

where σ_P is the polynomial map from $T^\star_M \to \mathrm{Hom}(E,F)$ defined by

$$(1.2)\quad T^\star_{M,x} \ni \xi \mapsto \sigma_P(x,\xi) \in \mathrm{Hom}(E_x,F_x), \qquad \sigma_P(x,\xi) = \sum_{|\alpha|=\delta} a_\alpha(x)\xi^\alpha.$$

The formula involving $e^{-tu}P(e^{tu})$ shows that $\sigma_P(x,\xi)$ actually does not depend on the choice of coordinates nor on the trivializations used for E, F. It is clear that $\sigma_P(x,\xi)$ is smooth on $T^\star_M$ as a function of (x,ξ), and is a homogeneous polynomial of degree δ in ξ. We say that σ_P is *the principal symbol* of P. Now, if E, F, G are vector bundles and

$$P : C^\infty(M,E) \to C^\infty(M,F), \qquad Q : C^\infty(M,F) \to C^\infty(M,G)$$

are differential operators of respective degrees δ_P, δ_Q, it is easy to check that $Q\circ P : C^\infty(M,E) \to C^\infty(M,G)$ is a differential operator of degree $\delta_P+\delta_Q$ and that

$$(1.3)\quad \sigma_{Q\circ P}(x,\xi) = \sigma_Q(x,\xi)\sigma_P(x,\xi).$$

Here the product of symbols is computed as a product of matrices.

Now, assume that M is oriented and is equipped with a smooth volume form $dV(x) = \gamma(x)dx_1\wedge\ldots dx_m$, where $\gamma(x) > 0$ is a smooth density. If E is a euclidean or hermitian vector bundle, we have a Hilbert space $L^2(M,E)$ of global sections u of E with measurable coefficients, satisfying the L^2 estimate

$$(1.4)\quad \|u\|^2 = \int_M |u(x)|^2\,dV(x) < +\infty.$$

We denote by

$$(1.4')\quad \langle\!\langle u,v\rangle\!\rangle = \int_M \langle u(x),v(x)\rangle\,dV(x), \qquad u,v\in L^2(M,E)$$

the corresponding L^2 inner product.

(1.5) Definition. *If $P : C^\infty(M,E)\to C^\infty(M,F)$ is a differential operator and both E, F are euclidean or hermitian, there exists a unique differential operator*

$$P^\star : C^\infty(M,F) \to C^\infty(M,E),$$

called the formal adjoint of P, such that for all sections $u\in C^\infty(M,E)$ and $v\in C^\infty(M,F)$ there is an identity

$$\langle\!\langle Pu,v\rangle\!\rangle = \langle\!\langle u,P^\star v\rangle\!\rangle, \qquad \text{whenever } \mathrm{Supp}\,u\cap\mathrm{Supp}\,v \subset\subset M.$$

Proof. The uniqueness is easy, using the density of the set of elements $u\in C^\infty(M,E)$ with compact support in $L^2(M,E)$. Since uniqueness is clear, it is enough, by a partition of unity argument, to show the existence of $P^\star$ locally. Now, let $Pu(x) = \sum_{|\alpha|\le\delta} a_\alpha(x) D^\alpha u(x)$ be the expansion of P with respect to trivializations of E, F given by orthonormal frames over some coordinate open set $\Omega \subset M$. When $\operatorname{Supp} u \cap \operatorname{Supp} v \subset\subset \Omega$ an integration by parts yields

$$\begin{aligned}
\langle\langle Pu, v\rangle\rangle &= \int_\Omega \sum_{|\alpha|\le\delta,\lambda,\mu} a_{\alpha\lambda\mu} D^\alpha u_\mu(x) \overline{v}_\lambda(x)\, \gamma(x)\, dx_1,\ldots,dx_m \\
&= \int_\Omega \sum_{|\alpha|\le\delta,\lambda,\mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha(\gamma(x)\,\overline{a}_{\alpha\lambda\mu} v_\lambda(x)}\, dx_1,\ldots,dx_m \\
&= \int_\Omega \langle u, \sum_{|\alpha|\le\delta} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha\big(\gamma(x)\,{}^t\overline{a}_\alpha v(x)\big)\rangle\, dV(x).
\end{aligned}$$

Hence we see that $P^\star$ exists and is uniquely defined by

$$P^\star v(x) = \sum_{|\alpha|\le\delta} (-1)^{|\alpha|} \gamma(x)^{-1} D^\alpha\big(\gamma(x)\,{}^t\overline{a}_\alpha v(x)\big). \tag{1.6}$$ □

It follows immediately from (1.6) that the principal symbol of $P^\star$ is

$$\sigma_{P^\star}(x,\xi) = (-1)^\delta \sum_{|\alpha|=\delta} {}^t\overline{a}_\alpha \xi^\alpha = (-1)^\delta \sigma_P(x,\xi)^\star. \tag{1.7}$$

(1.8) Definition. *A differential operator P is said to be elliptic if*

$$\sigma_P(x,\xi) \in \operatorname{Hom}(E_x, F_x)$$

is injective for every $x \in M$ and $\xi \in T^\star_{M,x} \smallsetminus \{0\}$.

§2. Formalism of PseudoDifferential Operators

We assume throughout this section that (M,g) is a compact Riemannian manifold. For any positive integer k and any hermitian bundle $F \to M$, we denote by $W^k(M,F)$ the Sobolev space of sections $s : M \to F$ whose derivatives up to order k are in L^2. Let $\|\ \ \|_k$ be the norm of the Hilbert space $W^k(M,F)$. Let P be an elliptic differential operator of order d acting on $C^\infty(M,F)$. We need the following basic facts of elliptic PDE theory, see e.g. (Hörmander 1963).

(2.1) Sobolev lemma. *For $k > l + \frac{m}{2}$, $W^k(M,F) \subset C^l(M,F)$.*

(2.2) Rellich lemma. *For every integer k, the inclusion*

$$W^{k+1}(M,F) \hookrightarrow W^k(M,F)$$

is a compact linear operator.

(2.3) Gårding's inequality. *Let $\widetilde{P}$ be the extension of P to sections with distribution coefficients. For any $u \in W^0(M,F)$ such that $\widetilde{P}u \in W^k(M,F)$, then $u \in W^{k+d}(M,F)$ and*

$$\|u\|_{k+d} \le C_k(\|\widetilde{P}u\|_k + \|u\|_0),$$

where C_k is a positive constant depending only on k.

(2.4) Corollary. *The operator $P : C^\infty(M,F) \to C^\infty(M,F)$ has the following properties:*

i) $\ker P$ *is finite dimensional.*

ii) $P\big(C^\infty(M,F)\big)$ *is closed and of finite codimension; furthermore, if $P^\star$ is the formal adjoint of P, there is a decomposition*

$$C^\infty(M,F) = P\big(C^\infty(M,F)\big) \oplus \ker P^\star$$

as an orthogonal direct sum in $W^0(M,F) = L^2(M,F)$.

Proof. (i) Gårding's inequality shows that $\|u\|_{k+d} \le C_k\|u\|_0$ for any u in $\ker P$. Thanks to the Sobolev lemma, this implies that $\ker P$ is closed in $W^0(M,F)$. Moreover, the unit closed $\|\ \ \|_0$-ball of $\ker P$ is contained in the $\|\ \ \|_d$-ball of radius C_0, thus compact by the Rellich lemma. Riesz' theorem implies that $\dim \ker P < +\infty$.

(ii) We first show that the extension

$$\widetilde{P} : W^{k+d}(M,F) \to W^k(M,F)$$

has a closed range for any k. For every $\varepsilon > 0$, there exists a finite number of elements $v_1, \ldots, v_N \in W^{k+d}(M,F)$, $N = N(\varepsilon)$, such that

$$\|u\|_0 \le \varepsilon\|u\|_{k+d} + \sum_{j=1}^{N} |\langle\langle u, v_j\rangle\rangle_0| \ ; \tag{2.5}$$

indeed the set

$$K_{(v_j)} = \Big\{u \in W^{k+d}(M,F) \ ; \ \varepsilon\|u\|_{k+d} + \sum_{j=1}^{N} |\langle\langle u, v_j\rangle\rangle_0| \le 1\Big\}$$

is relatively compact in $W^0(M,F)$ and $\bigcap_{(v_j)} \overline{K}_{(v_j)} = \{0\}$. It follows that there exist elements (v_j) such that $\overline{K}_{(v_j)}$ is contained in the unit ball of $W^0(M,F)$, *QED*. Substitute $||u||_0$ by the upper bound (2.5) in Gårding's inequality; we get

$$(1 - C_k\varepsilon)\|u\|_{k+d} \le C_k\Big(\|\widetilde{P}u\|_k + \sum_{j=1}^{N} |\langle\!\langle u, v_j\rangle\!\rangle_0|\Big).$$

Define $G = \big\{u \in W^{k+d}(M,F)\ ;\ u \perp v_j,\ 1 \le j \le n\big\}$ and choose $\varepsilon = 1/2C_k$. We obtain

$$\|u\|_{k+d} \le 2C_k\|\widetilde{P}u\|_k, \quad \forall u \in G.$$

This implies that $\widetilde{P}(G)$ is closed. Therefore

$$\widetilde{P}\big(W^{k+d}(M,F)\big) = \widetilde{P}(G) + \mathrm{Vect}\big(\widetilde{P}(v_1), \ldots, \widetilde{P}(v_N)\big)$$

is closed in $W^k(M,F)$. Take in particular $k = 0$. Since $C^\infty(M,F)$ is dense in $W^d(M,F)$, we see that in $W^0(M,F)$

$$\Big(\widetilde{P}\big(W^d(M,F)\big)\Big)^\perp = \Big(P\big(C^\infty(M,F)\big)\Big)^\perp = \ker \widetilde{P^\star}.$$

We have proved that

$$W^0(M,F) = \widetilde{P}\big(W^d(M,F)\big) \oplus \ker \widetilde{P^\star}. \tag{2.6}$$

Since $P^\star$ is also elliptic, it follows that $\ker \widetilde{P^\star}$ is finite dimensional and that $\ker \widetilde{P^\star} = \ker P^\star$ is contained in $C^\infty(M,F)$. Thanks to Gårding's inequality, the decomposition formula (2.6) yields

$$W^k(M,F) = \widetilde{P}\big(W^{k+d}(M,F)\big) \oplus \ker P^\star, \tag{2.7}$$

$$C^\infty(M,F) = P\big(C^\infty(M,F)\big) \oplus \ker P^\star. \tag{2.8}$$

§3. Hodge Theory of Compact Riemannian Manifolds

§3.1. Euclidean Structure of the Exterior Algebra

Let (M,g) be an oriented Riemannian C^∞-manifold, $\dim_{\mathbb{R}} M = m$, and $E \to M$ a hermitian vector bundle of rank r over M. We denote respectively by $(\xi_1, \ldots, \xi_m)$ and $(e_1, \ldots, e_r)$ orthonormal frames of T_M and E over an open subset $\Omega \subset M$, and by $(\xi_1^\star, \ldots, \xi_m^\star)$, $(e_1^\star, \ldots, e_r^\star)$ the corresponding dual frames of $T_M^\star$, $E^\star$. Let dV stand for the Riemannian volume form on M. The exterior algebra $\Lambda T_M^\star$ has a natural inner product $\langle \bullet, \bullet\rangle$ such that

$$\langle u_1 \wedge \ldots \wedge u_p, v_1 \wedge \ldots \wedge v_p\rangle = \det(\langle u_j, v_k\rangle)_{1 \le j,k \le p}, \qquad u_j, v_k \in T_M^\star \tag{3.1}$$

for all p, with $\Lambda T_M^\star = \bigoplus \Lambda^p T_M^\star$ as an orthogonal sum. Then the covectors $\xi_I^\star = \xi_{i_1}^\star \wedge \cdots \wedge \xi_{i_p}^\star$, $i_1 < i_2 < \cdots < i_p$, provide an orthonormal basis of $\Lambda T_M^\star$. We also denote by $\langle \bullet, \bullet\rangle$ the corresponding inner product on $\Lambda T_M^\star \otimes E$.

(3.2) Hodge Star Operator. *The Hodge-Poincaré-De Rham operator $\star$ is the collection of linear maps defined by*

$$\star : \Lambda^p T^\star_M \to \Lambda^{m-p} T^\star_M, \qquad u \wedge \star\, v = \langle u, v\rangle\, dV, \qquad \forall u, v \in \Lambda^p T^\star_M.$$

The existence and uniqueness of this operator is easily seen by using the duality pairing

$$\begin{aligned} &\Lambda^p T^\star_M \times \Lambda^{m-p} T^\star_M \longrightarrow \mathbb{R} \\ (3.3) \qquad & (u, v) \longmapsto u \wedge v / dV = \sum \varepsilon(I, \complement I)\, u_I v_{\complement I}, \end{aligned}$$

where $u = \sum_{|I|=p} u_I\, \xi^\star_I$, $v = \sum_{|J|=m-p} v_J\, \xi^\star_J$, where $\complement I$ stands for the (ordered) complementary multi-index of I and $\varepsilon(I, \complement I)$ for the signature of the permutation $(1, 2, \ldots, m) \longmapsto (I, \complement I)$. From this, we find

$$(3.4) \qquad \star\, v = \sum_{|I|=p} \varepsilon(I, \complement I) v_I\, \xi^\star_{\complement I}.$$

More generally, the sesquilinear pairing $\{\bullet, \bullet\}$ defined in (V-7.1) yields an operator $\star$ on vector valued forms, such that

$$(3.3') \qquad \star\ : \Lambda^p T^\star_M \otimes E \to \Lambda^{m-p} T^\star_M \otimes E, \qquad \{s, \star\, t\} = \langle s, t\rangle\, dV, \qquad s, t \in \Lambda^p T^\star_M \otimes E,$$

$$(3.4') \qquad \star\, t = \sum_{|I|=p,\lambda} \varepsilon(I, \complement I)\, t_{I,\lambda}\, \xi^\star_{\complement I} \otimes e_\lambda$$

for $t = \sum t_{I,\lambda}\, \xi^\star_I \otimes e_\lambda$. Since $\varepsilon(I, \complement I)\varepsilon(\complement I, I) = (-1)^{p(m-p)} = (-1)^{p(m-1)}$, we get immediately

$$(3.5) \qquad \star \star t = (-1)^{p(m-1)} t \quad \text{ on } \ \Lambda^p T^\star_M \otimes E.$$

It is clear that $\star$ is an isometry of $\Lambda^\bullet T^\star_M \otimes E$.

We shall also need a variant of the $\star$ operator, namely the conjugate-linear operator

$$\#\ :\ \ \Lambda^p T^\star_M \otimes E \longrightarrow \Lambda^{m-p} T^\star_M \otimes E^\star$$

defined by $s \wedge \#\, t = \langle s, t\rangle\, dV$, where the wedge product $\wedge$ is combined with the canonical pairing $E \times E^\star \to \mathbb{C}$. We have

$$(3.6) \qquad \#\, t = \sum_{|I|=p,\lambda} \varepsilon(I, \complement I)\, \overline{t}_{I,\lambda}\, \xi^\star_{\complement I} \otimes e^\star_\lambda.$$

(3.7) Contraction by a Vector Field.. *Given a tangent vector $\theta \in T_M$ and a form $u \in \Lambda^p T^\star_M$, the contraction $\theta \lrcorner\, u \in \Lambda^{p-1} T^\star_M$ is defined by*

$$\theta \lrcorner\ u\, (\eta_1, \ldots, \eta_{p-1}) = u(\theta, \eta_1, \ldots, \eta_{p-1}), \qquad \eta_j \in T_M.$$

In terms of the basis (ξ_j), $\bullet \lrcorner \bullet$ is the bilinear operation characterized by

$$\xi_l \lrcorner (\xi^\star_{i_1} \wedge \ldots \wedge \xi^\star_{i_p}) = \begin{cases} 0 & \text{if } l \notin \{i_1, \ldots, i_p\}, \\ (-1)^{k-1} \xi^\star_{i_1} \wedge \ldots \widehat{\xi^\star_{i_k}} \ldots \wedge \xi^\star_{i_p} & \text{if } l = i_k. \end{cases}$$

This formula is in fact valid even when (ξ_j) is non orthonormal. A rather easy computation shows that $\theta \lrcorner \bullet$ is a *derivation* of the exterior algebra, i.e. that

$$\theta \lrcorner (u \wedge v) = (\theta \lrcorner u) \wedge v + (-1)^{\deg u} u \wedge (\theta \lrcorner v).$$

Moreover, if $\widetilde{\theta} = \langle \bullet, \theta \rangle \in T^\star_M$, the operator $\theta \lrcorner \bullet$ is the adjoint map of $\widetilde{\theta} \wedge \bullet$, that is,

$$(3.8) \quad \langle \theta \lrcorner u, v \rangle = \langle u, \widetilde{\theta} \wedge v \rangle, \qquad u, v \in \Lambda T^\star_M.$$

Indeed, this property is immediately checked when $\theta = \xi_l$, $u = \xi^\star_I$, $v = \xi^\star_J$.

§3.2. Laplace-Beltrami Operators

Let us consider the Hilbert space $L^2(M, \Lambda^p T^\star_M)$ of p-forms u on M with measurable coefficients such that

$$\|u\|^2 = \int_M |u|^2 \, dV < +\infty.$$

We denote by $\langle\!\langle \;, \; \rangle\!\rangle$ the global inner product on L^2-forms. The Hilbert space $L^2(M, \Lambda^p T^\star_M \otimes E)$ is defined similarly.

(3.9) Theorem. *The operator $d^\star = (-1)^{mp+1} \star d \star$ is the formal adjoint of the exterior derivative d acting on $C^\infty(M, \Lambda^p T^\star_M \otimes E)$.*

Proof. If $u \in C^\infty(M, \Lambda^p T^\star_M)$, $v \in C^\infty(M, \Lambda^{p+1} T^\star_M \otimes)$ are compactly supported we get

$$\begin{aligned} \langle\!\langle du, v \rangle\!\rangle &= \int_M \langle du, v \rangle \, dV = \int_M du \wedge \star v \\ &= \int_M d(u \wedge \star v) - (-1)^p u \wedge d \star v = -(-1)^p \int_M u \wedge d \star v \end{aligned}$$

by Stokes' formula. Therefore (3.4) implies

$$\langle\!\langle du, v \rangle\!\rangle = -(-1)^p (-1)^{p(m-1)} \int_M u \wedge \star\star d \star v = (-1)^{mp+1} \langle\!\langle u, \star d \star v \rangle\!\rangle. \quad \square$$

(3.10) Remark. If m is even, the formula reduces to $d^\star = - \star d \star$.

(3.11) Definition. *The operator $\Delta = dd^\star + d^\star d$ is called the Laplace-Beltrami operator of M.*

Since $d^\star$ is the adjoint of d, the Laplace operator Δ is formally self-adjoint, i.e. $\langle\!\langle \Delta u, v\rangle\!\rangle = \langle\!\langle u, \Delta v\rangle\!\rangle$ when the forms u, v are of class C^∞ and compactly supported.

(3.12) Example. Let M be an open subset of $\mathbb{R}^m$ and $g = \sum_{i=1}^m dx_i^2$. In that case we get

$$u = \sum_{|I|=p} u_I dx_I, \qquad du = \sum_{|I|=p,j} \frac{\partial u_I}{\partial x_j} dx_j \wedge dx_I,$$
$$\langle\!\langle u, v\rangle\!\rangle = \int_M \langle u, v\rangle\, dV = \int_M \sum_I u_I v_I\, dV$$

One can write $dv = \sum dx_j \wedge (\partial v/\partial x_j)$ where $\partial v/\partial x_j$ denotes the form v in which all coefficients v_I are differentiated as $\partial v_I/\partial x_j$. An integration by parts combined with contraction gives

$$\begin{aligned}\langle\!\langle d^\star u, v\rangle\!\rangle = \langle\!\langle u, dv\rangle\!\rangle &= \int_M \langle u, \sum_j dx_j \wedge \frac{\partial v}{\partial x_j}\rangle\, dV\\ &= \int_M \sum_j \langle \frac{\partial}{\partial x_j} \lrcorner\ u,\ \frac{\partial v}{\partial x_j}\rangle\, dV = -\int_M \langle \sum_j \frac{\partial}{\partial x_j} \lrcorner\ \frac{\partial u}{\partial x_j}, v\rangle\, dV,\\ d^\star u = -\sum_j \frac{\partial}{\partial x_j} \lrcorner\ \frac{\partial u}{\partial x_j} &= -\sum_{I,j} \frac{\partial u_I}{\partial x_j} \frac{\partial}{\partial x_j} \lrcorner\ dx_I.\end{aligned}$$

We get therefore

$$dd^\star u = -\sum_{I,j,k} \frac{\partial^2 u_I}{\partial x_j \partial x_k} dx_k \wedge \Big(\frac{\partial}{\partial x_j} \lrcorner\ dx_I\Big),$$
$$d^\star du = -\sum_{I,j,k} \frac{\partial^2 u_I}{\partial x_j \partial x_k} \frac{\partial}{\partial x_j} \lrcorner\ (dx_k \wedge dx_I).$$

Since

$$\frac{\partial}{\partial x_j} \lrcorner\ (dx_k \wedge dx_I) = \Big(\frac{\partial}{\partial x_j} \lrcorner\ dx_k\Big) dx_I - dx_k \wedge \Big(\frac{\partial}{\partial x_j} \lrcorner\ dx_I\Big),$$

we obtain

$$\Delta u = -\sum_I \Big(\sum_j \frac{\partial^2 u_I}{\partial x_j^2}\Big) dx_I.$$

In the case of an arbitrary riemannian manifold (M, g) we have

$$\begin{aligned}
u &= \sum u_I\, \xi_I^\star,\\
du &= \sum_{I,j} (\xi_j \cdot u_I)\, \xi_j^\star \wedge \xi_I^\star + \sum_I u_I\, d\xi_I^\star,\\
d^\star u &= -\sum_{I,j} (\xi_j \cdot u_I)\, \xi_j \;\lrcorner\; \xi_I^\star + \sum_{I,K} \alpha_{I,K} u_I\, \xi_K^\star\,,
\end{aligned}$$

for some C^∞ coefficients $\alpha_{I,K}$, $|I| = p$, $|K| = p-1$. It follows that the principal part of Δ is the same as that of the second order operator

$$u \longmapsto -\sum_I \big(\sum_j \xi_j^2 \cdot u_I\big)\xi_I^\star.$$

As a consequence, Δ is *elliptic*.

Assume now that D_E is a hermitian connection on E. The formal adjoint operator of D_E acting on $C^\infty(M, \Lambda^p T_M^\star \otimes E)$ is

(3.13) $\quad D_E^\star = (-1)^{mp+1} \star D_E \star\,.$

Indeed, if $s \in C^\infty(M, \Lambda^p T_M^\star \otimes E)$, $t \in C^\infty(M, \Lambda^{p+1} T_M^\star \otimes E)$ have compact support, we get

$$\begin{aligned}
\langle\!\langle D_E s, t\rangle\!\rangle &= \int_M \langle D_E s, t\rangle\, dV = \int_M \{D_E s, \star t\}\\
&= \int_M d\{s, \star t\} - (-1)^p \{s, D_E \star t\} = (-1)^{mp+1} \langle\!\langle s, \star D_E \star t\rangle\!\rangle.
\end{aligned}$$

(3.14) Definition. *The Laplace-Beltrami operator associated to D_E is the second order operator $\Delta_E = D_E D_E^\star + D_E^\star D_E$.*

Δ_E is a self-adjoint elliptic operator with principal part

$$s \longmapsto -\sum_{I,\lambda} \Big(\sum_j \xi_j^2 \cdot s_{I,\lambda}\Big)\xi_I^\star \otimes e_\lambda.$$

§3.3. Harmonic Forms and Hodge Isomorphism

Let E be a hermitian vector bundle over a *compact* Riemannian manifold (M, g). We assume that E possesses a *flat* hermitian connection D_E (this means that $\Theta(D_E) = D_E^2 = 0$, or equivalently, that E is given by a representation $\pi_1(M) \to U(r)$, cf. § V-6). A fundamental example is of course the trivial bundle $E = M \times \mathbb{C}$ with the connection $D_E = d$. Thanks to our flatness assumption, D_E defines a generalized De Rham complex

$$D_E : C^\infty(M, \Lambda^p T_M^\star \otimes E) \longrightarrow C^\infty(M, \Lambda^{p+1} T_M^\star \otimes E).$$

The cohomology groups of this complex will be denoted by $H^p_{DR}(M,E)$.

The space of *harmonic forms of degree* p with respect to the Laplace-Beltrami operator $\Delta_E = D_E D_E^\star + D_E^\star D_E$ is defined by

$$\text{(3.15)}\quad \mathcal{H}^p(M,E) = \big\{s \in C^\infty(M, \Lambda^p T_M^\star \otimes E)\ ;\ \Delta_E s = 0\big\}.$$

Since $\langle\!\langle \Delta_E s, s\rangle\!\rangle = ||D_E s||^2 + ||D_E^\star s||^2$, we see that $s \in \mathcal{H}^p(M,E)$ if and only if $D_E s = D_E^\star s = 0$.

(3.16) Theorem. *For any p, there exists an orthogonal decomposition*

$$\begin{aligned}
&C^\infty(M, \Lambda^p T_M^\star \otimes E) = \mathcal{H}^p(M,E) \oplus \operatorname{Im} D_E \oplus \operatorname{Im} D_E^\star,\\
&\operatorname{Im} D_E = D_E\big(C^\infty(M, \Lambda^{p-1} T_M^\star \otimes E)\big),\\
&\operatorname{Im} D_E^\star = D_E^\star\big(C^\infty(M, \Lambda^{p+1} T_M^\star \otimes E)\big).
\end{aligned}$$

Proof. It is immediate that $\mathcal{H}^p(M,E)$ is orthogonal to both subspaces $\operatorname{Im} D_E$ and $\operatorname{Im} D_E^\star$. The orthogonality of these two subspaces is also clear, thanks to the assumption $D_E^2 = 0$:

$$\langle\!\langle D_E s, D_E^\star t\rangle\!\rangle = \langle\!\langle D_E^2 s, t\rangle\!\rangle = 0.$$

We apply now Cor. 2.4 to the elliptic operator $\Delta_E = \Delta_E^\star$ acting on p-forms, i.e. on the bundle $F = \Lambda^p T_M^\star \otimes E$. We get

$$\begin{aligned}
&C^\infty(M, \Lambda^p T_M^\star \otimes E) = \mathcal{H}^p(M,E) \oplus \Delta_E\big(C^\infty(M, \Lambda^p T_M^\star \otimes E)\big),\\
&\operatorname{Im} \Delta_E = \operatorname{Im}(D_E D_E^\star + D_E^\star D_E) \subset \operatorname{Im} D_E + \operatorname{Im} D_E^\star. \qquad \square
\end{aligned}$$

(3.17) Hodge isomorphism theorem. *The De Rham cohomology group $H^p_{DR}(M,E)$ is finite dimensional and $H^p_{DR}(M,E) \simeq \mathcal{H}^p(M,E)$.*

Proof. According to decomposition 3.16, we get

$$\begin{aligned}
&B^p_{DR}(M,E) = D_E\big(C^\infty(M, \Lambda^{p-1} T_M^\star \otimes E)\big),\\
&Z^p_{DR}(M,E) = \ker D_E = (\operatorname{Im} D_E^\star)^\perp = \mathcal{H}^p(M,E) \oplus \operatorname{Im} D_E.
\end{aligned}$$

This shows that every De Rham cohomology class contains a unique harmonic representative. $\square$

(3.18) Poincaré duality. *The bilinear pairing*

$$H^p_{DR}(M,E) \times H^{m-p}_{DR}(M,E^\star) \longrightarrow \mathbb{C}, \qquad (s,t) \longmapsto \int_M s \wedge t$$

is a non degenerate duality.

Proof. First note that there exists a naturally defined flat connection $D_{E^\star}$ such that for any $s_1 \in C^\infty_\bullet(M,E)$, $s_2 \in C^\infty_\bullet(M,E^\star)$ we have

$$(3.19)\quad d(s_1\wedge s_2) = (D_E s_1)\wedge s_2 + (-1)^{\deg s_1} s_1 \wedge D_{E^\star} s_2.$$

It is then a consequence of Stokes' formula that the map $(s,t)\mapsto \int_M s\wedge t$ can be factorized through cohomology groups. Let $s\in C^\infty(M,\Lambda^pT^\star_M\otimes E)$. We leave to the reader the proof of the following formulas (use (3.19) in analogy with the proof of Th. 3.9):

$$(3.20)\quad \begin{aligned} D_{E^\star}(\# s) &= (-1)^p \# D^\star_E s,\\ \delta_{E^\star}(\# s) &= (-1)^{p+1}\# D_E s,\\ \Delta_{E^\star}(\# s) &= \#\Delta_E s, \end{aligned}$$

Consequently $\#s\in\mathcal{H}^{m-p}(M,E^\star)$ if and only if $s\in\mathcal{H}^p(M,E)$. Since

$$\int_M s\wedge\# s = \int_M |s|^2\,dV = \|s\|^2,$$

we see that the Poincaré pairing has zero kernel in the left hand factor $\mathcal{H}^p(M,E)\simeq H^p_{DR}(M,E)$. By symmetry, it has also zero kernel on the right. The proof is achieved. □

§4. Hermitian and Kähler Manifolds

Let X be a complex n-dimensional manifold. A *hermitian metric* on X is a positive definite hermitian form of class C^∞ on T_X ; in a coordinate system $(z_1,\ldots,z_n)$, such a form can be written $h(z)=\sum_{1\le j,k\le n} h_{jk}(z)\,dz_j\otimes d\overline{z}_k$, where (h_{jk}) is a positive hermitian matrix with C^∞ coefficients. According to (III-1.8), the *fundamental* $(1,1)$*-form* associated to h is the positive form of type $(1,1)$

$$\omega = -\mathrm{Im}\, h = \frac{\mathrm{i}}{2}\sum h_{jk}dz_j\wedge d\overline{z}_k,\qquad 1\le j,k\le n.$$

(4.1) Definition.

a) *A hermitian manifold is a pair (X,ω) where ω is a C^∞ positive definite $(1,1)$-form on X.*

b) *The metric ω is said to be kähler if $d\omega=0$.*

c) *X is said to be a Kähler manifold if X carries at least one Kähler metric.*

Since ω is real, the conditions $d\omega=0$, $d'\omega=0$, $d''\omega=0$ are all equivalent. In local coordinates we see that $d'\omega=0$ if and only if

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j} \quad , \quad 1 \le j,k,l \le n.$$

A simple computation gives

$$\frac{\omega^n}{n!} = \det(h_{jk}) \bigwedge_{1\le j\le n} \Big(\frac{\mathrm{i}}{2} dz_j \wedge d\overline{z}_j\Big) = \det(h_{jk})\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

where $z_n = x_n + \mathrm{i}y_n$. Therefore the (n,n)-form

$$(4.2) \quad dV = \frac{1}{n!}\omega^n$$

is positive and coincides with the hermitian volume element of X. If X is compact, then $\int_X \omega^n = n!\ \mathrm{Vol}_\omega(X) > 0$. This simple remark already implies that compact Kähler manifolds must satisfy some restrictive topological conditions:

(4.3) Consequence.

a) *If (X,ω) is compact Kähler and if $\{\omega\}$ denotes the cohomology class of ω in $H^2(X,\mathbb{R})$, then $\{\omega\}^n \ne 0$.*

b) *If X is compact Kähler, then $H^{2k}(X,\mathbb{R}) \ne 0$ for $0 \le k \le n$. In fact, $\{\omega\}^k$ is a non zero class in $H^{2k}(X,\mathbb{R})$.*

(4.4) Example. The complex projective space $\mathbb{P}^n$ is Kähler. A natural Kähler metric ω on $\mathbb{P}^n$, called the *Fubini-Study metric*, is defined by

$$p^\star\omega = \frac{\mathrm{i}}{2\pi} d'd'' \log\big(|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2\big)$$

where $\zeta_0, \zeta_1, \ldots, \zeta_n$ are coordinates of $\mathbb{C}^{n+1}$ and where $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the projection. Let $z = (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$ be non homogeneous coordinates on $\mathbb{C}^n \subset \mathbb{P}^n$. Then (V-15.8) and (V-15.12) show that

$$\omega = \frac{\mathrm{i}}{2\pi} d'd'' \log(1+|z|^2) = \frac{\mathrm{i}}{2\pi} c\big(\mathcal{O}(1)\big), \qquad \int_{\mathbb{P}^n} \omega^n = 1.$$

Furthermore $\{\omega\} \in H^2(\mathbb{P}^n,\mathbb{Z})$ is a generator of the cohomology algebra $H^\bullet(\mathbb{P}^n,\mathbb{Z})$ in virtue of Th. V-15.10.

(4.5) Example. A *complex torus* is a quotient $X = \mathbb{C}^n/\Gamma$ by a lattice Γ of rank $2n$. Then X is a compact complex manifold. Any positive definite hermitian form $\omega = \mathrm{i}\sum h_{jk} dz_j \wedge d\overline{z}_k$ with constant coefficients defines a Kähler metric on X.

(4.6) Example. Every (complex) submanifold Y of a Kähler manifold (X,ω) is Kähler with metric $\omega_{\restriction Y}$. Especially, all submanifolds of $\mathbb{P}^n$ are Kähler.

(4.7) Example. Consider the complex surface

$$X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where $\Gamma = \{\lambda^n \ ; \ n \in \mathbb{Z}\}$, $\lambda < 1$, acts as a group of homotheties. Since $\mathbb{C}^2 \setminus \{0\}$ is diffeomorphic to $\mathbb{R}_+^\star \times S^3$, we have $X \simeq S^1 \times S^3$. Therefore $H^2(X, \mathbb{R}) = 0$ by Künneth's formula IV-15.10, and property 4.3 b) shows that X is not Kähler. More generally, one can obtaintake Γ to be an infinite cyclic group generated by a holomorphic contraction of $\mathbb{C}^2$, of the form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix}, \qquad \text{resp.} \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda z_1 \\ \lambda z_2 + z_1^p \end{pmatrix},$$

where $\lambda, \lambda_1, \lambda_2$ are complex numbers such that $0 < |\lambda_1| \le |\lambda_2| < 1$, $0 < |\lambda| < 1$, and p a positive integer. These non Kähler surfaces are called *Hopf surfaces.*

The following Theorem shows that a hermitian metric ω on X is Kähler if and only if the metric ω is tangent at order 2 to a hermitian metric with constant coefficients at every point of X.

(4.8) Theorem. *Let ω be a C^∞ positive definite $(1,1)$-form on X. In order that ω be Kähler, it is necessary and sufficient that to every point $x_0 \in X$ corresponds a holomorphic coordinate system $(z_1, \ldots, z_n)$ centered at x_0 such that*

$$(4.9) \quad \omega = \mathrm{i} \sum_{1 \le l,m \le n} \omega_{lm} \, dz_l \wedge d\overline{z}_m, \qquad \omega_{lm} = \delta_{lm} + O(|z|^2).$$

If ω is Kähler, the coordinates $(z_j)_{1 \le j \le n}$ can be chosen such that

$$(4.10) \quad \omega_{lm} = \langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \rangle = \delta_{lm} - \sum_{1 \le j,k \le n} c_{jklm} \, z_j \overline{z}_k + O(|z|^3),$$

where (c_{jklm}) are the coefficients of the Chern curvature tensor

$$(4.11) \quad \Theta(T_X)_{x_0} = \sum_{j,k,l,m} c_{jklm} \, dz_j \wedge d\overline{z}_k \otimes \Big(\frac{\partial}{\partial z_l}\Big)^\star \otimes \frac{\partial}{\partial z_m}$$

associated to (T_X, ω) at x_0. Such a system (z_j) will be called a geodesic coordinate system at x_0.

Proof. It is clear that (4.9) implies $d_{x_0}\omega = 0$, so the condition is sufficient. Assume now that ω is Kähler. Then one can choose local coordinates $(x_1, \ldots, x_n)$ such that $(dx_1, \ldots, dx_n)$ is an ω-orthonormal basis of $T^\star_{x_0} X$. Therefore

$$\omega = \mathrm{i} \sum_{1\le l,m\le n} \widetilde{\omega}_{lm}\, dx_l \wedge d\overline{x}_m, \qquad \text{where}$$

$$\widetilde{\omega}_{lm} = \delta_{lm} + O(|x|) = \delta_{lm} + \sum_{1\le j\le n} (a_{jlm}x_j + a'_{jlm}\overline{x}_j) + O(|x|^2). \tag{4.12}$$

Since ω is real, we have $a'_{jlm} = \overline{a}_{jml}$; on the other hand the Kähler condition $\partial\omega_{lm}/\partial x_j = \partial\omega_{jm}/\partial x_l$ at x_0 implies $a_{jlm} = a_{ljm}$. Set now

$$z_m = x_m + \frac{1}{2}\sum_{j,l} a_{jlm}x_jx_l, \qquad 1\le m\le n.$$

Then (z_m) is a coordinate system at x_0, and

$$dz_m = dx_m + \sum_{j,l} a_{jlm}x_j dx_l,$$

$$\begin{aligned}\mathrm{i}\sum_m dz_m\wedge d\overline{z}_m &= \mathrm{i}\sum_m dx_m\wedge d\overline{x}_m + \mathrm{i}\sum_{j,l,m} a_{jlm}x_j\, dx_l\wedge d\overline{x}_m\\ &\qquad + \mathrm{i}\sum_{j,l,m}\overline{a}_{jlm}\overline{x}_j\, dx_m\wedge d\overline{x}_l + O(|x|^2)\\ &= \mathrm{i}\sum_{l,m}\widetilde{\omega}_{lm}\, dx_l\wedge \overline{d}x_m + O(|x|^2) = \omega + O(|z|^2).\end{aligned}$$

Condition (4.9) is proved. Suppose the coordinates (x_m) chosen from the beginning so that (4.9) holds with respect to (x_m). Then the Taylor expansion (4.12) can be refined into

$$\begin{aligned}\widetilde{\omega}_{lm} &= \delta_{lm} + O(|x|^2)\\ &= \delta_{lm} + \sum_{j,k}\big(a_{jklm}x_j\overline{x}_k + a'_{jklm}x_jx_k + a''_{jklm}\overline{x}_j\overline{x}_k\big) + O(|x|^3).\end{aligned} \tag{4.13}$$

These new coefficients satisfy the relations

$$a'_{jklm} = a'_{kjlm}, \qquad a''_{jklm} = \overline{a}'_{jkml}, \qquad \overline{a}_{jklm} = a_{kjml}.$$

The Kähler condition $\partial\omega_{lm}/\partial x_j = \partial\omega_{jm}/\partial x_l$ at $x = 0$ gives the equality $a'_{jklm} = a'_{lkjm}$; in particular a'_{jklm} is invariant under all permutations of j,k,l. If we set

$$z_m = x_m + \frac{1}{3}\sum_{j,k,l} a'_{jklm}\, x_jx_kx_l, \qquad 1\le m\le n,$$

then by (4.13) we find

$$dz_m = dx_m + \sum_{j,k,l} a'_{jklm}\, x_j x_k\, dx_l, \qquad 1 \le m \le n,$$

$$\omega = \mathrm{i} \sum_{1\le m\le n} dz_m \wedge d\overline{z}_m + \mathrm{i} \sum_{j,k,l,m} a_{jklm}\, x_j \overline{x}_k\, dx_l \wedge d\overline{x}_m + O(|x|^3),$$

$$\omega = \mathrm{i} \sum_{1\le m\le n} dz_m \wedge d\overline{z}_m + \mathrm{i} \sum_{j,k,l,m} a_{jklm}\, z_j \overline{z}_k\, dz_l \wedge d\overline{z}_m + O(|z|^3). \tag{4.14}$$

It is now easy to compute the Chern curvature tensor $\Theta(T_X)_{x_0}$ in terms of the coefficients a_{jklm}. Indeed

$$\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \rangle = \delta_{lm} + \sum_{j,k} a_{jklm}\, z_j \overline{z}_k + O(|z|^3),$$

$$d'\langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \rangle = \Big\{ D' \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \Big\} = \sum_{j,k} a_{jklm}\, \overline{z}_k\, dz_j + O(|z|^2),$$

$$\Theta(T_X) \cdot \frac{\partial}{\partial z_l} = D''D'\Big(\frac{\partial}{\partial z_l}\Big) = - \sum_{j,k,m} a_{jklm}\, dz_j \wedge d\overline{z}_k \otimes \frac{\partial}{\partial z_m} + O(|z|),$$

therefore $c_{jklm} = -a_{jklm}$ and the expansion (4.10) follows from (4.14). □

(4.15) Remark. As a by-product of our computations, we find that on a Kähler manifold the coefficients of $\Theta(T_X)$ satisfy the symmetry relations

$$\overline{c}_{jklm} = c_{kjml}, \qquad c_{jklm} = c_{lkjm} = c_{jmlk} = c_{lmjk}.$$

§5. Basic Results of Kähler Geometry

§5.1. Operators of Hermitian Geometry

Let (X, ω) be a hermitian manifold and let $z_j = x_j + \mathrm{i} y_j$, $1 \le j \le n$, be analytic coordinates at a point $x \in X$ such that $\omega(x) = \mathrm{i} \sum dz_j \wedge d\overline{z}_j$ is diagonalized at this point. The associated hermitian form is the $h(x) = 2 \sum dz_j \otimes d\overline{z}_j$ and its real part is the euclidean metric $2 \sum (dx_j)^2 + (dy_j)^2$. It follows from this that $|dx_j| = |dy_j| = 1/\sqrt{2}$, $|dz_j| = |d\overline{z}_j| = 1$, and that $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ is an orthonormal basis of $(T^\star_x X, \omega)$. Formula (3.1) with u_j, v_k in the orthogonal sum $(\mathbb{C} \otimes T_X)^\star = T^\star_X \oplus \overline{T^\star_X}$ defines a natural inner product on the exterior algebra $\Lambda^\bullet(\mathbb{C} \otimes T_X)^\star$. The norm of a form

$$u = \sum_{I,J} u_{I,J} dz_I \wedge d\overline{z}_J \in \Lambda(\mathbb{C} \otimes T_X)^\star$$

at the given point x is then equal to

(5.1) $\quad |u(x)|^2 = \sum_{I,J} |u_{I,J}(x)|^2.$

The Hodge $\star$ operator (3.2) can be extended to $\mathbb{C}$-valued forms by the formula

(5.2) $\quad u \wedge \star\,\overline{v} = \langle u, v\rangle\, dV.$

It follows that $\star$ is a $\mathbb{C}$-linear isometry

$$\star\ :\ \Lambda^{p,q}T^\star_X \longrightarrow \Lambda^{n-q,n-p}T^\star_X.$$

The usual operators of hermitian geometry are the operators d, $\delta = -\star\, d\,\star$, $\Delta = d\delta + \delta d$ already defined, and their complex counterparts

$$(5.3)\qquad \begin{cases} d = d' + d'', \\ \delta = d'^\star + d''^\star, \qquad d'^\star = (d')^\star = -\star\, d''\star, \qquad d''^\star = (d'')^\star = -\star\, d'\star, \\ \Delta' = d'd'^\star + d'^\star d', \qquad \Delta'' = d''d''^\star + d''^\star d''. \end{cases}$$

Another important operator is the operator L of type (1,1) defined by

(5.4) $\quad Lu = \omega \wedge u$

and its adjoint $\Lambda = \star^{-1} L\,\star$:

(5.5) $\quad \langle u, \Lambda v\rangle = \langle Lu, v\rangle.$

§5.2. Commutation Identities

If A, B are endomorphisms of the algebra $C^\infty_{\bullet,\bullet}(X,\mathbb{C})$, their graded commutator (or graded Lie bracket) is defined by

(5.6) $\quad [A, B] = AB - (-1)^{ab} BA$

where a, b are the degrees of A and B respectively. If C is another endomorphism of degree c, the following *Jacobi identity* is easy to check:

(5.7) $\quad (-1)^{ca}\big[A, [B, C]\big] + (-1)^{ab}\big[B, [C, A]\big] + (-1)^{bc}\big[C, [A, B]\big] = 0.$

For any $\alpha \in \Lambda^{p,q}T^\star_X$, we still denote by α the endomorphism of type (p, q) on $\Lambda^{\bullet,\bullet}T^\star_X$ defined by $u \mapsto \alpha \wedge u$.

Let $\gamma \in \Lambda^{1,1}T^\star_X$ be a real (1,1)-form. There exists an ω-orthogonal basis $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ in T_X which diagonalizes both forms ω and γ:

$$\omega = \mathrm{i} \sum_{1\le j\le n} \zeta^\star_j \wedge \overline{\zeta}^\star_j, \qquad \gamma = \mathrm{i} \sum_{1\le j\le n} \gamma_j\, \zeta^\star_j \wedge \overline{\zeta}^\star_j, \quad \gamma_j \in \mathbb{R}.$$

(5.8) Proposition. *For every form* $u = \sum u_{J,K}\, \zeta^\star_J \wedge \overline{\zeta}^\star_K$, *one has*

$$[\gamma, \Lambda]u = \sum_{J,K} \Big(\sum_{j\in J} \gamma_j + \sum_{j\in K} \gamma_j - \sum_{1\le j\le n} \gamma_j \Big) u_{J,K}\, \zeta_J^\star \wedge \overline{\zeta}_K^\star.$$

Proof. If u is of type (p,q), a brute-force computation yields

$$\begin{aligned}
\Lambda u &= \mathrm{i}(-1)^p \sum_{J,K,l} u_{J,K}\, (\zeta_l \lrcorner \zeta_J^\star) \wedge (\overline{\zeta}_l \lrcorner \overline{\zeta}_K^\star), \qquad 1 \le l \le n,\\
\gamma \wedge u &= \mathrm{i}(-1)^p \sum_{J,K,m} \gamma_m u_{J,K}\, \zeta_m^\star \wedge \zeta_J^\star \wedge \overline{\zeta}_m^\star \wedge \overline{\zeta}_K^\star, \qquad 1 \le m \le n,\\
[\gamma, \Lambda]u &= \sum_{J,K,l,m} \gamma_m\, u_{J,K} \Big(\big(\zeta_l^\star \wedge (\zeta_m \lrcorner \zeta_J^\star)\big) \wedge \big(\overline{\zeta}_l^\star \wedge (\overline{\zeta}_m \lrcorner \overline{\zeta}_K^\star)\big)\\
&\qquad\qquad - \big(\zeta_m \lrcorner (\zeta_l^\star \wedge \zeta_J^\star)\big) \wedge \big(\overline{\zeta}_m \lrcorner (\overline{\zeta}_l^\star \wedge \overline{\zeta}_K^\star)\big)\Big)\\
&= \sum_{J,K,m} \gamma_m\, u_{J,K} \Big(\zeta_m^\star \wedge (\zeta_m \lrcorner \zeta_J^\star) \wedge \overline{\zeta}_K^\star\\
&\qquad\qquad + \zeta_J^\star \wedge \overline{\zeta}_m^\star \wedge (\overline{\zeta}_m \lrcorner \overline{\zeta}_K^\star) - \zeta_J^\star \wedge \overline{\zeta}_K^\star \Big)\\
&= \sum_{J,K} \Big(\sum_{m\in J} \gamma_m + \sum_{m\in K} \gamma_m - \sum_{1\le m\le n} \gamma_m \Big) u_{J,K}\, \zeta_J^\star \wedge \overline{\zeta}_K^\star. \qquad \square
\end{aligned}$$

(5.9) Corollary. *For every $u \in \Lambda^{p,q} T_X^\star$, we have*

$$[L, \Lambda]u = (p + q - n)u.$$

Proof. Indeed, if $\gamma = \omega$, we have $\gamma_1 = \cdots = \gamma_n = 1$. $\square$

This result can be generalized as follows: for every $u \in \Lambda^k(\mathbb{C} \otimes T_X)^\star$, we have

$$(5.10) \qquad [L^r, \Lambda]u = r(k - n + r - 1)\, L^{r-1} u.$$

In fact, it is clear that

$$\begin{aligned}
[L^r, \Lambda]u &= \sum_{0\le m\le r-1} L^{r-1-m} [L, \Lambda] L^m u\\
&= \sum_{0\le m\le r-1} (2m + k - n) L^{r-1-m} L^m u = \big(r(r-1) + r(k-n)\big) L^{r-1} u.
\end{aligned}$$

§5.3. Primitive Elements and Hard Lefschetz Theorem

In this subsection, we prove a fundamental decomposition theorem for the representation of the unitary group $U(T_X) \simeq U(n)$ acting on the spaces $\Lambda^{p,q}T^\star_X$ of (p,q)-forms. It turns out that the representation is never irreducible if $0 < p, q < n$.

(5.11) Definition. *A homogeneous element $u \in \Lambda^k(\mathbb{C}\otimes T_X)^\star$ is called primitive if $\Lambda u = 0$. The space of primitive elements of total degree k will be denoted*

$$\mathrm{Prim}^k T^\star_X = \bigoplus_{p+q=k} \mathrm{Prim}^{p,q} T^\star_X.$$

Let $u \in \mathrm{Prim}^k T^\star_X$. Then

$$\Lambda^s L^r u = \Lambda^{s-1}(\Lambda L^r - L^r\Lambda)u = r(n-k-r+1)\Lambda^{s-1}L^{r-1}u.$$

By induction, we get for $r \geq s$

$$(5.12)\quad \Lambda^s L^r u = r(r-1)\cdots(r-s+1)\cdot(n-k-r+1)\cdots(n-k-r+s)L^{r-s}u.$$

Apply (5.12) for $r = n+1$. Then $L^{n+1}u$ is of degree $> 2n$ and therefore we have $L^{n+1}u = 0$. This gives

$$(n+1)\cdots\big(n+1-(s-1)\big)\cdot(-k)(-k+1)\cdots(-k+s-1)L^{n+1-s}u = 0.$$

The integral coefficient is $\neq 0$ if $s \leq k$, hence:

(5.13) Corollary. *If $u \in \mathrm{Prim}^k T^\star_X$, then $L^s u = 0$ for $s \geq (n+1-k)_+$.*

(5.14) Corollary. $\mathrm{Prim}^k T^\star_X = 0$ *for $n+1 \leq k \leq 2n$.*

Proof. Apply Corollary 5.13 with $s = 0$. □

(5.15) Primitive decomposition formula. *For every $u \in \Lambda^k(\mathbb{C}\otimes T_X)^\star$, there is a unique decomposition*

$$u = \sum_{r \geq (k-n)_+} L^r u_r, \qquad u_r \in \mathrm{Prim}^{k-2r} T^\star_X.$$

Furthermore $u_r = \Phi_{k,r}(L,\Lambda)u$ where $\Phi_{k,r}$ is a non commutative polynomial in L, Λ with rational coefficients. As a consequence, there are direct sum decompositions of $U(n)$-representations

$$\Lambda^k(\mathbb{C}\otimes T_X)^\star = \bigoplus_{r\ge(k-n)_+} L^r \operatorname{Prim}^{k-2r} T_X^\star,$$

$$\Lambda^{p,q}T_X^\star = \bigoplus_{r\ge(p+q-n)_+} L^r \operatorname{Prim}^{p-r,q-r} T_X^\star.$$

Proof of the uniqueness of the decomposition Assume that $u = 0$ and that $u_r \neq 0$ for some r. Let s be the largest integer such that $u_s \neq 0$. Then

$$\Lambda^s u = 0 = \sum_{(k-n)_+\le r\le s} \Lambda^s L^r u_r = \sum_{(k-n)_+\le r\le s} \Lambda^{s-r}\Lambda^r L^r u_r.$$

But formula (5.12) shows that $\Lambda^r L^r u_r = c_{k,r} u_r$ for some non zero integral coefficient $c_{k,r} = r!(n-k+r+1)\cdots(n-k+2r)$. Since u_r is primitive we get $\Lambda^s L^r u_r = 0$ when $r < s$, hence $u_s = 0$, a contradiction.

Proof of the existence of the decomposition We prove by induction on $s \ge (k-n)_+$ that $\Lambda^s u = 0$ implies

$$(5.16)\qquad u = \sum_{(k-n)_+\le r<s} L^r u_r, \qquad u_r = \Phi_{k,r,s}(L,\Lambda)u \in \operatorname{Prim}^{k-2r} T_X^\star.$$

The Theorem will follow from the step $s = n+1$.

Assume that the result is true for s and that $\Lambda^{s+1}u = 0$. Then $\Lambda^s u$ is in $\operatorname{Prim}^{k-2s} T_X^\star$. Since $s \ge (k-n)_+$ we have $c_{k,s} \neq 0$ and we set

$$u_s = \frac{1}{c_{k,s}}\Lambda^s u \in \operatorname{Prim}^{k-2s} T_X^\star,$$

$$u' = u - L^s u_s = \Big(1 - \frac{1}{c_{k,s}} L^s\Lambda^s\Big)u.$$

By formula (5.12), we get

$$\Lambda^s u' = \Lambda^s u - \Lambda^s L^s u_s = \Lambda^s u - c_{k,s}u_s = 0.$$

The induction hypothesis implies

$$u' = \sum_{(k-n)_+\le r<s} L^r u'_r,\ \ u'_r = \Phi_{k,r,s}(L,\Lambda)u' \in \operatorname{Prim}^{k-2r} T_X^\star,$$

hence $u = \sum_{(k-n)_+\le r\le s} L^r u_r$ with

$$\begin{cases} u_r = u'_r = \Phi_{k,r,s}(L,\Lambda)\Big(1 - \frac{1}{c_{k,s}} L^s\Lambda^s\Big)u,\ \ r < s, \\ u_s = \frac{1}{c_{k,s}}\Lambda^s u. \end{cases}$$

It remains to prove the validity of the decomposition 5.16) for the initial step $s = (k-n)_+$, i.e. that $\Lambda^s u = 0$ implies $u = 0$. If $k \le n$, then $s = 0$ and there is nothing to prove. We are left with the case $k > n$, $\Lambda^{k-n}u = 0$. Then $v = \star u \in \Lambda^{2n-k}(\mathbb{C} \otimes T_X)^\star$ and $2n - k < n$. Since the decomposition exists in degree $\le n$ by what we have just proved, we get

$$v = \star u = \sum_{r \ge 0} L^r v_r, \ \ v_r \in \mathrm{Prim}^{2n-k-2r}\, T_X^\star,$$
$$0 = \star \Lambda^{k-n} u = L^{k-n} \star u = \sum_{r \ge 0} L^{r+k-n} v_r,$$

with degree $(L^{r+k-n}v_r) = 2n - k + 2(k-n) = k$. The uniqueness part shows that $v_r = 0$ for all r, hence $u = 0$. The Theorem is proved. □

(5.17) Corollary. *The linear operators*

$$L^{n-k} : \Lambda^k(\mathbb{C} \otimes T_X)^\star \longrightarrow \Lambda^{2n-k}(\mathbb{C} \otimes T_X)^\star,$$
$$L^{n-p-q} : \Lambda^{p,q}T_X^\star \longrightarrow \Lambda^{n-q,n-p}T_X^\star,$$

are isomorphisms for all integers $k \le n$, $p + q \le n$.

Proof. For every $u \in \Lambda^k_{\mathbb{C}} T_X^\star$, the primitive decomposition $u = \sum_{r \ge 0} L^r u_r$ is mapped bijectively onto that of $L^{n-k}u$:

$$L^{n-k}u = \sum_{r \ge 0} L^{r+n-k} u_r. \qquad \square$$

§6. Commutation Relations

§6.1. Commutation Relations on a Kähler Manifold

Assume first that $X = \Omega \subset \mathbb{C}^n$ is an open subset and that ω is the standard Kähler metric

$$\omega = \mathrm{i} \sum_{1 \le j \le n} dz_j \wedge d\overline{z}_j.$$

For any form $u \in C^\infty(\Omega, \Lambda^{p,q}T_X^\star)$ we have

$$(6.1') \qquad d'u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J,$$

$$(6.1'') \qquad d''u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.$$

Since the global L^2 inner product is given by

$$\langle\!\langle u, v\rangle\!\rangle = \int_\Omega \sum_{I,J} u_{I,J}\overline{v}_{I,J}\, dV,$$

easy computations analogous to those of Example 3.12 show that

$$(6.2') \qquad d'^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k}\frac{\partial}{\partial z_k} \lrcorner\ (dz_I \wedge d\overline{z}_J),$$

$$(6.2'') \qquad d''^\star u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k}\frac{\partial}{\partial \overline{z}_k} \lrcorner\ (dz_I \wedge d\overline{z}_J).$$

We first prove a lemma due to (Akizuki and Nakano 1954).

(6.3) Lemma. *In $\mathbb{C}^n$, we have $[d''^\star, L] = \mathrm{i}d'$.*

Proof. Formula (6.2″) can be written more briefly

$$d''^\star u = -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\frac{\partial u}{\partial z_k}\Big).$$

Then we get

$$[d''^\star, L]u = -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\frac{\partial}{\partial z_k}(\omega\wedge u)\Big) + \omega \wedge \sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\frac{\partial u}{\partial z_k}\Big).$$

Since ω has constant coefficients, we have $\dfrac{\partial}{\partial z_k}(\omega\wedge u) = \omega\wedge\dfrac{\partial u}{\partial z_k}$ and therefore

$$\begin{aligned}[d''^\star, L]\, u &= -\sum_k \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\ \Big(\omega\wedge\frac{\partial u}{\partial z_k}\Big) - \omega\wedge\Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\ \frac{\partial u}{\partial z_k}\Big)\Big)\\ &= -\sum_k \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\ \omega\Big)\wedge\frac{\partial u}{\partial z_k}.\end{aligned}$$

Clearly $\dfrac{\partial}{\partial \overline{z}_k} \lrcorner\ \omega = -\mathrm{i}dz_k$, so

$$[d''^\star, L]\, u = \mathrm{i}\sum_k dz_k \wedge \frac{\partial u}{\partial z_k} = \mathrm{i}d'u. \qquad \square$$

We are now ready to derive the basic commutation relations in the case of an arbitrary Kähler manifold (X,ω).

(6.4) Theorem. *If (X,ω) is Kähler, then*

$$\begin{aligned} &[d''^\star, L] = \mathrm{i}d', && [d'^\star, L] = -\mathrm{i}d'',\\ &[\Lambda, d''] = -\mathrm{i}d'^\star, && [\Lambda, d'] = \mathrm{i}d''^\star.\end{aligned}$$

Proof. It is sufficient to verify the first relation, because the second one is the conjugate of the first, and the relations of the second line are the adjoint of those of the first line. If (z_j) is a geodesic coordinate system at a point $x_0 \in X$, then for any (p,q)-forms u, v with compact support in a neighborhood of x_0, (4.9) implies

$$\langle\!\langle u, v\rangle\!\rangle = \int_M \Big(\sum_{I,J} u_{IJ}\overline{v}_{IJ} + \sum_{I,J,K,L} a_{IJKL}\, u_{IJ}\overline{v}_{KL}\Big)\, dV,$$

with $a_{IJKL}(z) = O(|z|^2)$ at x_0. An integration by parts as in (3.12) and (6.2″) yields

$$d''^{\star} u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} \frac{\partial}{\partial \overline{z}_k} \lrcorner\ (dz_I \wedge d\overline{z}_J) + \sum_{I,J,K,L} b_{IJKL}\, u_{IJ}\, dz_K \wedge d\overline{z}_L,$$

where the coefficients b_{IJKL} are obtained by derivation of the a_{IJKL}'s. Therefore $b_{IJKL} = O(|z|)$. Since $\partial\omega/\partial z_k = O(|z|)$, the proof of Lemma 6.3 implies here $[d''^{\star}, L]u = \mathrm{i}d'u + O(|z|)$, in particular both terms coincide at every given point $x_0 \in X$. □

(6.5) Corollary. *If (X, ω) is Kähler, the complex Laplace-Beltrami operators satisfy*

$$\Delta' = \Delta'' = \frac{1}{2}\Delta.$$

Proof. It will be first shown that $\Delta'' = \Delta'$. We have

$$\Delta'' = [d'', d''^{\star}] = -\mathrm{i}\big[d'', [\Lambda, d']\big].$$

Since $[d', d''] = 0$, Jacobi's identity (5.7) implies

$$-\big[d'', [\Lambda, d']\big] + \big[d', [d'', \Lambda]\big] = 0,$$

hence $\Delta'' = \big[d', -\mathrm{i}[d'', \Lambda]\big] = [d', d'^{\star}] = \Delta'$. On the other hand

$$\Delta = [d' + d'', d'^{\star} + d''^{\star}] = \Delta' + \Delta'' + [d', d''^{\star}] + [d'', d'^{\star}].$$

Thus, it is enough to prove:

(6.6) Lemma. $[d', d''^{\star}] = 0,\ [d'', d'^{\star}] = 0.$

Proof. We have $[d', d''^{\star}] = -\mathrm{i}\big[d', [\Lambda, d']\big]$ and (5.7) implies

$$-\big[d', [\Lambda, d']\big] + \big[\Lambda, [d', d']\big] + \big[d', [d', \Lambda]\big] = 0,$$

hence $-2\big[d', [\Lambda, d']\big] = 0$ and $[d', d''^{\star}] = 0$. The second relation $[d'', d'^{\star}] = 0$ is the adjoint of the first. □

(6.7) Theorem. *Δ commutes with all operators $\star, d', d'', d'^\star, d''^\star, L, \Lambda$.*

Proof. The identities $[d', \Delta'] = [d'^\star, \Delta'] = 0$, $[d'', \Delta''] = [d''^\star, \Delta''] = 0$ and $[\Delta, \star] = 0$ are immediate. Furthermore, the equality $[d', L] = d'\omega = 0$ together with the Jacobi identity implies

$$[L, \Delta'] = \big[L, [d', d'^\star]\big] = -\big[d', [d'^\star, L]\big] = \mathrm{i}[d', d''] = 0.$$

By adjunction, we also get $[\Delta', \Lambda] = 0$. □

§6.2 Commutation Relations on Hermitian Manifolds

We are going to extend the commutation relations of § 6.1 to an arbitrary hermitian manifold (X, ω). In that case ω is no longer tangent to a constant metric, and the commutation relations involve extra terms arising from the *torsion* of ω. Theorem 6.8 below is taken from (Demailly 1984), but the idea was already contained in (Griffiths 1966).

(6.8) Theorem. *Let τ be the operator of type $(1,0)$ and order 0 defined by $\tau = [\Lambda, d'\omega]$. Then*

a) $[d''^\star, L] = \mathrm{i}(d' + \tau)$,
b) $[d'^\star, L] = -\mathrm{i}(d'' + \overline{\tau})$,
c) $[\Lambda, d''] = -\mathrm{i}(d'^\star + \tau^\star)$,
d) $[\Lambda, d'] = \mathrm{i}(d''^\star + \overline{\tau}^\star)$;

$d'\omega$ will be called the torsion form of ω, and τ the torsion operator.

Proof. b) follows from a) by conjugation, whereas c), d) follow from a), b) by adjunction. It is therefore enough to prove relation a).

Let $(z_j)_{1\le j\le n}$ be complex coordinates centered at a point $x_0 \in X$, such that $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ is an orthonormal basis of $T_{x_0}X$ for the metric $\omega(x_0)$. Consider the metric with constant coefficients

$$\omega_0 = \mathrm{i} \sum_{1\le j\le n} dz_j \wedge d\overline{z}_j.$$

The metric ω can then be written

$$\omega = \omega_0 + \gamma \quad\text{with}\quad \gamma = O(|z|).$$

Denote by $\langle\ ,\ \rangle_0$, L_0, Λ_0, $d_0'^\star$, $d_0''^\star$ the inner product and the operators associated to the constant metric ω_0, and let $dV_0 = \omega_0^n/2^n n!$. The proof of relation a) is based on a Taylor expansion of L, Λ, $d'^\star$, $d''^\star$ in terms of the operators with constant coefficients L_0, Λ_0, $d_0'^\star$, $d_0''^\star$.

(6.9) Lemma. *Let $u, v \in C^\infty(X, \Lambda^{p,q}T_X^\star)$. Then in a neighborhood of x_0*

$$\langle u, v\rangle\, dV = \langle u - [\gamma, \Lambda_0]u, v\rangle_0\, dV_0 + O(|z|^2).$$

Proof. In a neighborhood of x_0, let

$$\gamma = \mathrm{i} \sum_{1\le j\le n} \gamma_j\, \zeta_j^\star \wedge \overline{\zeta}_j^\star, \qquad \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n,$$

be a diagonalization of the (1,1)-form $\gamma(z)$ with respect to an orthonormal basis $(\zeta_j)_{1\le j\le n}$ of T_zX for $\omega_0(z)$. We thus have

$$\omega = \omega_0 + \gamma = \mathrm{i} \sum \lambda_j\, \zeta_j^\star \wedge \overline{\zeta}_j^\star$$

with $\lambda_j = 1 + \gamma_j$ and $\gamma_j = O(|z|)$. Set now

$$J = \{j_1, \ldots, j_p\}, \qquad \zeta_J^\star = \zeta_{j_1}^\star \wedge \cdots \wedge \zeta_{j_p}^\star, \qquad \lambda_J = \lambda_{j_1} \cdots \lambda_{j_p},$$

$$u = \sum u_{J,K}\, \zeta_J^\star \wedge \overline{\zeta}_K^\star, \qquad v = \sum v_{J,K}\, \zeta_J^\star \wedge \overline{\zeta}_K^\star$$

where summations are extended to increasing multi-indices J, K such that $|J| = p$, $|K| = q$. With respect to ω we have $\langle \zeta_j^\star, \zeta_j^\star\rangle = \lambda_j^{-1}$, hence

$$\begin{aligned}
\langle u, v\rangle\, dV &= \sum_{J,K} \lambda_J^{-1}\lambda_K^{-1}\, u_{J,K}\overline{v}_{J,K}\, \lambda_1 \cdots \lambda_n\, dV_0 \\
&= \sum_{J,K} \Big(1 - \sum_{j\in J} \gamma_j - \sum_{j\in K} \gamma_j + \sum_{1\le j\le n} \gamma_j\Big) u_{J,K}\overline{v}_{J,K}\, dV_0 + O(|z|^2).
\end{aligned}$$

Lemma 6.9 follows if we take Prop. 5.8 into account. □

(6.10) Lemma. *$d''^\star = d_0''^\star + \big[\Lambda_0, [d_0''^\star, \gamma]\big]$ at point x_0, i.e. at this point both operators have the same formal expansion.*

Proof. Since $d''^\star$ is an operator of order 1, Lemma 6.9 shows that $d''^\star$ coincides at point x_0 with the formal adjoint of d'' for the metric

$$\langle\!\langle u, v\rangle\!\rangle_1 = \int_X \langle u - [\gamma, \Lambda_0]u, v\rangle_0\, dV_0.$$

For any compactly supported $u \in C^\infty(X, \Lambda^{p,q}T_X^\star)$, $v \in C^\infty(X, \Lambda^{p,q-1}T_X^\star)$ we get by definition

$$\langle\!\langle u, d''v\rangle\!\rangle_1 = \int_X \langle u - [\gamma, \Lambda_0]u, d''v\rangle_0\, dV_0 = \int_X \langle d_0''^\star u - d_0''^\star[\gamma, \Lambda_0]u, v\rangle_0\, dV_0.$$

Since ω and ω_0 coincide at point x_0 and since $\gamma(x_0) = 0$ we obtain at this point

$$\begin{aligned}
&d''^\star u = d_0''^\star u - d_0''^\star[\gamma, \Lambda_0]u = d_0''^\star u - \big[d_0''^\star, [\gamma, \Lambda_0]\big]\, u\ ; \\
&\quad d''^\star = d_0''^\star - \big[d_0''^\star, [\gamma, \Lambda_0]\big].
\end{aligned}$$

We have $[\Lambda_0, d_0''^\star] = [d'', L_0]^\star = 0$ since $d''\omega_0 = 0$. The Jacobi identity (5.7) implies

$$\big[d_0''^\star, [\gamma, \Lambda_0]\big] + \big[\Lambda_0, [d_0''^\star, \gamma]\big] = 0,$$

and Lemma 6.10 follows. □

Proof Proof of formula 6.8 a) The equality $L = L_0 + \gamma$ and Lemma 6.10 yield

$$(6.11)\quad [L, d''^\star] = [L_0, d_0''^\star] + \Big[L_0, \big[\Lambda_0, [d_0''^\star, \gamma]\big]\Big] + [\gamma, d_0''^\star]$$

at point x_0, because the triple bracket involving γ twice vanishes at x_0. From the Jacobi identity applied to $C = [d_0''^\star, \gamma]$, we get

$$(6.12)\quad \begin{cases} \big[L_0, [\Lambda_0, C]\big] = -[\Lambda_0, [C, L_0]] - \big[C, [L_0, \Lambda_0]\big], \\ [C, L_0] = \big[L_0, [d_0''^\star, \gamma]\big] = \big[\gamma, [L_0, d_0''^\star]\big] \quad (\text{since } \ [\gamma, L_0] = 0). \end{cases}$$

Lemma 6.3 yields $[L_0, d_0''^\star] = -\mathrm{i}d'$, hence

$$(6.13)\quad [C, L_0] = -[\gamma, \mathrm{i}d'] = \mathrm{i}d'\gamma = \mathrm{i}d'\omega.$$

On the other hand, C is of type $(1,0)$ and Cor. 5.9 gives

$$(6.14)\quad \big[C, [L_0, \Lambda_0]\big] = -C = -[d_0''^\star, \gamma].$$

From (6.12), (6.13), (6.14) we get

$$\Big[L_0, \big[\Lambda_0, [d_0''^\star, \gamma]\big]\Big] = -[\Lambda_0, \mathrm{i}d'\omega] + [d_0''^\star, \gamma].$$

This last equality combined with (6.11) implies

$$[L, d''^\star] = [L_0, d_0''^\star] - [\Lambda_0, \mathrm{i}d'\omega] = -\mathrm{i}(d' + \tau)$$

at point x_0. Formula 6.8 a) is proved. □

(6.15) Corollary. *The complex Laplace-Beltrami operators satisfy*

$$\begin{aligned} \Delta'' &= \Delta' + [d', \tau^\star] - [d'', \overline{\tau}^\star], \\ [d', d''^\star] &= -[d', \overline{\tau}^\star], \qquad [d'', d'^\star] = -[d'', \tau^\star], \\ \Delta &= \Delta' + \Delta'' - [d', \overline{\tau}^\star] - [d'', \tau^\star]. \end{aligned}$$

Therefore Δ', Δ'' and $\frac{1}{2}\Delta$ no longer coincide, but they differ by linear differential operators of order 1 only.

Proof. As in the Kähler case (Cor. 6.5 and Lemma 6.6), we find

$$\begin{aligned}\Delta'' &= [d'', d''^\star] = \big[d'', -i[\Lambda, d'] - \overline{\tau}^\star\big] \\ &= \big[d', -\mathrm{i}[d'', \Lambda]\big] - [d'', \overline{\tau}^\star\big] = \Delta' + [d', \tau^\star] - [d'', \overline{\tau}^\star], \\ [d', d''^\star + \overline{\tau}^\star] &= -\mathrm{i}\big[d', [\Lambda, d']\big] = 0,\end{aligned}$$

and the first two lines are proved. The third one is an immediate consequence of the second. □

§7. Groups $\mathcal{H}^{p,q}(X, E)$ and Serre Duality

Let (X, ω) be a *compact hermitian* manifold and E a holomorphic hermitian vector bundle of rank r over X. We denote by D_E the Chern connection of E, by $D_E^\star = - \star D_E \star$ the formal adjoint of D_E, and by $D_E'^\star$, $D_E''^\star$ the components of $D_E^\star$ of type $(-1, 0)$ and $(0, -1)$.

Corollary 6.8 implies that the principal part of the operator $\Delta''_E = D''D''^\star_E + D''^\star_E D''$ is one half that of Δ_E. Consequently, the operator Δ''_E acting on each space $C^\infty(X, \Lambda^{p,q}T^\star_X \otimes E)$ is a self-adjoint elliptic operator. Since $D''^2 = 0$, the following results can be obtained in a way similar to those of § 3.3.

(7.1) Theorem. *For every bidegree (p, q), there exists an orthogonal decomposition*

$$C^\infty(X, \Lambda^{p,q}T^\star_X \otimes E) = \mathcal{H}^{p,q}(X, E) \oplus \operatorname{Im} D''_E \oplus \operatorname{Im} D''^\star_E$$

where $\mathcal{H}^{p,q}(X, E)$ is the space of Δ''_E-harmonic forms in $C^\infty(X, \Lambda^{p,q}T^\star_X \otimes E)$.

The above decomposition shows that the subspace of d''-cocycles in $C^\infty(X, \Lambda^{p,q}T^\star_X \otimes E)$ is $\mathcal{H}^{p,q}(X, E) \oplus \operatorname{Im} D''_E$. From this, we infer

(7.2) Hodge isomorphism theorem. *The Dolbeault cohomology group $H^{p,q}(X, E)$ is finite dimensional, and there is an isomorphism*

$$H^{p,q}(X, E) \simeq \mathcal{H}^{p,q}(X, E).$$

(7.3) Serre duality theorem. *The bilinear pairing*

$$H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^\star) \longrightarrow \mathbb{C}, \qquad (s, t) \longmapsto \int_M s \wedge t$$

is a non degenerate duality.

Proof. Let $s_1 \in C^\infty(X, \Lambda^{p,q}T^\star_X \otimes E)$, $s_2 \in C^\infty(X, \Lambda^{n-p,n-q-1}T^\star_X \otimes E)$. Since $s_1 \wedge s_2$ is of bidegree $(n, n-1)$, we have

$$(7.4) \quad d(s_1 \wedge s_2) = d''(s_1 \wedge s_2) = d''s_1 \wedge s_2 + (-1)^{p+q} s_1 \wedge d''s_2.$$

Stokes' formula implies that the above bilinear pairing can be factorized through Dolbeault cohomology groups. The # operator defined in § 3.1 is such that

$$\# : C^\infty(X, \Lambda^{p,q}T_X^\star \otimes E) \longrightarrow C^\infty(X, \Lambda^{n-p,n-q}T_X^\star \otimes E^\star).$$

Furthermore, (3.20) implies

$$d''(\# s) = (-1)^{\deg s} \# D_E''^\star s, \qquad D_{E^\star}''^\star(\# s) = (-1)^{\deg s+1} \# D_E''^\star s,$$
$$\Delta_{E^\star}''(\# s) = \# \Delta_E'' s,$$

where $D_{E^\star}$ is the Chern connection of $E^\star$. Consequently, $s \in \mathcal{H}^{p,q}(X, E)$ if and only if $\# s \in \mathcal{H}^{n-p,n-q}(X, E^\star)$. Theorem 7.3 is then a consequence of the fact that the integral $\|s\|^2 = \int_X s \wedge \# s$ does not vanish unless $s = 0$. □

§8. Cohomology of Compact Kähler Manifolds

§8.1. Bott-Chern Cohomology Groups

Let X be for the moment an arbitrary complex manifold. The following "cohomology" groups are helpful to describe Hodge theory on compact complex manifolds which are not necessarily Kähler.

(8.1) Definition. *We define the Bott-Chern cohomology groups of X to be*

$$H_{\mathrm{BC}}^{p,q}(X, \mathbb{C}) = \big(C^\infty(X, \Lambda^{p,q}T_X^\star) \cap \ker d\big) / d'd''C^\infty(X, \Lambda^{p-1,q-1}T_X^\star).$$

Then $H_{\mathrm{BC}}^{\bullet,\bullet}(X, \mathbb{C})$ has the structure of a bigraded algebra, which we call the Bott-Chern cohomology algebra of X.

As the group $d'd''C^\infty(X, \Lambda^{p-1,q-1}T_X^\star)$ is contained in the coboundary groups $d''C^\infty(X, \Lambda^{p,q-1}T_X^\star)$ or $dC^\infty(X, \Lambda^{p+q-1}(\mathbb{C} \otimes T_X)^\star)$, there are canonical morphisms

$$H_{\mathrm{BC}}^{p,q}(X, \mathbb{C}) \longrightarrow H^{p,q}(X, \mathbb{C}), \tag{8.2}$$

$$H_{\mathrm{BC}}^{p,q}(X, \mathbb{C}) \longrightarrow H_{\mathrm{DR}}^{p+q}(X, \mathbb{C}), \tag{8.3}$$

of the Bott-Chern cohomology to the Dolbeault or De Rham cohomology. These morphisms are homomorphisms of $\mathbb{C}$-algebras. It is also clear from the definition that we have the symmetry property $H_{\mathrm{BC}}^{q,p}(X, \mathbb{C}) = \overline{H_{\mathrm{BC}}^{p,q}(X, \mathbb{C})}$. It can be shown from the Hodge-Frölicher spectral sequence (see § 11 and Exercise 13.??) that $H_{\mathrm{BC}}^{p,q}(X, \mathbb{C})$ is always finite dimensional if X is compact.

§8.2. Hodge Decomposition Theorem

We suppose from now on that (X, ω) is a *compact Kähler* manifold. The equality $\Delta = 2\Delta''$ shows that Δ is homogeneous with respect to bidegree and that there is an orthogonal decomposition

$$(8.4) \quad \mathcal{H}^k(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X,\mathbb{C}).$$

As $\overline{\Delta''} = \Delta' = \Delta''$, we also have $\mathcal{H}^{q,p}(X,\mathbb{C}) = \overline{\mathcal{H}^{p,q}(X,\mathbb{C})}$. Using the Hodge isomorphism theorems for the De Rham and Dolbeault cohomology, we get:

(8.5) Hodge decomposition theorem. *On a compact Kähler manifold, there are canonical isomorphisms*

$$\begin{aligned} H^k(X,\mathbb{C}) &\simeq \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}) && (\textit{Hodge decomposition}),\\ H^{q,p}(X,\mathbb{C}) &\simeq \overline{H^{p,q}(X,\mathbb{C})} && (\textit{Hodge symmetry}). \end{aligned}$$

The only point which is not a priori completely clear is that this decomposition is independent of the Kähler metric. In order to show that this is the case, one can use the following Lemma, which allows us to compare all three types of cohomology groups considered in § 8.1.

(8.6) Lemma. *Let u be a d-closed (p,q)-form. The following properties are equivalent:*

a) *u is d-exact;*

b$'$) *u is d'-exact;*

b$''$) *u is d''-exact;*

c) *u is $d'd''$-exact, i.e. u can be written $u = d'd''v$.*

d) *u is orthogonal to $\mathcal{H}^{p,q}(X,\mathbb{C})$.*

Proof. It is obvious that c) implies a), b$'$), b$''$) and that a) or b$'$) or b$''$) implies d). It is thus sufficient to prove that d) implies c). As $du = 0$, we have $d'u = d''u = 0$, and as u is supposed to be orthogonal to $\mathcal{H}^{p,q}(X,\mathbb{C})$, Th. 7.1 implies $u = d''s$, $s \in C^\infty(X, \Lambda^{p,q-1}T^\star_X)$. By the analogue of Th. 7.1 for d', we have $s = h + d'v + d'^\star w$, with $h \in \mathcal{H}^{p,q-1}(X,\mathbb{C})$, $v \in C^\infty(X, \Lambda^{p-1,q-1}T^\star_X)$ and $w \in C^\infty(X, \Lambda^{p+1,q-1}T^\star_X)$. Therefore

$$u = d''d'v + d''d'^\star w = -d'd''v - d'^\star d''w$$

in view of Lemma 6.6. As $d'u = 0$, the component $d'^\star d''w$ orthogonal to $\ker d'$ must be zero. □

From Lemma 8.6 we infer the following Corollary, which in turn implies that the Hodge decomposition does not depend on the Kähler metric.

(8.7) Corollary. *Let X be a compact Kähler manifold. Then the natural morphisms*

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}), \qquad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k_{\mathrm{DR}}(X,\mathbb{C})$$

are isomorphisms.

Proof. The surjectivity of $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C})$ comes from the fact that every class in $H^{p,q}(X,\mathbb{C})$ can be represented by a harmonic (p,q)-form, thus by a d-closed (p,q)-form; the injectivity means nothing more than the equivalence $(8.5\,\mathrm{b}'') \Leftrightarrow (8.5\,\mathrm{c})$. Hence $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \simeq H^{p,q}(X,\mathbb{C}) \simeq \mathcal{H}^{p,q}(X,\mathbb{C})$, and the isomorphism $\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k_{\mathrm{DR}}(X,\mathbb{C})$ follows from (8.4). □

Let us quote now two simple applications of Hodge theory. The first of these is a computation of the Dolbeault cohomology groups of $\mathbb{P}^n$. As $H^{2p}_{\mathrm{DR}}(\mathbb{P}^n,\mathbb{C}) = \mathbb{C}$ and $H^{p,p}(\mathbb{P}^n,\mathbb{C}) \ni \{\omega^p\} \neq 0$, the Hodge decomposition formula implies:

(8.8) Application. *The Dolbeault cohomology groups of $\mathbb{P}^n$ are*

$$H^{p,p}(\mathbb{P}^n,\mathbb{C}) = \mathbb{C} \quad \text{for } 0 \le p \le n, \qquad H^{p,q}(\mathbb{P}^n,\mathbb{C}) = 0 \quad \text{for } p \neq q. \qquad \square$$

(8.9) Proposition. *Every holomorphic p-form on a compact Kähler manifold X is d-closed.*

Proof. If u is a holomorphic form of type $(p,0)$ then $d''u = 0$. Furthermore $d''^{\star}u$ is of type $(p,-1)$, hence $d''^{\star}u = 0$. Therefore $\Delta u = 2\Delta'' u = 0$, which implies $du = 0$. □

(8.10) Example. Consider the *Heisenberg group* $G \subset \mathrm{Gl}_3(\mathbb{C})$, defined as the subgroup of matrices

$$M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad (x,y,z) \in \mathbb{C}^3.$$

Let Γ be the discrete subgroup of matrices with entries $x,y,z \in \mathbb{Z}[\mathrm{i}]$ (or more generally in the ring of integers of an imaginary quadratic field). Then $X = G/\Gamma$ is a compact complex 3-fold, known as the *Iwasawa manifold*. The equality

$$M^{-1}dM = \begin{pmatrix} 0 & dx & dz - xdy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}$$

shows that dx, dy, $dz - xdy$ are left invariant holomorphic 1-forms on G. These forms induce holomorphic 1-forms on the quotient $X = G/\Gamma$. Since $dz - xdy$ is not d-closed, we see that X cannot be Kähler.

§8.3. Primitive Decomposition and Hard Lefschetz Theorem

We first introduce some standard notation. The *Betti numbers* and *Hodge numbers* of X are by definition

(8.11) $\quad b_k = \dim_{\mathbb{C}} H^k(X,\mathbb{C}), \qquad h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X,\mathbb{C}).$

Thanks to Hodge decomposition, these numbers satisfy the relations

(8.12) $$b_k = \sum_{p+q=k} h^{p,q}, \qquad h^{q,p} = h^{p,q}.$$

As a consequence, the Betti numbers b_{2k+1} of a compact Kähler manifold are even. Note that the Serre duality theorem gives the additional relation $h^{p,q} = h^{n-p,n-q}$, which holds as soon as X is compact. The existence of primitive decomposition implies other interesting specific features of the cohomology algebra of compact Kähler manifolds.

(8.13) Lemma. *If $u = \sum_{r\ge(k-n)_+} L^r u_r$ is the primitive decomposition of a harmonic k-form u, then all components u_r are harmonic.*

Proof. Since $[\Delta, L] = 0$, we get $0 = \Delta u = \sum_r L^r \Delta u_r$, hence $\Delta u_r = 0$ by uniqueness. □

Let us denote by $\operatorname{Prim}\mathcal{H}^k(X,\mathbb{C}) = \bigoplus_{p+q=k}\operatorname{Prim}\mathcal{H}^{p,q}(X,\mathbb{C})$ the spaces of primitive harmonic k-forms and let $b_{k,\mathrm{prim}}$, $h^{p,q}_{\mathrm{prim}}$ be their respective dimensions. Lemma 8.13 yields

(8.14) $$\mathcal{H}^{p,q}(X,\mathbb{C}) = \bigoplus_{r\ge(p+q-n)_+} L^r \operatorname{Prim}\mathcal{H}^{p-r,q-r}(X,\mathbb{C}),$$

(8.15) $$h^{p,q} = \sum_{r\ge(p+q-n)_+} h^{p-r,q-r}_{\mathrm{prim}}.$$

Formula (8.15) can be rewritten

(8.15′) $$\begin{cases} \text{if } \ p+q\le n, \quad h^{p,q} = h^{p,q}_{\mathrm{prim}} + h^{p-1,q-1}_{\mathrm{prim}} + \cdots \\ \text{if } \ p+q\ge n, \quad h^{p,q} = h^{n-q,n-p}_{\mathrm{prim}} + h^{n-q-1,n-p-1}_{\mathrm{prim}} + \cdots . \end{cases}$$

(8.16) Corollary. *The Hodge and Betti numbers satisfy the inequalities*
a) *if* $k = p + q \le n$, *then* $h^{p,q} \ge h^{p-1,q-1}$, $b_k \ge b_{k-2}$,
b) *if* $k = p + q \ge n$, *then* $h^{p,q} \ge h^{p+1,q+1}$, $b_k \ge b_{k+2}$. □

Another important result of Hodge theory (which is in fact a direct consequence of Cor. 5.17) is the

(8.17) Hard Lefschetz theorem. *The mappings*

$$\begin{aligned} L^{n-k} &: \quad H^k(X,\mathbb{C}) \longrightarrow H^{2n-k}(X,\mathbb{C}), & k \le n,\\ L^{n-p-q} &: \quad H^{p,q}(X,\mathbb{C}) \longrightarrow H^{n-q,n-p}(X,\mathbb{C}), & p+q \le n, \end{aligned}$$

are isomorphisms. □

§9. Jacobian and Albanese Varieties

§9.1. Description of the Picard Group

An important application of Hodge theory is a description of the Picard group $H^1(X, \mathcal{O}^\star)$ of a compact Kähler manifold. We assume here that X is connected. The exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\star \to 1$ gives

$$\begin{aligned} (9.1)\qquad 0 \longrightarrow & H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}) \longrightarrow H^1(X,\mathcal{O}^\star)\\ \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}) \end{aligned}$$

because the map $\exp(2\pi i \bullet) : H^0(X,\mathcal{O}) = \mathbb{C} \longrightarrow H^0(X,\mathcal{O}^\star) = \mathbb{C}^\star$ is onto. We have $H^1(X,\mathcal{O}) \simeq H^{0,1}(X,\mathbb{C})$ by (V-11.6). The dimension of this group is called the *irregularity of* X and is usually denoted

$$(9.2)\qquad q = q(X) = h^{0,1} = h^{1,0}.$$

Therefore we have $b_1 = 2q$ and

$$(9.3)\qquad H^1(X,\mathcal{O}) \simeq \mathbb{C}^q, \qquad H^0(X,\Omega^1_X) = H^{1,0}(X,\mathbb{C}) \simeq \mathbb{C}^q.$$

(9.4) Lemma. *The image of* $H^1(X,\mathbb{Z})$ *in* $H^1(X,\mathcal{O})$ *is a lattice.*

Proof. Consider the morphisms

$$H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathbb{R}) \longrightarrow H^1(X,\mathbb{C}) \longrightarrow H^1(X,\mathcal{O})$$

induced by the inclusions $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C} \subset \mathcal{O}$. Since the Čech cohomology groups with values in $\mathbb{Z}$, $\mathbb{R}$ can be computed by finite acyclic coverings, we see that $H^1(X,\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module and that the image of $H^1(X,\mathbb{Z})$ in $H^1(X,\mathbb{R})$ is a lattice. It is enough to check that the map $H^1(X,\mathbb{R}) \longrightarrow H^1(X,\mathcal{O})$ is an isomorphism. However, the commutative diagram

$$\begin{array}{ccccccccc}
0\longrightarrow \mathbb{C}\longrightarrow & \mathcal{E}^0 & \xrightarrow{d} & \mathcal{E}^1 & \xrightarrow{d} & \mathcal{E}^2 & \longrightarrow\cdots \\
\downarrow \quad & \downarrow & & \downarrow & & \downarrow & \\
0\longrightarrow \mathcal{O}\longrightarrow & \mathcal{E}^{0,0} & \xrightarrow{d''} & \mathcal{E}^{0,1} & \xrightarrow{d''} & \mathcal{E}^{0,2} & \longrightarrow\cdots
\end{array}$$

shows that the map $H^1(X,\mathbb{R}) \longrightarrow H^1(X,\mathcal{O})$ corresponds in De Rham and Dolbeault cohomologies to the composite mapping

$$H^1_{DR}(X,\mathbb{R}) \subset H^1_{DR}(X,\mathbb{C}) \longrightarrow H^{0,1}(X,\mathbb{C}).$$

Since $H^{1,0}(X,\mathbb{C})$ and $H^{0,1}(X,\mathbb{C})$ are complex conjugate subspaces in $H^1_{DR}(X,\mathbb{C})$, we conclude that $H^1_{DR}(X,\mathbb{R}) \longrightarrow H^{0,1}(X,\mathbb{C})$ is an isomorphism. □

As a consequence of this lemma, $H^1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2q}$. The q-dimensional complex torus

(9.5) $\quad \mathrm{Jac}(X) = H^1(X,\mathcal{O})/H^1(X,\mathbb{Z})$

is called the *Jacobian variety of* X and is isomorphic to the subgroup of $H^1(X,\mathcal{O}^\star)$ corresponding to line bundles of zero first Chern class. On the other hand, the kernel of

$$H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}) = H^{0,2}(X,\mathbb{C})$$

which consists of integral cohomology classes of type $(1,1)$, is equal to the image of c_1 in $H^2(X,\mathbb{Z})$. This subgroup is called the *Neron-Severi group* of X, and is denoted $NS(X)$. The exact sequence (9.1) yields

(9.6) $\quad 0 \longrightarrow \mathrm{Jac}(X) \longrightarrow H^1(X,\mathcal{O}^\star) \xrightarrow{c_1} NS(X) \longrightarrow 0.$

The Picard group $H^1(X,\mathcal{O}^\star)$ is thus an extension of the complex torus $\mathrm{Jac}(X)$ by the finitely generated $\mathbb{Z}$-module $NS(X)$.

(9.7) Corollary. *The Picard group of $\mathbb{P}^n$ is $H^1(\mathbb{P}^n,\mathcal{O}^\star) \simeq \mathbb{Z}$, and every line bundle over $\mathbb{P}^n$ is isomorphic to one of the line bundles $\mathcal{O}(k)$, $k \in \mathbb{Z}$.*

Proof. We have $H^k(\mathbb{P}^n,\mathcal{O}) = H^{0,k}(\mathbb{P}^n,\mathbb{C}) = 0$ for $k \geq 1$ by Appl. 8.8, thus $\mathrm{Jac}(\mathbb{P}^n) = 0$ and $NS(\mathbb{P}^n) = H^2(\mathbb{P}^n,\mathbb{Z}) \simeq \mathbb{Z}$. Moreover, $c_1\big(\mathcal{O}(1)\big)$ is a generator of $H^2(\mathbb{P}^n,\mathbb{Z})$ in virtue of Th. V-15.10. □

§9.2. Albanese Variety

A proof similar to that of Lemma 9.4 shows that the image of $H^{2n-1}(X,\mathbb{Z})$ in $H^{n-1,n}(X,\mathbb{C})$ via the composite map

(9.8) $\quad H^{2n-1}(X,\mathbb{Z}) \to H^{2n-1}(X,\mathbb{R}) \to H^{2n-1}(X,\mathbb{C}) \to H^{n-1,n}(X,\mathbb{C})$

is a lattice. The q-dimensional complex torus

$$(9.9)\quad \mathrm{Alb}(X) = H^{n-1,n}(X,\mathbb{C})/\operatorname{Im} H^{2n-1}(X,\mathbb{Z})$$

is called the *Albanese variety of* X. We first give a slightly different description of $\mathrm{Alb}(X)$, based on the Serre duality isomorphism

$$H^{n-1,n}(X,\mathbb{C}) \simeq \big(H^{1,0}(X,\mathbb{C})\big)^\star \simeq \big(H^0(X,\Omega^1_X)\big)^\star.$$

(9.10) Lemma. *For any compact oriented differentiable manifold M with $\dim_{\mathbb{R}} M = m$, there is a natural isomorphism*

$$H_1(M,\mathbb{Z}) \to H^{m-1}(M,\mathbb{Z})$$

where $H_1(M,\mathbb{Z})$ is the first homology group of M, that is, the abelianization of $\pi_1(M)$.

Proof. This is a well known consequence of Poincaré duality, see e.g. (Spanier 1966). We will content ourselves with a description of the morphism. Fix a base point $a \in M$. Every homotopy class $[\gamma] \in \pi_1(M,a)$ can be represented by as a composition of closed loops diffeomorphic to S^1. Let γ be such a loop. As every oriented vector bundle over S^1 is trivial, the normal bundle to γ is trivial. Hence $\gamma(S^1)$ has a neighborhood U diffeomorphic to $S^1 \times \mathbb{R}^{m-1}$, and there is a diffeomorphism $\varphi : S^1 \times \mathbb{R}^{m-1} \to U$ with $\varphi_{\restriction S^1\times\{0\}} = \gamma$. Let $\{\delta_0\} \in H^{m-1}_c(\mathbb{R}^{m-1},\mathbb{Z})$ be the fundamental class represented by the Dirac measure $\delta_0 \in \mathcal{D}'_0(\mathbb{R}^{m-1})$ in De Rham cohomology. Then the cartesian product $1 \times \{\delta_0\} \in H^{m-1}_c(S^1 \times \mathbb{R}^{m-1},\mathbb{Z})$ is represented by the current $[S^1] \otimes \{\delta_0\} \in \mathcal{D}'_1(S^1 \times \mathbb{R}^{m-1})$ and the current of integration over γ is precisely the direct image current

$$I_\gamma := \varphi_\star([S^1] \otimes \delta_0) = (\varphi^{-1})^\star([S^1] \otimes \delta_0).$$

Its cohomology class $\{I_\gamma\} \in H^{m-1}_c(U,\mathbb{R})$ is thus the image of the class $(\varphi^{-1})^\star\big(1 \times \{\delta_0\}\big) \in H^{m-1}_c(U,\mathbb{Z})$. Therefore, we have obtained a well defined morphism

$$\pi_1(M,a) \longrightarrow H^{m-1}_c(U,\mathbb{Z}) \longrightarrow H^{m-1}(M,\mathbb{Z}), \qquad [\gamma] \longmapsto (\varphi^{-1})^\star\big(1 \times \{\delta_0\}\big)$$

and the image of $[\gamma]$ in $H^{m-1}(M,\mathbb{R})$ is the De Rham cohomology class of the integration current I_γ. □

Thanks to Lemma 9.10, we can reformulate the definition of the Albanese variety as

$$(9.11)\quad \mathrm{Alb}(X) = \big(H^0(X,\Omega^1_X)\big)^\star/\operatorname{Im} H_1(X,\mathbb{Z})$$

where $H_1(X,\mathbb{Z})$ is mapped to $\big(H^0(X,\Omega^1_X)\big)^\star$ by

$$[\gamma] \longmapsto \widetilde{I}_\gamma = \Big(u \mapsto \int_\gamma u\Big).$$

Observe that the integral only depends on the homotopy class $[\gamma]$ because all holomorphic 1-forms u on X are closed by Prop. 8.9.

We are going to show that there exists a canonical holomorphic map $\alpha : X \to \mathrm{Alb}(X)$. Let a be a base point in X. For any $x \in X$, we select a path ξ from a to x and associate to x the linear form in $\big(H^0(X,\Omega_X^1)\big)^\star$ defined by $\widetilde{I}_\xi$. By construction the class of this linear form mod $\mathrm{Im}\, H_1(X,\mathbb{Z})$ does not depend on ξ, since $\widetilde{I}_{\xi'^{-1}\xi}$ is in the image of $H_1(X,\mathbb{Z})$ for any other path ξ'. It is thus legitimate to define the *Albanese map* as

$$(9.12)\quad \alpha : X \longrightarrow \mathrm{Alb}(X), \qquad x \longmapsto \Big(u \mapsto \int_a^x u\Big) \ \text{ mod } \ \mathrm{Im}\ H_1(X,\mathbb{Z}).$$

Of course, if we fix a basis $(u_1,\ldots,u_q)$ of $H^0(X,\Omega_X^1)$, the Albanese map can be seen in coordinates as the map

$$(9.13)\quad \alpha : X \longrightarrow \mathbb{C}^q/\Lambda, \qquad x \longmapsto \Big(\int_a^x u_1,\ldots,\int_a^x u_q\Big) \ \text{ mod } \ \Lambda,$$

where $\Lambda \subset \mathbb{C}^q$ is the *group of periods* of $(u_1,\ldots,u_q)$:

$$(9.13')\quad \Lambda = \Big\{\Big(\int_\gamma u_1,\ldots,\int_\gamma u_q\Big)\ ;\ [\gamma]\in\pi_1(X,a)\Big\}.$$

It is then clear that α is a holomorphic map. With the original definition (9.9) of the Albanese variety, it is not difficult to see that α is the map given by

$$(9.14)\quad \alpha : X \longrightarrow \mathrm{Alb}(X), \qquad x \longmapsto \{I_\xi^{n-1,n}\} \ \text{ mod } \ H^{2n-1}(X,\mathbb{Z}),$$

where $\{I_\xi^{n-1,n}\} \in H^{n-1,n}(X,\mathbb{C})$ denotes the $(n-1,n)$-component of the De Rham cohomology class $\{I_\xi\}$.

§10. Complex Curves

We show here how Hodge theory can be used to derive quickly the basic properties of compact manifolds of complex dimension 1 (also called *complex curves* or *Riemann surfaces*). Let X be such a curve. We shall always assume in this section that X is compact and connected. Since every positive $(1,1)$-form on a curve defines a Kähler metric, the results of § 8 and § 9 can be applied.

§10.1. Riemann-Roch Formula

Denoting $g = h^1(X,\mathcal{O})$, we find

$$\text{(10.1)}\quad H^1(X,\mathcal{O})\simeq\mathbb{C}^g,\qquad H^0(X,\Omega^1_X)\simeq\mathbb{C}^g,$$
$$\text{(10.2)}\quad H^0(X,\mathbb{Z})=\mathbb{Z},\qquad H^1(X,\mathbb{Z})=\mathbb{Z}^{2g},\qquad H^2(X,\mathbb{Z})=\mathbb{Z}.$$

The classification of oriented topological surfaces shows that X is homeomorphic to a sphere with g handles (= torus with g holes), but this property will not be used in the sequel. The number g is called the *genus* of X.

Any divisor on X can be written $\Delta=\sum m_ja_j$ where (a_j) is a finite sequence of points and $m_j\in\mathbb{Z}$. Let E be a line bundle over X. We shall identify E and the associated locally free sheaf $\mathcal{O}(E)$. According to V-13.2, we denote by $E(\Delta)$ the sheaf of germs of meromorphic sections f of E such that $\operatorname{div} f+\Delta\geq 0$, i.e. which have a pole of order $\leq m_j$ at a_j if $m_j>0$, and which have a zero of order $\geq|m_j|$ at a_j if $m_j<0$. Clearly

$$\text{(10.3)}\quad E(\Delta)=E\otimes\mathcal{O}(\Delta),\qquad \mathcal{O}(\Delta+\Delta')=\mathcal{O}(\Delta)\otimes\mathcal{O}(\Delta').$$

For any point $a\in X$ and any integer $m>0$, there is an exact sequence

$$0\longrightarrow E\longrightarrow E(m[a])\longrightarrow\mathcal{S}\longrightarrow 0$$

where $\mathcal{S}=E(m[a])/E$ is a sheaf with only one non zero stalk $\mathcal{S}_a$ isomorphic to $\mathbb{C}^m$. Indeed, if z is a holomorphic coordinate near a, the stalk $\mathcal{S}_a$ corresponds to the polar parts $\sum_{-m\leq k<0}c_kz^k$ in the power series expansions of germs of meromorphic sections at point a. We get an exact sequence

$$H^0\big(X,E(m[a])\big)\longrightarrow\mathbb{C}^m\longrightarrow H^1(X,E).$$

When m is chosen larger than $\dim H^1(X,E)$, we see that $E(m[a])$ has a non zero section and conclude:

(10.4) Theorem. *Let a be a given point on a curve. Then every line bundle E has non zero meromorphic sections f with a pole at a and no other poles.*

If Δ is the divisor of a meromorphic section f of E, we have $E\simeq\mathcal{O}(\Delta)$, so the map

$$\operatorname{Div}(X)\longrightarrow H^1(X,\mathcal{O}^\star),\qquad \Delta\longmapsto\mathcal{O}(\Delta)$$

is onto (cf. (V-13.8)). On the other hand, Div is clearly a soft sheaf, thus $H^1(X,\operatorname{Div})=0$. The long cohomology sequence associated to the exact sequence $1\to\mathcal{O}^\star\to\mathcal{M}^\star\to\operatorname{Div}\to 0$ implies:

(10.5) Corollary. *On any complex curve, one has $H^1(X,\mathcal{M}^\star)=0$ and there is an exact sequence*

$$0\longrightarrow\mathbb{C}^\star\longrightarrow\mathcal{M}^\star(X)\longrightarrow\operatorname{Div}(X)\longrightarrow H^1(X,\mathcal{O}^\star)\longrightarrow 0.$$

The first Chern class $c_1(E)\in H^2(X,\mathbb{Z})$ can be interpreted as an integer. This integer is called the *degree* of E. If $E\simeq\mathcal{O}(\Delta)$ with $\Delta=\sum m_ja_j$, formula V-13.6 shows that the image of $c_1(E)$ in $H^2(X,\mathbb{R})$ is the De Rham cohomology class of the associated current $[\Delta] = \sum m_j \delta_{a_j}$, hence

$$(10.6)\quad c_1(E) = \int_X [\Delta] = \sum m_j.$$

If $\sum m_j a_j$ is the divisor of a meromorphic function, we have $\sum m_j = 0$ because the associated bundle $E = \mathcal{O}(\sum m_j a_j)$ is trivial.

(10.7) Theorem. *Let E be a line bundle on a complex curve X. Then*

a) $H^0(X,E) = 0$ *if* $c_1(E) < 0$ *or if* $c_1(E) = 0$ *and E is non trivial*;

b) *For every positive $(1,1)$-form ω on X with $\int_X \omega = 1$, E has a hermitian metric such that $\frac{\mathrm{i}}{2\pi}\Theta(E) = c_1(E)\,\omega$. In particular, E has a metric of positive (resp. negative) curvature if and only if $c_1(E) > 0$ (resp. if and only if $c_1(E) < 0$).*

Proof. a) If E has a non zero holomorphic section f, then its degree is $c_1(E) = \int_X \operatorname{div} f \geq 0$. In fact, we even have $c_1(E) > 0$ unless f does not vanish, in which case E is trivial.

b) Select an arbitrary hermitian metric h on E. Then $c_1(E)\,\omega - \frac{\mathrm{i}}{2\pi}\Theta_h(E)$ is a real $(1,1)$-form cohomologous to zero (the integral over X is zero), so Lemma 8.6 c) implies

$$c_1(E)\,\omega - \frac{\mathrm{i}}{2\pi}\Theta_h(E) = \mathrm{i}d'd''\varphi$$

for some real function $\varphi \in C^\infty(X,\mathbb{R})$. If we replace the initial metric of E by $h' = h\,e^{-\varphi}$, we get a metric of constant curvature $c_1(E)\,\omega$. □

(10.8) Riemann-Roch formula. *Let E be a holomorphic line bundle and let $h^q(E) = \dim H^q(X,E)$. Then*

$$h^0(E) - h^1(E) = c_1(E) - g + 1.$$

Moreover $h^1(E) = h^0(K\otimes E^\star)$, where $K = \Omega^1_X$ is the canonical line bundle of X.

Proof. We claim that for every line bundle F and every divisor Δ we have the equality

$$(10.9)\quad h^0\big(F(\Delta)\big) - h^1\big(F(\Delta)\big) = h^0(F) - h^1(F) + \int_X [\Delta].$$

If we write $E = \mathcal{O}(\Delta)$ and apply the above equality with $F = \mathcal{O}$, the Riemann-Roch formula results from (10.6), (10.9) and from the equalities

$$h^0(\mathcal{O}) = \dim H^0(X,\mathcal{O}) = 1, \qquad h^1(\mathcal{O}) = \dim H^1(X,\mathcal{O}) = g.$$

However, (10.9) need only be proved when $\Delta \geq 0$: otherwise we are reduced to this case by writing $\Delta = \Delta_1 - \Delta_2$ with $\Delta_1, \Delta_2 \geq 0$ and by applying the result to the pairs (F, Δ_1) and $\big(F(\Delta), \Delta_2\big)$. If $\Delta = \sum m_j a_j \geq 0$, there is an exact sequence

$$0 \longrightarrow F \longrightarrow F(\Delta) \longrightarrow \mathcal{S} \longrightarrow 0$$

where $\mathcal{S}_{a_j} \simeq \mathbb{C}^{m_j}$ and the other stalks are zero. Let $m = \sum m_j = \int_X [\Delta]$. The sheaf $\mathcal{S}$ is acyclic, because its support $\{a_j\}$ is of dimension 0. Hence there is an exact sequence

$$0 \longrightarrow H^0(F) \longrightarrow H^0\big(F(\Delta)\big) \longrightarrow \mathbb{C}^m \longrightarrow H^1(F) \longrightarrow H^1\big(F(\Delta)\big) \longrightarrow 0$$

and (10.9) follows. The equality $h^1(E) = h^0(K \otimes E^\star)$ is a consequence of the Serre duality theorem

$$\big(H^{0,1}(X,E)\big)^\star \simeq H^{1,0}(X, E^\star), \quad \text{i.e.} \quad \big(H^1(X,E)\big)^\star \simeq H^0(X, K \otimes E^\star). \ \square$$

(10.10) Corollary (Hurwitz' formula). $c_1(K) = 2g - 2$.

Proof. Apply Riemann-Roch to $E = K$ and observe that

$$(10.11) \qquad \begin{aligned} h^0(K) &= \dim H^0(X, \Omega^1_X) = g \\ h^1(K) &= \dim H^1(X, \Omega^1_X) = h^{1,1} = b_2 = 1 \end{aligned}$$

(10.12) Corollary. *For every $a \in X$ and every $m \in \mathbb{Z}$*

$$h^0\big(K(-m[a])\big) = h^1\big(\mathcal{O}(m[a])\big) = h^0\big(\mathcal{O}(m[a])\big) - m + g - 1.$$

§10.2. Jacobian of a Curve

By the Neron-Severi sequence (9.6), there is an exact sequence

$$(10.13) \quad 0 \longrightarrow \mathrm{Jac}(X) \longrightarrow H^1(X, \mathcal{O}^\star) \xrightarrow{c_1} \mathbb{Z} \longrightarrow 0,$$

where the Jacobian $\mathrm{Jac}(X)$ is a g-dimensional torus. Choose a base point $a \in X$. For every point $x \in X$, the line bundle $\mathcal{O}([x] - [a])$ has zero first Chern class, so we have a well-defined map

$$(10.14) \quad \Phi_a : X \longrightarrow \mathrm{Jac}(X), \qquad x \longmapsto \mathcal{O}([x] - [a]).$$

Observe that the Jacobian $\mathrm{Jac}(X)$ of a curve coincides by definition with the Albanese variety $\mathrm{Alb}(X)$.

(10.15) Lemma. *The above map Φ_a coincides with the Albanese map $\alpha : X \longrightarrow \mathrm{Alb}(X)$ defined in* (9.12).

Proof. By holomorphic continuation, it is enough to prove that $\Phi_a(x) = \alpha(x)$ when x is near a. Let z be a complex coordinate and let $D' \subset\subset D$ be open disks centered at a. Relatively to the covering

$$U_1 = D, \qquad U_2 = X \setminus \overline{D'},$$

the line bundle $\mathcal{O}([x]-[a])$ is defined by the Čech cocycle $c \in C^1(\mathcal{U}, \mathcal{O}^\star)$ such that

$$c_{12}(z) = \frac{z-x}{z-a} \quad \text{on} \quad U_{12} = D \setminus \overline{D'}.$$

On the other hand, we compute $\alpha(x)$ by Formula (9.14). The path integral current $I_{[a,x]} \in \mathcal{D}'_1(X)$ is equal to 0 on U_2. Lemma I-2.10 implies $d''(dz/2\pi\mathrm{i}z) = \delta_0\, d\overline{z} \wedge dz/2i = \delta_0$ according to the usual identification of distributions and currents of degree 0, thus

$$I^{0,1}_{[a,x]} = d''\Big(\frac{dz}{2\pi iz} \star I^{0,1}_{[a,x]}\Big) \quad \text{on} \quad U_1.$$

Therefore $\{I^{0,1}_{[a,x]}\} \in H^{0,1}(X, \mathbb{C})$ is equal to the Čech cohomology class $\{c'\}$ in $H^1(X, \mathcal{O})$ represented by the cocycle

$$c'_{12}(z) = \frac{dw}{2\pi iw} \star I^{0,1}_{[a,x]} = \frac{1}{2\pi i}\int_a^x \frac{dw}{w-z} = \frac{1}{2\pi \mathrm{i}} \log \frac{z-x}{z-a} \quad \text{on} \quad U_{12}$$

and we have $c = \exp(2\pi \mathrm{i} c')$ in $H^1(X, \mathcal{O}^\star)$. □

The nature of Φ_a depends on the value of the genus g. A careful examination of Φ_a will enable us to determine all curves of genus 0 and 1.

(10.16) Theorem. *The following properties are equivalent:*

a) $g = 0$;

b) *X has a meromorphic function f having only one simple pole p;*

c) *X is biholomorphic to $\mathbb{P}^1$.*

Proof. c) $\Longrightarrow$ a) is clear.

a) $\Longrightarrow$ b). Since $g = 0$, we have $\mathrm{Jac}(X) = 0$. If $p, p' \in X$ are distinct points, the bundle $\mathcal{O}([p'] - [p])$ has zero first Chern class, therefore it is trivial and there exists a meromorphic function f with $\operatorname{div} f = [p'] - [p]$. In particular p is the only pole of f, and this pole is simple.

b) $\Longrightarrow$ c). We may consider f as a map $X \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. For every value $w \in \mathbb{C}$, the function $f - w$ must have exactly one simple zero $x \in X$ because $\int_X \operatorname{div}(f - w) = 0$ and p is a simple pole. Therefore $f : X \to \mathbb{P}^1$ is bijective and X is biholomorphic to $\mathbb{P}^1$. □

(10.17) Theorem. *The map Φ_a is always injective for $g \geq 1$.*

a) *If $g = 1$, Φ_a is a biholomorphism. In particular every curve of genus* 1 *is biholomorphic to a complex torus $\mathbb{C}/\Gamma$.*

b) *If $g \geq 2$, Φ_a is an embedding.*

Proof. If Φ_a is not injective, there exist points $x_1 \neq x_2$ such that $\mathcal{O}([x_1]-[x_2])$ is trivial; then there is a meromorphic function f such that $\operatorname{div} f = [x_1]-[x_2]$ and Th. 10.16 implies that $g = 0$.

When $g = 1$, Φ_a is an injective map $X \longrightarrow \mathrm{Jac}(X) \simeq \mathbb{C}/\Gamma$, thus Φ_a is open. It follows that $\Phi_a(X)$ is a compact open subset of $\mathbb{C}/\Gamma$, so $\Phi_a(X) = \mathbb{C}/\Gamma$ and Φ_a is a biholomorphism of X onto $\mathbb{C}/\Gamma$.

In order to prove that Φ_a is an embedding when $g \geq 2$, it is sufficient to show that the holomorphic 1-forms $u_1, \ldots, u_g$ do not all vanish at a given point $x \in X$. In fact, X has no non constant meromorphic function having only a simple pole at x, thus $h^0\big(\mathcal{O}([x])\big) = 1$ and Cor. 10.12 implies

$$h^0\big(K(-[x])\big) = g - 1 < h^0(K) = g.$$

Hence K has a section u which does not vanish at x. □

§10.3. Weierstrass Points of a Curve

We want to study how many meromorphic functions have a unique pole of multiplicity $\leq m$ at a given point $a \in X$, i.e. we want to compute $h^0\big(\mathcal{O}(m[a])\big)$. As we shall see soon, these numbers may depend on a only if m is small. We have $c_1\big(K(-m[a])\big) = 2g - 2 - m$, so the degree is < 0 and $h^0\big(K(-m[a])\big) = 0$ for $m \geq 2g - 1$ by 10.7 a). Cor. 10.12 implies

$$(10.18)\quad h^0\big(\mathcal{O}(m[a])\big) = m - g + 1 \quad \text{for} \quad m \geq 2g - 1.$$

It remains to compute $h^0\big(K(-m[a])\big)$ for $0 \leq m \leq 2g-2$ and $g \geq 1$. Let $u_1, \ldots, u_g$ be a basis of $H^0(X, K)$ and let z be a complex coordinate centered at a. Any germ $u \in \mathcal{O}(K)_a$ can be written $u = U(z)\,dz$ with $U(z) = \sum_{m \in \mathbb{N}} \frac{1}{m!} U^{(m)}(a) z^m\,dz$. The unique non zero stalk of the quotient sheaf $\mathcal{O}\big(K(-m[a])\big)/\mathcal{O}\big(K(-(m+1)[a])\big)$ is canonically isomorphic to K_a^{m+1} via the map $u \mapsto U^{(m)}(a)(dz)^{m+1}$, which is independant of the choice of z. Hence $\bigwedge^g \big(\mathcal{O}(K)/\mathcal{O}(K - g[a])\big) \simeq K_a^{1+2+\cdots+g}$ and the *Wronskian*

$$(10.19)\quad W(u_1, \ldots, u_g) = \begin{vmatrix} U_1(z) & \ldots & U_g(z) \\ U_1'(z) & \ldots & U_g'(z) \\ \vdots & & \vdots \\ U_1^{(g-1)}(z) & \ldots & U_g^{(g-1)}(z) \end{vmatrix} dz^{1+2+\cdots+g}$$

defines a global section $W(u_1, \ldots, u_g) \in H^0(X, K^{g(g+1)/2})$. At the given point a, we can find linear combinations $\widetilde{u}_1, \ldots, \widetilde{u}_g$ of $u_1, \ldots, u_g$ such that

$$\widetilde{u}_j(z) = \big(z^{s_j-1} + \mathrm{O}(z^{s_j})\big)dz, \qquad s_1 < \ldots < s_g.$$

We know that not all sections of K vanish at a and that $c_1(K) = 2g-2$, thus $s_1 = 1$ and $s_g \le 2g-1$. We have $W(\widetilde{u}_1, \ldots, \widetilde{u}_g) \sim W(z^{s_1-1}dz, \ldots, z^{s_g-1}dz)$ at point a, and an easy induction on $\sum s_j$ combined with differentiation in z yields

$$W(z^{s_1-1}dz, \ldots, z^{s_g-1}dz) = C\, z^{s_1+\ldots+s_g-g(g+1)/2}\, dz^{g(g+1)/2}$$

for some positive integer constant C. In particular, $W(u_1, \ldots, u_g)$ is not identically zero and vanishes at a with multiplicity

$$\text{(10.20)} \qquad \mu_a = s_1 + \ldots + s_g - g(g+1)/2 > 0$$

unless $s_1 = 1$, $s_2 = 2$, …, $s_g = g$. Now, we have

$$h^0\big(K(-m[a])\big) = \mathrm{card}\{j\,;\, s_j > m\} = g - \mathrm{card}\{j\,;\, s_j \le m\}$$

and Cor. 10.12 gives

$$\text{(10.21)} \qquad h^0\big(\mathcal{O}(m[a])\big) = m + 1 - \mathrm{card}\{j\,;\, s_j \le m\}.$$

If a is not a zero of $W(u_1, \ldots, u_g)$, we find

$$\text{(10.22)} \qquad \begin{cases} h^0\big(\mathcal{O}(m[a])\big) = 1 & \text{for } \ m \le g, \\ h^0\big(\mathcal{O}(m[a])\big) = m + 1 - g & \text{for } \ m > g. \end{cases}$$

The zeroes of $W(u_1, \ldots, u_g)$ are called the *Weierstrass points* of X, and the associated *Weierstrass sequence* is the sequence $w_m = h^0\big(\mathcal{O}(m[a])\big)$, $m \in \mathbb{N}$. We have $w_{m-1} \le w_m \le w_{m-1}+1$ and $s_1 < \ldots < s_g$ are precisely the integers $m \ge 1$ such that $w_m = w_{m-1}$. The numbers $s_j \in \{1, 2, \ldots, 2g-1\}$ are called the *gaps* and μ_a the *weight* of the Weierstrass point a. Since $W(u_1, \ldots, u_g)$ is a section of $K^{g(g+1)/2}$, Hurwitz' formula implies

$$\text{(10.23)} \qquad \sum_{a \in X} \mu_a = c_1(K^{g(g+1)/2}) = g(g+1)(g-1).$$

In particular, a curve of genus g has at most $g(g+1)(g-1)$ Weierstrass points.

§11. Hodge-Frölicher Spectral Sequence

Let X be a *compact* complex n-dimensional manifold. We consider the double complex $K^{p,q} = C^\infty(X, \Lambda^{p,q}T_X^\star)$, $d = d' + d''$. The Hodge-Frölicher spectral sequence is by definition the spectral sequence associated to this double complex (cf. IV-11.9). It starts with

$$(11.1)\quad E_1^{p,q} = H^{p,q}(X, \mathbb{C})$$

and the limit term $E_\infty^{p,q}$ is the graded module associated to a filtration of the De Rham cohomology group $H^k(X, \mathbb{C})$, $k = p + q$. In particular, if the numbers b_k and $h^{p,q}$ are still defined as in (8.11), we have

$$(11.2)\quad b_k = \sum_{p+q=k} \dim E_\infty^{p,q} \le \sum_{p+q=k} \dim E_1^{p,q} = \sum_{p+q=k} h^{p,q}.$$

The equality is equivalent to the degeneration of the spectral sequence at $E_1^\bullet$. As a consequence, the Hodge-Frölicher spectral sequence of a compact Kähler manifold degenerates in $E_1^\bullet$.

(11.3) Theorem and Definition. *The existence of an isomorphism*

$$H^k_{\mathrm{DR}}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$$

is equivalent to the degeneration of the Hodge-Frölicher spectral sequence at E_1. In this case, the isomorphism is canonically defined and we say that X admits a Hodge decomposition. □

In general, interesting informations can be deduced from the spectral sequence. Theorem IV-11.8 shows in particular that

$$(11.4)\quad b_1 \ge \dim E_2^{1,0} + (\dim E_2^{0,1} - \dim E_2^{2,0})_+.$$

However, $E_2^{1,0}$ is the central cohomology group in the sequence

$$d_1 = d' : E_1^{0,0} \longrightarrow E_1^{1,0} \longrightarrow E_1^{2,0},$$

and as $E_1^{0,0}$ is the space of holomorphic functions on X, the first map d_1 is zero (by the maximum principle, holomorphic functions are constant on each connected component of X). Hence $\dim E_2^{1,0} \ge h^{1,0} - h^{2,0}$. Similarly, $E_2^{0,1}$ is the kernel of a map $E_1^{0,1} \to E_1^{1,1}$, thus $\dim E_2^{0,1} \ge h^{0,1} - h^{1,1}$. By (11.4) we obtain

$$(11.5)\quad b_1 \ge (h^{1,0} - h^{2,0})_+ + (h^{0,1} - h^{1,1} - h^{2,0})_+.$$

Another interesting relation concerns the topological Euler-Poincaré characteristic

$$\chi_{\rm top}(X) = b_0 - b_1 + \ldots - b_{2n-1} + b_{2n}.$$

We need the following simple lemma.

(11.6) Lemma. *Let $(C^\bullet, d)$ a bounded complex of finite dimensional vector spaces over some field. Then, the Euler characteristic*

$$\chi(C^\bullet) = \sum (-1)^q \dim C^q$$

is equal to the Euler characteristic $\chi\big(H^\bullet(C^\bullet)\big)$ of the cohomology module.

Proof. Set

$$c_q = \dim C^q, \qquad z_q = \dim Z^q(C^\bullet), \qquad b_q = \dim B^q(C^\bullet), \qquad h_q = \dim H^q(C^\bullet).$$

Then

$$c_q = z_q + b_{q+1}, \qquad h_q = z_q - b_q.$$

Therefore we find

$$\sum (-1)^q\, c_q = \sum (-1)^q\, z_q - \sum (-1)^q\, b_q = \sum (-1)^q\, h_q. \qquad \square$$

In particular, if the term $E_r^\bullet$ of the spectral sequence of a filtered complex $K^\bullet$ is a bounded and finite dimensional complex, we have

$$\chi(E_r^\bullet) = \chi(E_{r+1}^\bullet) = \ldots = \chi(E_\infty^\bullet) = \chi\big(H^\bullet(K^\bullet)\big)$$

because $E_{r+1}^\bullet = H^\bullet(E_r^\bullet)$ and $\dim E_\infty^l = \dim H^l(K^\bullet)$. In the Hodge-Frölicher spectral sequence, we have $\dim E_1^l = \sum_{p+q=l} h^{p,q}$, hence:

(11.7) Theorem. *For any compact complex manifold X, one has*

$$\chi_{\rm top}(X) = \sum_{0\le k\le 2n} (-1)^k b_k = \sum_{0\le p,q\le n} (-1)^{p+q} h^{p,q}.$$

§12. Effect of a Modification on Hodge Decomposition

In this section, we show that the existence of a Hodge decomposition on a compact complex manifold X is guaranteed as soon as there exists such a decomposition on a modification $\widetilde X$ of X (see II-??.?? for the Definition). This leads us to extend Hodge theory to a class of manifolds which are non necessarily Kähler, the so called Fujiki class $(\mathcal{C})$ of manifolds bimeromorphic to Kähler manifolds.

§12.1. Sheaf Cohomology Reinterpretation of $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$

We first give a description of $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ in terms of the hypercohomology of a suitable complex of sheaves. This interpretation, combined with the analogue of the Hodge-Frölicher spectral sequence, will imply in particular that $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ is always finite dimensional when X is compact. Let us denote by $\mathcal{E}^{p,q}$ the sheaf of germs of C^∞ forms of bidegree (p,q), and by Ω^p the sheaf of germs of holomorphic p-forms on X. For a fixed bidegree (p_0,q_0), we let $k_0=p_0+q_0$ and we introduce a complex of sheaves $(\mathcal{L}^\bullet_{p_0,q_0},\delta)$, also denoted $\mathcal{L}^\bullet$ for simplicity, such that

$$\begin{aligned}\mathcal{L}^k &= \bigoplus_{p+q=k,p<p_0,q<q_0} \mathcal{E}^{p,q} \quad \text{for} \quad k\le k_0-2,\\ \mathcal{L}^{k-1} &= \bigoplus_{p+q=k,p\ge p_0,q\ge q_0} \mathcal{E}^{p,q} \quad \text{for} \quad k\ge k_0.\end{aligned}$$

The differential δ^k on $\mathcal{L}^k$ is chosen equal to the exterior derivative d for $k\ne k_0-2$ (in the case $k\le k_0-3$, we neglect the components which fall outside $\mathcal{L}^{k+1}$), and we set

$$\delta^{k_0-2}=d'd'':\mathcal{L}^{k_0-2}=\mathcal{E}^{p_0-1,q_0-1}\longrightarrow\mathcal{L}^{k_0-1}=\mathcal{E}^{p_0,q_0}.$$

We find in particular $H^{p_0,q_0}_{\mathrm{BC}}(X,\mathbb{C}) = H^{k_0-1}\big(\mathcal{L}^\bullet(X)\big)$. We observe that $\mathcal{L}^\bullet$ has subcomplexes $(\mathcal{S}'^\bullet,d')$ and $(\mathcal{S}''^\bullet,d'')$ defined by

$$\begin{aligned}&\mathcal{S}'^k=\Omega^k_X \quad \text{for} \quad 0\le k\le p_0-1, \qquad \mathcal{S}'^k=0 \quad \text{otherwise},\\ &\mathcal{S}''^k=\overline{\Omega^k_X} \quad \text{for} \quad 0\le k\le q_0-1, \qquad \mathcal{S}''^k=0 \quad \text{otherwise}.\end{aligned}$$

If $p_0=0$ or $q_0=0$ we set instead $\mathcal{S}'^0=\mathbb{C}$ or $\mathcal{S}''^0=\mathbb{C}$, and take the other components to be zero. Finally, we let $\mathcal{S}^\bullet=\mathcal{S}'^\bullet+\mathcal{S}''^\bullet\subset\mathcal{L}^\bullet$ (the sum is direct except for $\mathcal{S}^0$); we denote by $\mathcal{M}^\bullet$ the sheaf complex defined in the same way as $\mathcal{L}^\bullet$, except that the sheaves $\mathcal{E}^{p,q}$ are replaced by the sheaves of currents $\mathcal{D}'_{n-p,n-q}$.

(12.1) Lemma. *The inclusions $\mathcal{S}^\bullet\subset\mathcal{L}^\bullet\subset\mathcal{M}^\bullet$ induce isomorphisms*

$$\mathcal{H}^k(\mathcal{S}^\bullet)\simeq\mathcal{H}^k(\mathcal{L}^\bullet)\simeq\mathcal{H}^k(\mathcal{M}^\bullet),$$

and these cohomology sheaves vanish for $k\ne 0,p_0-1,q_0-1$.

Proof. We will prove the result only for the inclusion $\mathcal{S}^\bullet\subset\mathcal{L}^\bullet$, the other case $\mathcal{S}^\bullet\subset\mathcal{M}^\bullet$ is identical. Let us denote by $\mathcal{Z}^{p,q}$ the sheaf of d''-closed differential forms of bidegree (p,q). We consider the filtration

$$F_p(\mathcal{L}^k)=\mathcal{L}^k\cap\bigoplus_{r\ge p}\mathcal{E}^{r,\bullet}$$

and the induced filtration on $\mathcal{S}^\bullet$. In the case of $\mathcal{L}^\bullet$, the first spectral sequence has the following terms $E_0^\bullet$ and $E_1^\bullet$:

$$\begin{array}{lll}
\text{if } p < p_0 & E_0^{p,\bullet} : & 0 \longrightarrow \mathcal{E}^{p,0} \xrightarrow{d''} \mathcal{E}^{p,1} \longrightarrow \cdots \xrightarrow{d''} \mathcal{E}^{p,q_0-1} \longrightarrow 0,\\
\text{if } p \ge p_0 & E_0^{p,\bullet} : & 0 \longrightarrow \mathcal{E}^{p,q_0} \xrightarrow{d''} \mathcal{E}^{p,q_0+1} \longrightarrow \cdots \longrightarrow \mathcal{E}^{p,q} \xrightarrow{d''} \cdots,\\
\text{if } p < p_0 & \multicolumn{2}{l}{E_1^{p,0} = \Omega_X^p,\ \ E_1^{p,q_0-1} \simeq \mathcal{Z}^{p,q_0},\ \ E_1^{p,q} = 0 \quad \text{for} \quad q \ne 0, q_0-1,}\\
\text{if } p \ge p_0 & \multicolumn{2}{l}{E_1^{p,q_0-1} = \mathcal{Z}^{p,q_0},\ \ E_1^{p,q} = 0 \quad \text{for} \quad q \ne q_0-1.}
\end{array}$$

The isomorphism in the third line is given by

$$\mathcal{E}^{p,q_0-1}/d''\mathcal{E}^{p,q_0-2} \simeq d''\mathcal{E}^{p,q_0-1} \simeq \mathcal{Z}^{p,q_0}.$$

The map $d_1 : E_1^{p_0-1,q_0-1} \longrightarrow E_1^{p_0,q_0-1}$ is induced by $d'd''$ acting on $\mathcal{E}^{p_0-1,q_0-1}$, but thanks to the previous identification, this map becomes d' acting on $\mathcal{Z}^{p_0-1,q_0}$. Hence $E_1^\bullet$ consists of two sequences

$$E_1^{\bullet,0} : 0 \longrightarrow \Omega_X^0 \xrightarrow{d'} \Omega_X^1 \longrightarrow \cdots \xrightarrow{d'} \Omega_X^{p_0-1} \longrightarrow 0,$$
$$E_1^{\bullet,q_0-1} : 0 \longrightarrow \mathcal{Z}^{0,q_0} \xrightarrow{d'} \mathcal{Z}^{1,q_0} \longrightarrow \cdots \longrightarrow \mathcal{Z}^{p,q_0} \xrightarrow{d'} \cdots ;$$

if these sequences overlap ($q_0 = 1$), only the second one has to be considered. The term $E_1^\bullet$ in the spectral sequence of $\mathcal{S}^\bullet$ has the same first line, but the second is reduced to $E_1^{0,q_0-1} = \overline{d\Omega_X^{q_0-2}}$ (resp. $= \mathbb{C}$ for $q_0 = 1$). Thanks to Lemma 12.2 below, we see that the two spectral sequences coincide in $E_2^\bullet$, with at most three non zero terms:

$$E_2^{0,0} = \mathbb{C}, \qquad E_2^{p_0-1,0} = d\Omega_X^{p_0-2} \ \text{ for } \ p_0 \ge 2, \qquad E_2^{0,q_0-1} = \overline{d\Omega_X^{q_0-2}} \ \text{ for } \ q_0 \ge 2.$$

Hence $\mathcal{H}^k(\mathcal{S}^\bullet) \simeq \mathcal{H}^k(\mathcal{L}^\bullet)$ and these sheaves vanish for $k \ne 0, p_0-1, q_0-1$. □

(12.2) Lemma. *The complex of sheaves*

$$0 \longrightarrow \mathcal{Z}^{0,q_0} \xrightarrow{d'} \mathcal{Z}^{1,q_0} \longrightarrow \cdots \longrightarrow \mathcal{Z}^{p,q_0} \xrightarrow{d'} \cdots$$

is a resolution of $\overline{d\Omega_X^{q_0-1}}$ *for* $q_0 \ge 1$, *resp. of* $\mathbb{C}$ *for* $q_0 = 0$.

Proof. Embed $\mathcal{Z}^{\bullet,q_0}$ in the double complex

$$K^{p,q} = \mathcal{E}^{p,q} \quad \text{for} \quad q < q_0, \qquad K^{p,q} = 0 \quad \text{for} \quad q \ge q_0.$$

For the first fitration of $K^\bullet$, we find

$$E_1^{p,q_0-1} = \mathcal{Z}^{p,q_0}, \qquad E_1^{p,q} = 0 \quad \text{for} \quad q \ne q_0 - 1$$

The second fitration gives $\widetilde{E}_1^{p,q} = 0$ for $q \ge 1$ and

$$\widetilde{E}_1^{p,0} = H^0(K^{\bullet,p}) = \begin{cases} \overline{H^0(\mathcal{E}^{p,\bullet})} = \overline{\Omega_X^p} & \text{for } p \le q_0 - 1\\ 0 & \text{for } p \ge q_0, \end{cases}$$

thus the cohomology of $\mathcal{Z}^{\bullet,q_0}$ coincides with that of $(\overline{\Omega^p_X}, d)_{0\le p<q_0}$. □

Lemma IV-11.10 and formula (IV-12.9) imply

$$\begin{aligned}(12.3)\quad \mathbb{H}^k(X,\mathcal{S}^\bullet) &\simeq \mathbb{H}^k(X,\mathcal{L}^\bullet) \simeq \mathbb{H}^k(X,\mathcal{M}^\bullet)\\ &\simeq H^k\big(\mathcal{L}^\bullet(X)\big) \simeq H^k\big(\mathcal{M}^\bullet(X)\big)\end{aligned}$$

because the sheaves $\mathcal{L}^k$ and $\mathcal{M}^k$ are soft. In particular, the group $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ can be computed either by means of C^∞ differential forms or by means of currents. This property also holds for the De Rham or Dolbeault groups $H^k(X,\mathbb{C})$, $H^{p,q}(X,\mathbb{C})$, as was already remarked in §IV-6. Another important consequence of (12.3) is:

(12.4) Theorem. *If X is compact, then* $\dim H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) < +\infty$.

Proof. We show more generally that the hypercohomology groups $\mathbb{H}^k(X,\mathcal{S}^\bullet)$ are finite dimensional. As there is an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{S}'^\bullet \oplus \mathcal{S}''^\bullet \longrightarrow \mathcal{S}^\bullet \longrightarrow 0$$

and a corresponding long exact sequence for hypercohomology groups, it is enough to show that the groups $\mathbb{H}^k(X,\mathcal{S}'^\bullet)$ are finite dimensional. This property is proved for $\mathcal{S}'^\bullet = \mathcal{S}'^\bullet_{p_0}$ by induction on p_0. For $p_0 = 0$ or 1, the complex $\mathcal{S}'^\bullet$ is reduced to its term $\mathcal{S}'^0$, thus

$$\mathbb{H}^k(X,\mathcal{S}^\bullet) = H^k(X,\mathcal{S}'^0) = \begin{cases} H^k(X,\mathbb{C}) & \text{for } p_0 = 0\\ H^k(X,\mathcal{O}) & \text{for } p_0 = 1\end{cases}$$

and this groups are finite dimensional. In general, we have an exact sequence

$$0 \longrightarrow \Omega^{p_0}_X \longrightarrow \mathcal{S}^\bullet_{p_0+1} \longrightarrow \mathcal{S}^\bullet_{p_0} \longrightarrow 0$$

where $\Omega^{p_0}_X$ denotes the subcomplex of $\mathcal{S}^\bullet_{p_0+1}$ reduced to one term in degree p_0. As

$$\mathbb{H}^k(X,\Omega^{p_0}_X) = H^{k-p_0}(X,\Omega^{p_0}_X) = H^{p_0,k-p_0}(X,\mathbb{C})$$

is finite dimensional, the Theorem follows. □

(12.5) Definition. *We say that a compact manifold admits a strong Hodge decomposition if the natural maps*

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}),\qquad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k(X,\mathbb{C})$$

are isomorphisms.

This implies of course that there are natural isomorphisms

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}), \qquad H^{q,p}(X,\mathbb{C}) \simeq \overline{H^{p,q}(X,\mathbb{C})}$$

and that the Hodge-Frölicher spectral sequence degenerates in $E_1^\bullet$. It follows from § 8 that all Kähler manifolds admit a strong Hodge decomposition.

§12.2. Direct and Inverse Image Morphisms

Let $F : X \longrightarrow Y$ be a holomorphic map between complex analytic manifolds of respective dimensions n, m, and $r = n - m$. We have pull-back morphisms

$$(12.6)\quad \begin{aligned} F^\star &: \ H^k(Y,\mathbb{C}) \longrightarrow H^k(X,\mathbb{C}),\\ F^\star &: \ H^{p,q}(Y,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}),\\ F^\star &: \ H^{p,q}_{\mathrm{BC}}(Y,\mathbb{C}) \longrightarrow H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}),\end{aligned}$$

commuting with the natural morphisms (8.2), (8.3).

Assume now that F is *proper*. Theorem I-1.14 shows that one can define direct image morphisms

$$F_\star \ : \ \mathcal{D}'_k(X) \longrightarrow \mathcal{D}'_k(Y), \qquad F_\star \ : \ \mathcal{D}'_{p,q}(X) \longrightarrow \mathcal{D}'_{p,q}(Y),$$

commuting with d', d''. To $F_\star$ therefore correspond cohomology morphisms

$$(12.7)\quad \begin{aligned} F_\star &: \ H^k(X,\mathbb{C}) \longrightarrow H^{k-2r}(Y,\mathbb{C}),\\ F_\star &: \ H^{p,q}(X,\mathbb{C}) \longrightarrow H^{p-r,q-r}(Y,\mathbb{C}),\\ F_\star &: \ H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p-r,q-r}_{\mathrm{BC}}(Y,\mathbb{C}),\end{aligned}$$

which commute also with (8.2), (8.3). In addition, I-1.14 c) implies the *adjunction formula*

$$(12.8)\quad F_\star(\alpha \smile F^\star\beta) = (F_\star\alpha) \smile \beta$$

whenever α is a cohomology class (of any of the three above types) on X, and β a cohomology class (of the same type) on Y.

§12.3. Modifications and the Fujiki Class $(\mathcal{C})$

Recall that a modification of a compact manifold X is a holomorphic map $\mu : \widetilde{X} \longrightarrow X$ such that

i) $\widetilde{X}$ is a compact complex manifold of the same dimension as X ;

ii) there exists an analytic subset $S \subset X$ of codimension ≥ 1 such that $\mu : \widetilde{X} \setminus \mu^{-1}(S) \longrightarrow X \setminus S$ is a biholomorphism.

(12.9) Theorem. *If $\widetilde{X}$ admits a strong Hodge decomposition, and if $\mu : \widetilde{X} \longrightarrow X$ is a modification, then X also admits a strong Hodge decomposition.*

Proof. We first observe that $\mu_\star\mu^\star f = f$ for every smooth form f on Y. In fact, this property is equivalent to the equality

$$\int_Y (\mu_\star\mu^\star f)\wedge g = \int_X \mu^\star(f\wedge g) = \int_Y f\wedge g$$

for every smooth form g on Y, and this equality is clear because μ is a biholomorphism outside sets of Lebesgue measure 0. Consequently, the induced cohomology morphism $\mu_\star$ is surjective and $\mu^\star$ is injective (but these maps need not be isomorphisms). Now, we have commutative diagrams

$$\begin{array}{ccc} H^{p,q}_{\mathrm{BC}}(\widetilde X,\mathbb{C}) & \longrightarrow & H^{p,q}(\widetilde X,\mathbb{C}), \\ \mu_\star\big\downarrow\big\uparrow\mu^\star & & \mu_\star\big\downarrow\big\uparrow\mu^\star \\ H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) & \longrightarrow & H^{p,q}(X,\mathbb{C}), \end{array} \qquad \begin{array}{ccc} \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(\widetilde X,\mathbb{C}) & \longrightarrow & H^k(\widetilde X,\mathbb{C}) \\ \mu_\star\big\downarrow\big\uparrow\mu^\star & & \mu_\star\big\downarrow\big\uparrow\mu^\star \\ \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) & \longrightarrow & H^k(X,\mathbb{C}) \end{array}$$

with either upward or downward vertical arrows. Hence the surjectivity or injectivity of the top horizontal arrows implies that of the bottom horizontal arrows. □

(12.10) Definition. *A manifold X is said to be in the Fujiki class $(\mathcal{C})$ if X admits a Kähler modification $\widetilde X$.*

By Th. 12.9, Hodge decomposition still holds for a manifold in the class $(\mathcal{C})$. We will see later that there exist non-Kähler manifolds in $(\mathcal{C})$, for example all non projective Moišezon manifolds (cf. §?.?). The class $(\mathcal{C})$ has been first introduced in (Fujiki 1978).

Chapter VII
Positive Vector Bundles and Vanishing Theorems

In this chapter, we prove a few vanishing theorems for hermitian vector bundles over *compact* complex manifolds. All these theorems are based on an a priori inequality for (p,q)-forms with values in a vector bundle, known as the Bochner-Kodaira-Nakano inequality. This inequality naturally leads to several positivity notions for the curvature of a vector bundle (Kodaira 1953, 1954), (Griffiths 1969) and (Nakano 1955, 1973). The corresponding algebraic notion of ampleness introduced by (Grothendieck 196?) and (Hartshorne 1966) is also discussed. The differential geometric techniques yield optimal vanishing results in the case of line bundles (Kodaira-Akizuki-Nakano and Girbau vanishing theorems) and also some partial results in the case of vector bundles (Nakano vanishing theorem). As an illustration, we compute the cohomology groups $H^{p,q}(\mathbb{P}^n, \mathcal{O}(k))$; much finer results will be obtained in chapters 8–11. Finally, the Kodaira vanishing theorem is combined with a blowing-up technique in order to establish the projective embedding theorem for manifolds admitting a Hodge metric.

1. Bochner-Kodaira-Nakano Identity

Let (X,ω) be a hermitian manifold, $\dim_{\mathbb{C}} X = n$, and let E be a hermitian holomorphic vector bundle of rank r over X. We denote by $D = D' + D''$ its Chern connection (or D_E if we want to specify the bundle), and by $\delta = \delta' + \delta''$ the formal adjoint operator of D. The operators L, Λ of chapter 6 are extended to vector valued forms in $\Lambda^{p,q}T^\star X \otimes E$ by taking their tensor product with Id_E. The following result extends the commutation relations of chapter 6 to the case of bundle valued operators.

(1.1) Theorem. *If τ is the operator of type $(1,0)$ defined by $\tau = [\Lambda, d'\omega]$ on $C^\infty_{\bullet,\bullet}(X,E)$, then*

a) $[\delta''_E, L] = \mathrm{i}(D'_E + \tau)$,
b) $[\delta'_E, L] = -\mathrm{i}(D''_E + \overline{\tau})$,
c) $[\Lambda, D''_E] = -\mathrm{i}(\delta'_E + \tau^\star)$,
d) $[\Lambda, D'_E] = \mathrm{i}(\delta''_E + \overline{\tau}^\star)$.

Proof. Fix a point x_0 in X and a coordinate system $z = (z_1, \ldots, z_n)$ centered at x_0. Then Prop. V-12.?? shows the existence of a normal coordinate frame

(e_λ) at x_0. Given any section $s = \sum_\lambda \sigma_\lambda \otimes e_\lambda \in C^\infty_{p,q}(X,E)$, it is easy to check that the operators D_E, δ''_E, ... have Taylor expansions of the type

$$D_E s = \sum_\lambda d\sigma_\lambda \otimes e_\lambda + O(|z|), \qquad \delta''_E s = \sum_\lambda \delta''\sigma_\lambda \otimes e_\lambda + O(|z|), \ \ldots$$

in terms of the scalar valued operators d, δ, ... Here the terms $O(|z|)$ depend on the curvature coefficients of E. The proof of Th. 1.1 is then reduced to the case of scalar valued operators, which is granted by Th. VI-10.1. □

The Bochner-Kodaira-Nakano identity expresses the antiholomorphic Laplace operator $\Delta'' = D''\delta'' + \delta''D''$ acting on $C^\infty_{\bullet,\bullet}(X,E)$ in terms of its conjugate operator $\Delta' = D'\delta' + \delta'D'$, plus some extra terms involving the curvature of E and the torsion of the metric ω (in case ω is not Kähler). Such identities appear frequently in riemannian geometry (Weitzenböck formula).

(1.2) Theorem. $\Delta'' = \Delta' + [\mathrm{i}\Theta(E), \Lambda] + [D', \tau^\star] - [D'', \overline{\tau}^\star]$.

Proof. Equality 1.1 d) yields $\delta'' = -\mathrm{i}[\Lambda, D'] - \overline{\tau}^\star$, hence

$$\Delta'' = [D'', \delta''] = -\mathrm{i}[D'', [\Lambda, D']] - [D'', \overline{\tau}^\star].$$

The Jacobi identity VI-10.2 and relation 1.1 c) imply

$$\big[D'', [\Lambda, D']\big] = \big[\Lambda, [D', D'']\big] + \big[D', [D'', \Lambda]\big] = [\Lambda, \Theta(E)] + \mathrm{i}[D', \delta' + \tau^\star],$$

taking into account that $[D', D''] = D^2 = \Theta(E)$. Theorem 1.2 follows. □

(1.3) Corollary (Akizuki-Nakano 1955). *If ω is Kähler, then*

$$\Delta'' = \Delta' + [\mathrm{i}\Theta(E), \Lambda].$$

In the latter case, $\Delta'' - \Delta'$ is therefore an operator of order 0 closely related to the curvature of E. When ω is not Kähler, Formula 1.2 is not really satisfactory, because it involves the first order operators $[D', \tau^\star]$ and $[D'', \overline{\tau}^\star]$. In fact, these operators can be combined with Δ' in order to yield a new positive self-adjoint operator Δ'_τ.

(1.4) Theorem (Demailly 1985). *The operator $\Delta'_\tau = [D' + \tau, \delta' + \tau^\star]$ is a positive and formally self-adjoint operator with the same principal part as the Laplace operator Δ'. Moreover*

$$\Delta'' = \Delta'_\tau + [\mathrm{i}\Theta(E), \Lambda] + T_\omega,$$

where T_ω is an operator of order 0 depending only on the torsion of the hermitian metric ω :

$$T_\omega = \Big[\Lambda, \big[\Lambda, \frac{\mathrm{i}}{2} d'd''\omega\big]\Big] - \big[d'\omega, (d'\omega)^\star\big].$$

Proof. The first assertion is clear, because the equality $(D' + \tau)^\star = \delta' + \tau^\star$ implies the self-adjointness of Δ'_τ and

$$\langle\!\langle \Delta'_\tau u, u\rangle\!\rangle = \|D'u + \tau u\|^2 + \|\delta' u + \tau^\star u\|^2 \geq 0$$

for any compactly supported form $u \in C^\infty_{p,q}(X, E)$. In order to prove the formula, we need two lemmas.

(1.5) Lemma. a) $\quad [L, \tau] = 3d'\omega, \qquad$ b) $\quad [\Lambda, \tau] = -2\mathrm{i}\overline{\tau}^\star.$

Proof. a) Since $[L, d'\omega] = 0$, the Jacobi identity implies

$$[L, \tau] = \big[L, [\Lambda, d'\omega]\big] = -\big[d'\omega, [L, \Lambda]\big] = 3d'\omega,$$

taking into account Cor. VI-10.4 and the fact that $d'\omega$ is of degree 3.

b) By 1.1 a) we have $\tau = -\mathrm{i}[\delta'', L] - D'$, hence

$$[\Lambda, \tau] = -i\big[\Lambda, [\delta'', L]\big] - [\Lambda, D'] = -i\big(\big[\Lambda, [\delta'', L]\big] + \delta'' + \overline{\tau}^\star\big).$$

Using again VI-10.4 and the Jacobi identity, we get

$$\begin{aligned}\big[\Lambda, [\delta'', L]\big] &= -\big[L, [\Lambda, \delta'']\big] - \big[\delta'', [L, \Lambda]\big] \\ &= -\big[[d'', L], \Lambda\big]^\star - \delta'' = -[d''\omega, \Lambda]^\star - \delta'' = \overline{\tau}^\star - \delta''.\end{aligned}$$

A substitution in the previous equality gives $[\Lambda, \tau] = -2\mathrm{i}\overline{\tau}^\star$. □

(1.6) Lemma. *The following identities hold:*

a) $[D', \overline{\tau}^\star] = -[D', \delta''] = [\tau, \delta'']$,

b) $-[D'', \overline{\tau}^\star] = [\tau, \delta' + \tau^\star] + T_\omega$.

Proof. a) The Jacobi identity implies

$$-\big[D', [\Lambda, D']\big] + \big[D', [D', \Lambda]\big] + \big[\Lambda, [D', D']\big] = 0,$$

hence $-2\big[D', [\Lambda, D']\big] = 0$ and likewise $\big[\delta'', [\delta'', L]\big] = 0$. Assertion a) is now a consequence of 1.1 a) and d).

b) In order to verify b), we start from the equality $\overline{\tau}^\star = \frac{\mathrm{i}}{2}[\Lambda, \tau]$ provided by Lemma 1.5 b). It follows that

$$(1.7) \quad [D'', \overline{\tau}^\star] = \frac{\mathrm{i}}{2}\big[D'', [\Lambda, \tau]\big].$$

The Jacobi identity will now be used several times. One obtains

$$\begin{aligned}
&(1.8) && \big[D'', [\Lambda, \tau]\big] = \big[\Lambda, [\tau, D'']\big] + \big[\tau, [D'', \Lambda]\big]\ ;\\
&(1.9) && [\tau, D''] = [D'', \tau] = \big[D'', [\Lambda, d'\omega]\big] = \big[\Lambda, [d'\omega, D'']\big] + \big[d'\omega, [D'', \Lambda]\big]\\
& && \qquad = [\Lambda, d''d'\omega] + [d'\omega, A]
\end{aligned}$$

with $A = [D'', \Lambda] = \mathrm{i}(\delta' + \tau^\star)$. From (1.9) we deduce

$$(1.10)\quad \big[\Lambda, [\tau, D'']\big] = \big[\Lambda, [\Lambda, d''d'\omega]\big] + \big[\Lambda, [d'\omega, A]\big].$$

Let us compute now the second Lie bracket in the right hand side of (1.10:

$$\begin{aligned}
&(1.11) && \big[\Lambda, [d'\omega, A]\big] = \big[A, [\Lambda, d'\omega]\big] - \big[d'\omega, [A, \Lambda]\big] = [\tau, A] + \big[d'\omega, [\Lambda, A]\big]\ ;\\
&(1.12) && [\Lambda, A] = \mathrm{i}[\Lambda, \delta' + \tau^\star] = \mathrm{i}[D' + \tau, L]^\star.
\end{aligned}$$

Lemma 1.5 b) provides $[\tau, L] = -3d'\omega$, and it is clear that $[D', L] = d'\omega$. Equalities (1.12) and (1.11) yield therefore

$$\begin{aligned}
& && [\Lambda, A] = -2\mathrm{i}(d'\omega)^\star,\\
&(1.13) && \big[\Lambda, [d'\omega, A]\big] = \big[\tau, [D'', \Lambda]\big] - 2\mathrm{i}[d'\omega, (d'\omega)^\star].
\end{aligned}$$

Substituting (1.10) and (1.13) in (1.8) we get

$$\begin{aligned}
(1.14)\quad \big[D'', [\Lambda, \tau]\big] &= \big[\Lambda, [\Lambda, d''d'\omega]\big] + 2\big[\tau, [D'', \Lambda]\big] - 2\mathrm{i}\big[d'\omega, (d'\omega)^\star\big]\\
&= 2\mathrm{i}\big(T_\omega + [\tau, \delta' + \tau^\star]\big).
\end{aligned}$$

Formula b) is a consequence of (1.7) and (1.14). □

Theorem 1.4 follows now from Th. 1.2 if Formula 1.6 b) is rewritten

$$\Delta' + [D', \tau^\star] - [D'', \overline{\tau}^\star] = [D' + \tau, \delta' + \tau^\star] + T_\omega.$$

When ω is Kähler, then $\tau = T_\omega = 0$ and Lemma 1.6 a) shows that $[D', \delta''] = 0$. Together with the adjoint relation $[D'', \delta'] = 0$, this equality implies

$$(1.15)\quad \Delta = \Delta' + \Delta''.$$

When ω is not Kähler, Lemma 1.6 a) can be written $[D' + \tau, \delta''] = 0$ and we obtain more generally

$$[D + \tau, \delta + \tau^\star] = \big[(D' + \tau) + D'', (\delta' + \tau^\star) + \delta''\big] = \Delta'_\tau + \Delta''.$$

(1.16) Proposition. *Set $\Delta_\tau = [D + \tau, \delta + \tau^\star]$. Then $\Delta_\tau = \Delta'_\tau + \Delta''$.*

2. Basic a Priori Inequality

Let (X, ω) be a *compact* hermitian manifold, $\dim_{\mathbb{C}} X = n$, and E a hermitian holomorphic vector bundle over X. For any section $u \in C^\infty_{p,q}(X, E)$ we have $\langle\!\langle \Delta'' u, u\rangle\!\rangle = \|D''u\|^2 + \|\delta'' u\|^2$ and the similar formula for Δ'_τ gives $\langle\!\langle \Delta'_\tau u, u\rangle\!\rangle \geq 0$. Theorem 1.4 implies therefore

$$\text{(2.1)}\quad \|D''u\|^2 + \|\delta''u\|^2 \geq \int_X \big(\langle[\mathrm{i}\Theta(E), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle\big) dV.$$

This inequality is known as the *Bochner-Kodaira-Nakano* inequality. When u is Δ''-harmonic, we get in particular

$$\text{(2.2)}\quad \int_X \big(\langle[\mathrm{i}\Theta(E), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle\big) dV \leq 0.$$

These basic a priori estimates are the starting point of all vanishing theorems. Observe that $[\mathrm{i}\Theta(E), \Lambda] + T_\omega$ is a hermitian operator acting pointwise on $\Lambda^{p,q}T^\star X \otimes E$ (the hermitian property can be seen from the fact that this operator coincides with $\Delta'' - \Delta'_\tau$ on smooth sections). Using Hodge theory (Cor. VI-11.2), we get:

(2.3) Corollary. *If the hermitian operator* $[\mathrm{i}\Theta(E), \Lambda] + T_\omega$ *is positive definite on* $\Lambda^{p,q}T^\star X \otimes E$*, then* $H^{p,q}(X, E) = 0$. □

In some circumstances, one can improve Cor. 2.3 thanks to the following "analytic continuation lemma" due to (Aronszajn 1957):

(2.4) Lemma. *Let M be a connected C^∞-manifold, F a vector bundle over M, and P a second order elliptic differential operator acting on $C^\infty(M, F)$. Then any section $\alpha \in \ker P$ vanishing on a non-empty open subset of M vanishes identically on M.*

(2.5) Corollary. *Assume that X is compact and connected. If*

$$[\mathrm{i}\Theta(E), \Lambda] + T_\omega \in \mathrm{Herm}\big(\Lambda^{p,q}T^\star X \otimes E\big)$$

is semi-positive on X and positive definite in at least one point $x_0 \in X$, then $H^{p,q}(X, E) = 0$.

Proof. By (2.2) every Δ''-harmonic (p, q)-form u must vanish in the neighborhood of x_0 where $[\mathrm{i}\Theta(E), \Lambda] + T_\omega > 0$, thus $u \equiv 0$. Hodge theory implies $H^{p,q}(X, E) = 0$. □

3. Kodaira-Akizuki-Nakano Vanishing Theorem

The main goal of vanishing theorems is to find natural geometric or algebraic conditions on a bundle E that will ensure that some cohomology groups with values in E vanish. In the next three sections, we prove various vanishing theorems for cohomology groups of a hermitian *line bundle* E over a *compact* complex manifold X.

(3.1) Definition. *A hermitian holomorphic line bundle E on X is said to be positive (resp. negative) if the hermitian matrix $\big(c_{jk}(z)\big)$ of its Chern curvature form*

$$\mathrm{i}\Theta(E) = \mathrm{i} \sum_{1\le j,k\le n} c_{jk}(z)\, dz_j \wedge d\overline{z}_k$$

is positive (resp. negative) definite at every point $z \in X$.

Assume that X has a Kähler metric ω. Let

$$\gamma_1(x) \le \ldots \le \gamma_n(x)$$

be the eigenvalues of $\mathrm{i}\Theta(E)_x$ with respect to ω_x at each point $x \in X$, and let

$$\mathrm{i}\Theta(E)_x = \mathrm{i} \sum_{1\le j\le n} \gamma_j(x)\, \zeta_j \wedge \overline{\zeta}_j, \qquad \zeta_j \in T^\star_x X$$

be a diagonalization of $\mathrm{i}\Theta(E)_x$. By Prop. VI-8.3 we have

$$\begin{aligned} \langle[\mathrm{i}\Theta(E), \Lambda]u, u\rangle &= \sum_{J,K} \Big(\sum_{j\in J} \gamma_j + \sum_{j\in K} \gamma_j - \sum_{1\le j\le n} \gamma_j\Big)|u_{J,K}|^2 \\ &\ge (\gamma_1 + \ldots + \gamma_q - \gamma_{p+1} - \ldots - \gamma_n)|u|^2 \end{aligned} \tag{3.2}$$

for any form $u = \sum_{J,K} u_{J,K}\, \zeta_J \wedge \overline{\zeta}_K \in \Lambda^{p,q} T^\star X$.

(3.3) Akizuki-Nakano vanishing theorem (1954). *Let E be a holomorphic line bundle on X.*

a) *If E is positive, then* $H^{p,q}(X, E) = 0$ *for* $p + q \ge n + 1$.

b) *If E is negative, then* $H^{p,q}(X, E) = 0$ *for* $p + q \le n - 1$.

Proof. In case a), choose $\omega = \mathrm{i}\Theta(E)$ as a Kähler metric on X. Then we have $\gamma_j(x) = 1$ for all j and x, so that

$$\langle\!\langle[\mathrm{i}\Theta(E), \Lambda]u, u\rangle\!\rangle \ge (p + q - n)||u||^2$$

for any $u \in \Lambda^{p,q} T^\star X \otimes E$. Assertion a) follows now from Corollary 2.3. Property b) is proved similarly, by taking $\omega = -\mathrm{i}\Theta(E)$. One can also derive b) from a) by Serre duality (Theorem VI-11.3). □

When $p = 0$ or $p = n$, Th. 3.3 can be generalized to the case where $\mathrm{i}\Theta(E)$ degenerates at some points. We use here the standard notations

$$(3.4) \quad \Omega^p_X = \Lambda^p T^\star X, \qquad K_X = \Lambda^n T^\star X, \qquad n = \dim_{\mathbb{C}} X \ ;$$

K_X is called the *canonical line bundle* of X.

(3.5) Theorem (Grauert-Riemenschneider 1970). *Let (X, ω) be a compact and connected Kähler manifold and E a line bundle on X.*

a) *If $\mathrm{i}\Theta(E) \geq 0$ on X and $\mathrm{i}\Theta(E) > 0$ in at least one point $x_0 \in X$, then*
$$H^q(X, K_X \otimes E) = 0 \quad \text{for} \quad q \geq 1.$$

b) *If $\mathrm{i}\Theta(E) \leq 0$ on X and $\mathrm{i}\Theta(E) < 0$ in at least one point $x_0 \in X$, then*
$$H^q(X, E) = 0 \quad \text{for} \quad q \leq n - 1.$$

It will be proved in Volume II, by means of holomorphic Morse inequalities, that the Kähler assumption is in fact unnecessary. This improvement is a deep result first proved by (Siu 1984) with a different ad hoc method.

Proof. For $p = n$, formula (3.2) gives

$$(3.6) \quad \langle\!\langle [\mathrm{i}\Theta(E), \Lambda]u, u \rangle\!\rangle \geq (\gamma_1 + \ldots + \gamma_q)|u|^2$$

and a) follows from Cor. 2.5. Now b) is a consequence of a) by Serre duality. □

4. Girbau's Vanishing Theorem

Let E be a line bundle over a compact connected Kähler manifold (X, ω). Girbau's theorem deals with the (possibly everywhere) degenerate semi-positive case. We first state the corresponding generalization of Th. 4.5.

(4.1) Theorem. *If $\mathrm{i}\Theta(E)$ is semi-positive and has at least $n - s + 1$ positive eigenvalues at a point $x_0 \in X$ for some integer $s \in \{1, \ldots, n\}$, then*

$$H^q(X, K_X \otimes E) = 0 \quad \text{for} \quad q \geq s.$$

Proof. Apply 2.5 and inequality (3.6), and observe that $\gamma_q(x_0) > 0$ for all $q \geq s$. □

(4.2) Theorem (Girbau 1976). *If $\mathrm{i}\Theta(E)$ is semi-positive and has at least $n - s + 1$ positive eigenvalues at every point $x \in X$, then*

$$H^{p,q}(X, E) = 0 \quad \text{for} \quad p + q \geq n + s.$$

Proof. Let us consider on X the new Kähler metric

$$\omega_\varepsilon = \varepsilon\omega + \mathrm{i}\Theta(E), \qquad \varepsilon > 0,$$

and let $\mathrm{i}\Theta(E) = \mathrm{i}\sum \gamma_j\,\zeta_j \wedge \overline{\zeta}_j$ be a diagonalization of $\mathrm{i}\Theta(E)$ with respect to ω and with $\gamma_1 \le \ldots \le \gamma_n$. Then

$$\omega_\varepsilon = \mathrm{i}\sum\,(\varepsilon + \gamma_j)\,\zeta_j \wedge \overline{\zeta}_j.$$

The eigenvalues of $\mathrm{i}\Theta(E)$ with respect to ω_ε are given therefore by

$$(4.3)\quad \gamma_{j,\varepsilon} = \gamma_j/(\varepsilon + \gamma_j) \in [0,1[, \qquad 1 \le j \le n.$$

On the other hand, the hypothesis is equivalent to $\gamma_s > 0$ on X. For $j \ge s$ we have $\gamma_j \ge \gamma_s$, thus

$$(4.4)\quad \gamma_{j,\varepsilon} = \frac{1}{1+\varepsilon/\gamma_j} \ge \frac{1}{1+\varepsilon/\gamma_s} \ge 1 - \varepsilon/\gamma_s, \qquad s \le j \le n.$$

Let us denote the operators and inner products associated to ω_ε with ε as an index. Then inequality (3.2) combined with (4.4) implies

$$\begin{aligned}\langle[\mathrm{i}\Theta(E), \Lambda_\varepsilon]u, u\rangle_\varepsilon &\ge \Big(\big(q-s+1\big)\big)(1-\varepsilon/\gamma_s) - (n-p)\Big)|u|^2 \\ &= \big(p+q-n-s+1-(q-s+1)\varepsilon/\gamma_s\big)|u|^2.\end{aligned}$$

Theorem 4.2 follows now from Cor. 2.3 if we choose

$$\varepsilon < \frac{p+q-n-s+1}{q-s+1}\,\min_{x\in X}\gamma_s(x). \qquad \square$$

(4.5) Remark. The following example due to (Ramanujam 1972, 1974) shows that Girbau's result is no longer true for $p < n$ when $\mathrm{i}\Theta(E)$ is only assumed to have $n-s+1$ positive eigenvalues on a dense open set.

Let V be a hermitian vector space of dimension $n+1$ and X the manifold obtained from $P(V) \simeq \mathbb{P}^n$ by blowing-up one point a. The manifold X may be described as follows: if $P(V/\mathbb{C}a)$ is the projective space of lines ℓ containing a, then

$$X = \big\{(x,\ell) \in P(V) \times P(V/\mathbb{C}a)\ ;\ x \in \ell\big\}.$$

We have two natural projections

$$\begin{aligned}\pi_1 &: X \longrightarrow P(V) \simeq \mathbb{P}^n, \\ \pi_2 &: X \longrightarrow Y = P(V/\mathbb{C}a) \simeq \mathbb{P}^{n-1}.\end{aligned}$$

It is clear that the preimage $\pi_1^{-1}(x)$ is the single point $\big(x, \ell = (ax)\big)$ if $x \ne a$ and that $\pi_1^{-1}(a) = \{a\} \times Y \simeq \mathbb{P}^{n-1}$, therefore

$$\pi_1 \ : \ X \setminus (\{a\} \times Y) \longrightarrow P(V) \setminus \{a\}$$

is an isomorphism. On the other hand, π_2 is a locally trivial fiber bundle over Y with fiber $\pi_2^{-1}(\ell) = \ell \simeq \mathbb{P}^1$, in particular X is smooth and n-dimensional. Consider now the line bundle $E = \pi_1^\star \mathcal{O}(1)$ over X, with the hermitian metric induced by that of $\mathcal{O}(1)$. Then E is semi-positive and $\mathrm{i}\Theta(E)$ has n positive eigenvalues at every point of $X \setminus (\{a\} \times Y)$, hence the assumption of Th. 4.2 is satisfied on $X \setminus (\{a\} \times Y)$. However, we will see that

$$H^{p,p}(X, E) \neq 0, \qquad 0 \leq p \leq n-1,$$

in contradiction with the expected generalization of (4.2) when $2p \geq n+1$. Let $j : Y \simeq \{a\} \times Y \longrightarrow X$ be the inclusion. Then $\pi_1 \circ j \ : \ Y \to \{a\}$ and $\pi_2 \circ j = \mathrm{Id}_Y$; in particular $j^\star E = (\pi_1 \circ j)^\star \mathcal{O}(1)$ is the trivial bundle $Y \times \mathcal{O}(1)_a$. Consider now the composite morphism

$$\begin{array}{ccc} H^{p,p}(Y,\mathbb{C}) \otimes H^0\big(P(V), \mathcal{O}(1)\big) & \longrightarrow H^{p,p}(X,E) & \xrightarrow{\ j^\star\ } H^{p,p}(Y,\mathbb{C}) \otimes \mathcal{O}(1)_a \\ u \otimes s & \longmapsto \pi_2^\star u \otimes \pi_1^\star s, & \end{array}$$

given by $u \otimes s \longmapsto (\pi_2 \circ j)^\star u \otimes (\pi_1 \circ j)^\star s = u \otimes s(a)$; it is surjective and $H^{p,p}(Y,\mathbb{C}) \neq 0$ for $0 \leq p \leq n-1$, so we have $H^{p,p}(X,E) \neq 0$. $\square$

5. Vanishing Theorem for Partially Positive Line Bundles

Even in the case when the curvature form $\mathrm{i}\Theta(E)$ is not semi-positive, some cohomology groups of high tensor powers E^k still vanish under suitable assumptions. The prototype of such results is the following assertion, which can be seen as a consequence of the Andreotti-Grauert theorem (Andreotti-Grauert 1962), see IX-?.?; the special case where E is > 0 (that is, $s = 1$) is due to (Kodaira 1953) and (Serre 1956).

(5.1) Theorem. *Let F be a holomorphic vector bundle over a compact complex manifold X, s a positive integer and E a hermitian line bundle such that $\mathrm{i}\Theta(E)$ has at least $n-s+1$ positive eigenvalues at every point $x \in X$. Then there exists an integer $k_0 \geq 0$ such that*

$$H^q(X, E^k \otimes F) = 0 \qquad \text{for} \quad q \geq s \quad \text{and} \quad k \geq k_0.$$

Proof. The main idea is to construct a hermitian metric ω_ε on X in such a way that all negative eigenvalues of $\mathrm{i}\Theta(E)$ with respect to ω_ε will be of small absolute value. Let ω denote a fixed hermitian metric on X and let $\gamma_1 \leq \ldots \leq \gamma_n$ be the corresponding eigenvalues of $\mathrm{i}\Theta(E)$.

(5.2) Lemma. *Let $\psi \in C^\infty(\mathbb{R},\mathbb{R})$. If A is a hermitian $n\times n$ matrix with eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ and corresponding eigenvectors $v_1,\ldots,v_n$, we define $\psi[A]$ as the hermitian matrix with eigenvalues $\psi(\lambda_j)$ and eigenvectors v_j, $1\le j\le n$. Then the map $A \longmapsto \psi[A]$ is C^∞ on $\mathrm{Herm}(\mathbb{C}^n)$.*

Proof. Although the result is very well known, we give here a short proof. Without loss of generality, we may assume that ψ is compactly supported. Then we have

$$\psi[A] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{\psi}(t) e^{itA} dt$$

where $\widehat{\psi}$ is the rapidly decreasing Fourier transform of ψ. The equality $\int_0^t (t-u)^p u^q\,du = p!\,q!/(p+q+1)!$ and obvious power series developments yield

$$D_A(e^{itA})\cdot B = i\int_0^t e^{\mathrm{i}(t-u)A}\,B\,e^{iuA}du.$$

Since e^{iuA} is unitary, we get $\|D_A(e^{itA})\| \le |t|$. A differentiation under the integral sign and Leibniz' formula imply by induction on k the bound $\|D_A^k(e^{itA})\| \le |t|^k$. Hence $A\longmapsto \psi[A]$ is smooth. □

Let us consider now the positive numbers

$$t_0 = \inf_X \gamma_s > 0, \qquad M = \sup_X \max_j |\gamma_j| > 0.$$

We select a function $\psi_\varepsilon \in C^\infty(\mathbb{R},\mathbb{R})$ such that

$$\psi_\varepsilon(t) = t \ \text{ for } t\ge t_0,\quad \psi_\varepsilon(t)\ge t \ \text{ for } 0\le t\le t_0,\quad \psi_\varepsilon(t) = M/\varepsilon \ \text{ for } t\le 0.$$

By Lemma 5.2, $\omega_\varepsilon := \psi_\varepsilon[\mathrm{i}\Theta(E)]$ is a smooth hermitian metric on X. Let us write

$$\mathrm{i}\Theta(E) = \mathrm{i}\sum_{1\le j\le n} \gamma_j\,\zeta_j\wedge\overline{\zeta}_j,\qquad \omega_\varepsilon = \mathrm{i}\sum_{1\le j\le n} \psi_\varepsilon(\gamma_j)\,\zeta_j\wedge\overline{\zeta}_j$$

in an orthonormal basis $(\zeta_1,\ldots,\zeta_n)$ of $T^\star X$ for ω. The eigenvalues of $\mathrm{i}\Theta(E)$ with respect to ω_ε are given by $\gamma_{j,\varepsilon} = \gamma_j/\psi_\varepsilon(\gamma_j)$ and the construction of ψ_ε shows that $-\varepsilon \le \gamma_{j,\varepsilon}\le 1$, $1\le j\le n$, and $\gamma_{j,\varepsilon} = 1$ for $s\le j\le n$. Now, we have

$$H^q(X, E^k\otimes F) \simeq H^{n,q}(X, E^k\otimes G)$$

where $G = F\otimes K_X^\star$. Let e, $(g_\lambda)_{1\le\lambda\le r}$ and $(\zeta_j)_{1\le j\le n}$ denote orthonormal frames of E, G and $(T^\star X,\omega_\varepsilon)$ respectively. For

$$u = \sum_{|J|=q,\lambda} u_{J,\lambda}\,\zeta_1\wedge\ldots\wedge\zeta_n\wedge\overline{\zeta}_J\otimes e^k\otimes g_\lambda \in \Lambda^{n,q}T^\star X\otimes E^k\otimes G,$$

inequality (3.2) yields

$$\langle[\mathrm{i}\Theta(E),\Lambda_\varepsilon]u,u\rangle_\varepsilon=\sum_{J,\lambda}\Big(\sum_{j\in J}\gamma_{j,\varepsilon}\Big)|u_{J,\lambda}|^2\ge\big(q-s+1-(s-1)\varepsilon\big)|u|^2.$$

Choosing $\varepsilon=1/s$ and $q\ge s$, the right hand side becomes $\ge(1/s)|u|^2$. Since $\Theta(E^k\otimes G)=k\Theta(E)\otimes\mathrm{Id}_G+\Theta(G)$, there exists an integer k_0 such that

$$\big[\mathrm{i}\Theta(E^k\otimes G),\Lambda_\varepsilon\big]+T_{\omega_\varepsilon}\quad\text{acting on}\quad\Lambda^{n,q}T^\star X\otimes E^k\otimes G$$

is positive definite for $q\ge s$ and $k\ge k_0$. The proof is complete. □

6. Positivity Concepts for Vector Bundles

Let E be a hermitian holomorphic vector bundle of rank r over X, where $\dim_{\mathbb{C}}X=n$. Denote by $(e_1,\ldots,e_r)$ an orthonormal frame of E over a coordinate patch $\Omega\subset X$ with complex coordinates $(z_1,\ldots,z_n)$, and

$$\text{(6.1)}\quad \mathrm{i}\Theta(E)=\mathrm{i}\sum_{1\le j,k\le n,\ 1\le\lambda,\mu\le r}c_{jk\lambda\mu}\,dz_j\wedge d\overline{z}_k\otimes e_\lambda^\star\otimes e_\mu,\qquad \overline{c}_{jk\lambda\mu}=c_{kj\mu\lambda}$$

the Chern curvature tensor. To $\mathrm{i}\Theta(E)$ corresponds a natural hermitian form θ_E on $TX\otimes E$ defined by

$$\theta_E=\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}(dz_j\otimes e_\lambda^\star)\otimes(\overline{dz_k\otimes e_\mu^\star}),$$

and such that

$$\theta_E(u,u)=\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}(x)\,u_{j\lambda}\overline{u}_{k\mu},\qquad u\in T_xX\otimes E_x.(6.2)$$

(6.3) Definition (Nakano 1955). *E is said to be Nakano positive (resp. Nakano semi-negative) if θ_E is positive definite (resp. semi-negative) as a hermitian form on $TX\otimes E$, i.e. if for every $u\in TX\otimes E$, $u\ne0$, we have*

$$\theta_E(u,u)>0\quad(\textit{resp.}\le 0).$$

We write $>_{\mathrm{Nak}}$ (resp. $\le_{\mathrm{Nak}}$) for Nakano positivity (resp. semi-negativity).

(6.4) Definition (Griffiths 1969). *E is said to be Griffiths positive (resp. Griffiths semi-negative) if for all $\xi\in T_xX$, $\xi\ne0$ and $s\in E_x$, $s\ne0$ we have*

$$\theta_E(\xi\otimes s,\xi\otimes s)>0\quad(\textit{resp.}\le 0).$$

We write $>_{\mathrm{Grif}}$ (resp. $\le_{\mathrm{Grif}}$) for Griffiths positivity (resp. semi-negativity).

It is clear that Nakano positivity implies Griffiths positivity and that both concepts coincide if $r = 1$ (in the case of a line bundle, E is merely said to be positive). One can generalize further by introducing additional concepts of positivity which interpolate between Griffiths positivity and Nakano positivity.

(6.5) Definition. *Let T and E be complex vector spaces of dimensions n, r respectively, and let Θ be a hermitian form on $T \otimes E$.*

a) *A tensor $u \in T \otimes E$ is said to be of rank m if m is the smallest ≥ 0 integer such that u can be written*

$$u = \sum_{j=1}^{m} \xi_j \otimes s_j, \qquad \xi_j \in T, \ s_j \in E.$$

b) *Θ is said to be m-positive (resp. m-semi-negative) if $\Theta(u,u) > 0$ (resp. $\Theta(u,u) \leq 0$) for every tensor $u \in T \otimes E$ of rank $\leq m$, $u \neq 0$. In this case, we write*

$$\Theta >_m 0 \qquad (\textit{resp.}\ \ \Theta \leq_m 0).$$

We say that the bundle E is m-positive if $\theta_E >_m 0$. Griffiths positivity corresponds to $m = 1$ and Nakano positivity to $m \geq \min(n, r)$.

(6.6) Proposition. *A bundle E is Griffiths positive if and only if $E^\star$ is Griffiths negative.*

Proof. By (V-4.3′) we get $\mathrm{i}\Theta(E^\star) = -\mathrm{i}\Theta(E)^\dagger$, hence

$$\theta_{E^\star}(\xi_1 \otimes \overline{s}_2, \xi_2 \otimes \overline{s}_1) = -\theta_E(\xi_1 \otimes s_1, \xi_2 \otimes s_2), \qquad \forall \xi_1, \xi_2 \in TX,\ \forall s_1, s_2 \in E,$$

where $\overline{s}_j = \langle \bullet, s_j \rangle \in E^\star$. Proposition 6.6 follows immediately. □

It should be observed that the corresponding duality property for Nakano positive bundles is *not true.* In fact, using (6.1) we get

$$\mathrm{i}\Theta(E^\star) = -\mathrm{i} \sum_{j,k,\lambda,\mu} c_{jk\mu\lambda} dz_j \wedge d\overline{z}_k \otimes e_\lambda^{\star\star} \otimes e_\mu^\star,$$

$$\theta_{E^\star}(v, v) = - \sum_{j,k,\mu,\lambda} c_{jk\mu\lambda} v_{j\lambda} \overline{v}_{k\mu}, \tag{6.7}$$

for any $v = \sum v_{j\lambda}\,(\partial/\partial z_j) \otimes e_\lambda^\star \in TX \otimes E^\star$. The following example shows that Nakano positivity or negativity of θ_E and $\theta_{E^\star}$ are unrelated.

(6.8) Example. Let H be the rank n bundle over $\mathbb{P}^n$ defined in § V-15. For any $u = \sum u_{j\lambda}(\partial/\partial z_j) \otimes \widetilde{e}_\lambda \in TX \otimes H$, $v = \sum v_{j\lambda}(\partial/\partial z_j) \otimes \widetilde{e}_\lambda^\star \in TX \otimes H^\star$, $1 \leq j, \lambda \leq n$, formula (V-15.9) implies

$$(6.9)\qquad \begin{cases} \theta_H(u,u) = \sum u_{j\lambda}\overline{u}_{\lambda j} \\ \theta_{H^\star}(v,v) = \sum v_{jj}\overline{v}_{\lambda\lambda} = \big|\sum v_{jj}\big|^2. \end{cases}$$

It is then clear that $H \geq_{\rm Grif} 0$ and $H^\star \leq_{\rm Nak} 0$, but H is neither $\geq_{\rm Nak} 0$ nor $\leq_{\rm Nak} 0$.

(6.10) Proposition. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then*

a) $E \geq_{\rm Grif} 0 \implies Q \geq_{\rm Grif} 0$,
b) $E \leq_{\rm Grif} 0 \implies S \leq_{\rm Grif} 0$,
c) $E \leq_{\rm Nak} 0 \implies S \leq_{\rm Nak} 0$,

and analogous implications hold true for strict positivity.

Proof. If β is written $\sum dz_j \otimes \beta_j$, $\beta_j \in \hom(S,Q)$, then formulas (V-14.6) and (V-14.7) yield

$$\mathrm{i}\Theta(S) = \mathrm{i}\Theta(E)_{\restriction S} - \sum dz_j \wedge d\overline{z}_k \otimes \beta_k^\star \beta_j,$$
$$\mathrm{i}\Theta(Q) = \mathrm{i}\Theta(E)_{\restriction Q} + \sum dz_j \wedge d\overline{z}_k \otimes \beta_j \beta_k^\star.$$

Since $\beta \cdot (\xi \otimes s) = \sum \xi_j \beta_j \cdot s$ and $\beta^\star \cdot (\xi \otimes s) = \sum \overline{\xi}_k \beta_k^\star \cdot s$ we get

$$\theta_S(\xi \otimes s, \xi' \otimes s') = \theta_E(\xi \otimes s, \xi' \otimes s') - \sum_{j,k} \xi_j \overline{\xi}'_k \langle \beta_j \cdot s, \beta_k \cdot s' \rangle,$$

$$\theta_S(u,u) = \theta_E(u,u) - |\beta \cdot u|^2,$$
$$\theta_Q(\xi \otimes s, \xi' \otimes s') = \theta_E(\xi \otimes s, \xi' \otimes s') + \sum_{j,k} \xi_j \overline{\xi}'_k \langle \beta_k^\star \cdot s, \beta_j^\star \cdot s' \rangle,$$
$$\theta_Q(\xi \otimes s, \xi \otimes s) = \theta_E(\xi \otimes s, \xi \otimes s) + |\beta^\star \cdot (\xi \otimes s)|^2. \qquad \square$$

Since H is a quotient bundle of the trivial bundle $\underline{V}$, Example 6.8 shows that $E \geq_{\rm Nak} 0$ *does not imply* $Q \geq_{\rm Nak} 0$.

7. Nakano Vanishing Theorem

Let (X,ω) be a compact Kähler manifold, $\dim_{\mathbb{C}} X = n$, and $E \longrightarrow X$ a hermitian vector bundle of rank r. We are going to compute explicitly the hermitian operator $[\mathrm{i}\Theta(E),\Lambda]$ acting on $\Lambda^{p,q}T^\star X\otimes E$. Let $x_0\in X$ and $(z_1,\ldots,z_n)$ be local coordinates such that $(\partial/\partial z_1,\ldots,\partial/\partial z_n)$ is an orthonormal basis of (TX,ω) at x_0. One can write

$$\omega_{x_0} = \mathrm{i}\sum_{1\le j\le n} dz_j\wedge d\overline{z}_j,$$

$$\mathrm{i}\Theta(E)_{x_0} = \mathrm{i}\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, dz_j\wedge d\overline{z}_k\otimes e_\lambda^\star\otimes e_\mu$$

where $(e_1,\ldots,e_r)$ is an orthonormal basis of E_{x_0}. Let

$$u = \sum_{|J|=p,\,|K|=q,\,\lambda} u_{J,K,\lambda}\, dz_J\wedge d\overline{z}_K\otimes e_\lambda \in \big(\Lambda^{p,q}T^\star X\otimes E\big)_{x_0}.$$

A simple computation as in the proof of Prop. VI-8.3 gives

$$\Lambda u = \mathrm{i}(-1)^p\sum_{J,K,\lambda,s} u_{J,K,\lambda}\Big(\frac{\partial}{\partial z_s}\lrcorner\, dz_J\Big)\wedge\Big(\frac{\partial}{\partial\overline{z}_s}\lrcorner\, d\overline{z}_K\Big)\otimes e_\lambda,$$

$$\mathrm{i}\Theta(E)\wedge u = \mathrm{i}(-1)^p\sum_{j,k,\lambda,\mu,J,K} c_{jk\lambda\mu}\, u_{J,K,\lambda}\, dz_j\wedge dz_J\wedge d\overline{z}_k\wedge d\overline{z}_K\otimes e_\mu,$$

$$\begin{aligned}[\mathrm{i}\Theta(E),\Lambda]u = &\sum_{j,k,\lambda,\mu,J,K} c_{jk\lambda\mu}\, u_{J,K,\lambda}\, dz_j\wedge\Big(\frac{\partial}{\partial z_k}\lrcorner\, dz_J\Big)\wedge d\overline{z}_K\otimes e_\mu\\ &+\sum_{j,k,\lambda,\mu,J,K} c_{jk\lambda\mu}\, u_{J,K,\lambda}\, dz_J\wedge d\overline{z}_k\wedge\Big(\frac{\partial}{\partial\overline{z}_j}\lrcorner\, d\overline{z}_K\Big)\otimes e_\mu\\ &-\sum_{j,\lambda,\mu,J,K} c_{jj\lambda\mu}\, u_{J,K,\lambda}\, dz_J\wedge d\overline{z}_K\otimes e_\mu.\end{aligned}$$

We extend the definition of $u_{J,K,\lambda}$ to non increasing multi-indices $J=(j_s)$, $K=(k_s)$ by deciding that $u_{J,K,\lambda}=0$ if J or K contains identical components repeated and that $u_{J,K,\lambda}$ is alternate in the indices (j_s), (k_s). Then the above equality can be written

$$\begin{aligned}\langle[\mathrm{i}\Theta(E),\Lambda]u,u\rangle = &\sum c_{jk\lambda\mu}\, u_{J,jS,\lambda}\overline{u}_{J,kS,\mu}\\ &+\sum c_{jk\lambda\mu}\, u_{kR,K,\lambda}\overline{u}_{jR,K,\mu}\\ &-\sum c_{jj\lambda\mu}\, u_{J,K,\lambda}\overline{u}_{J,K,\mu},\end{aligned}$$

extended over all indices j,k,λ,μ,J,K,R,S with $|R|=p-1$, $|S|=q-1$. This hermitian form appears rather difficult to handle for general (p,q) because of sign compensation. Two interesting cases are $p=n$ and $q=n$.

• For $u = \sum u_{K,\lambda}\, dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_K \otimes e_\lambda$ of type (n,q), we get

$$(7.1)\quad \langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle = \sum_{|S|=q-1} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, u_{jS,\lambda} \overline{u}_{kS,\mu},$$

because of the equality of the second and third summations in the general formula. Since $u_{jS,\lambda} = 0$ for $j \in S$, the rank of the tensor $(u_{jS,\lambda})_{j,\lambda} \in \mathbb{C}^n \otimes \mathbb{C}^r$ is in fact $\le \min\{n-q+1, r\}$. We obtain therefore:

(7.2) Lemma. *Assume that $E >_m 0$ in the sense of Def. 6.5. Then the hermitian operator $[\mathrm{i}\Theta(E), \Lambda]$ is positive definite on $\Lambda^{n,q}T^\star X \otimes E$ for $q \ge 1$ and $m \ge \min\{n-q+1, r\}$.*

(7.3) Theorem. *Let X be a compact connected Kähler manifold of dimension n and E a hermitian vector bundle of rank r. If $\theta_E \ge_m 0$ on X and $\theta_E >_m 0$ in at least one point, then*

$$H^{n,q}(X,E) = H^q(X, K_X \otimes E) = 0 \quad \text{for} \quad q \ge 1 \quad \text{and} \quad m \ge \min\{n-q+1, r\}.$$

• Similarly, for $u = \sum u_{J,\lambda}\, dz_J \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n \otimes e_\lambda$ of type (p,n), we get

$$\langle [\mathrm{i}\Theta(E), \Lambda]u, u\rangle = \sum_{|R|=p-1} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, u_{kR,\lambda} \overline{u}_{jR,\mu},$$

because of the equality of the first and third summations in the general formula. The indices j,k are twisted, thus $[\mathrm{i}\Theta(E), \Lambda]$ defines a positive hermitian form under the assumption $\mathrm{i}\Theta(E)^\dagger >_m 0$, i.e. $\mathrm{i}\Theta(E^\star) <_m 0$, with $m \ge \min\{n-p+1, r\}$. Serre duality $\big(H^{p,0}(X,E)\big)^\star = H^{n-p,n}(X,E^\star)$ gives:

(7.4) Theorem. *Let X and E be as above. If $\theta_E \le_m 0$ on X and $\theta_E <_m 0$ in at least one point, then*

$$H^{p,0}(X,E) = H^0(X, \Omega_X^p \otimes E) = 0 \quad \text{for} \quad p < n \quad \text{and} \quad m \ge \min\{p+1, r\}.$$

The special case $m = r$ yields:

(7.5) Corollary. *For X and E as above:*

a) *Nakano vanishing theorem (1955):*

$E \ge_{\mathrm{Nak}} 0$, *strictly in one point* $\Longrightarrow$ $H^{n,q}(X,E) = 0$ *for* $q \ge 1$.

b) $E \le_{\mathrm{Nak}} 0$, *strictly in one point* $\Longrightarrow$ $H^{p,0}(X,E) = 0$ *for* $p < n$.

8. Relations Between Nakano and Griffiths Positivity

It is clear that Nakano positivity implies Griffiths positivity. The main result of § 8 is the following "converse" to this property (Demailly-Skoda 1979).

(8.1) Theorem. *For any hermitian vector bundle E,*

$$E >_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad E\otimes\det E >_{\mathrm{Nak}} 0.$$

To prove this result, we first use (V-4.2′) and (V-4.6). If $\mathrm{End}(E\otimes\det E)$ is identified to $\hom(E,E)$, one can write

$$\Theta(E\otimes\det E) = \Theta(E) + \mathrm{Tr}_E(\Theta(E))\otimes \mathrm{Id}_E,$$

$$\theta_{E\otimes\det E} = \theta_E + \mathrm{Tr}_E\,\theta_E\otimes h,$$

where h denotes the hermitian metric on E and where $\mathrm{Tr}_E\,\theta_E$ is the hermitian form on TX defined by

$$\mathrm{Tr}_E\,\theta_E(\xi,\xi) = \sum_{1\le\lambda\le r}\theta_E(\xi\otimes e_\lambda,\xi\otimes e_\lambda),\ \ \xi\in TX,$$

for any orthonormal frame $(e_1,\ldots,e_r)$ of E. Theorem 8.1 is now a consequence of the following simple property of hermitian forms on a tensor product of complex vector spaces.

(8.2) Proposition. *Let T, E be complex vector spaces of respective dimensions n, r, and h a hermitian metric on E. Then for every hermitian form Θ on $T\otimes E$*

$$\Theta >_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad \Theta + \mathrm{Tr}_E\,\Theta\otimes h >_{\mathrm{Nak}} 0.$$

We first need a lemma analogous to Fourier inversion formula for discrete Fourier transforms.

(8.3) Lemma. *Let q be an integer ≥ 3, and x_λ, y_μ, $1\le\lambda,\mu\le r$, be complex numbers. Let σ describe the set U_q^r of r-tuples of q-th roots of unity and put*

$$x'_\sigma = \sum_{1\le\lambda\le r} x_\lambda\overline{\sigma}_\lambda,\qquad y'_\sigma = \sum_{1\le\mu\le r} y_\mu\overline{\sigma}_\mu,\qquad \sigma\in U_q^r.$$

Then for every pair (α,β), $1\le\alpha,\beta\le r$, the following identity holds:

$$q^{-r}\sum_{\sigma\in U_q^r} x'_\sigma\overline{y}'_\sigma\sigma_\alpha\overline{\sigma}_\beta = \begin{cases} x_\alpha\overline{y}_\beta & \text{if }\ \alpha\ne\beta,\\ \displaystyle\sum_{1\le\mu\le r} x_\mu\overline{y}_\mu & \text{if }\ \alpha=\beta.\end{cases}$$

Proof. The coefficient of $x_\lambda \overline{y}_\mu$ in the summation $q^{-r}\sum_{\sigma\in U_q^r} x'_\sigma \overline{y}'_\sigma \sigma_\alpha \overline{\sigma}_\beta$ is given by

$$q^{-r}\sum_{\sigma\in U_q^r} \sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu .$$

This coefficient equals 1 when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are equal (in which case $\sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu = 1$ for any one of the q^r elements of U_q^r). Hence, it is sufficient to prove that

$$\sum_{\sigma\in U_q^r} \sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu = 0$$

when the pairs $\{\alpha,\mu\}$ and $\{\beta,\lambda\}$ are distinct.

If $\{\alpha,\mu\} \neq \{\beta,\lambda\}$, then one of the elements of one of the pairs does not belong to the other pair. As the four indices α,β,λ,μ play the same role, we may suppose for example that $\alpha \notin \{\beta,\lambda\}$. Let us apply to σ the substitution $\sigma \mapsto \tau$, where τ is defined by

$$\tau_\alpha = e^{2\pi\mathrm{i}/q}\sigma_\alpha,\ \ \tau_\nu = \sigma_\nu \quad \text{for} \quad \nu \neq \alpha.$$

We get

$$\sum_\sigma \sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu = \sum_\tau = \begin{cases} e^{2\pi\mathrm{i}/q}\sum_\sigma & \text{if } \ \alpha \neq \mu, \\ e^{4\pi\mathrm{i}/q}\sum_\sigma & \text{if } \ \alpha = \mu, \end{cases}$$

Since $q \geq 3$ by hypothesis, it follows that

$$\sum_\sigma \sigma_\alpha \overline{\sigma}_\beta \overline{\sigma}_\lambda \sigma_\mu = 0.$$

Proof of Proposition 8.2. Let $(t_j)_{1\leq j\leq n}$ be a basis of T, $(e_\lambda)_{1\leq\lambda\leq r}$ an orthonormal basis of E and $\xi = \sum_j \xi_j t_j \in T$, $u = \sum_{j,\lambda} u_{j\lambda}\, t_j \otimes e_\lambda \in T\otimes E$. The coefficients $c_{jk\lambda\mu}$ of Θ with respect to the basis $t_j \otimes e_\lambda$ satisfy the symmetry relation $\overline{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$, and we have the formulas

$$\begin{aligned} \Theta(u,u) &= \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda} \overline{u}_{k\mu}, \\ \mathrm{Tr}_E\, \Theta(\xi,\xi) &= \sum_{j,k,\lambda} c_{jk\lambda\lambda} \xi_j \overline{\xi}_k, \\ (\Theta + \mathrm{Tr}_E\, \Theta\otimes h)(u,u) &= \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda}\overline{u}_{k\mu} + c_{jk\lambda\lambda} u_{j\mu}\overline{u}_{k\mu}. \end{aligned}$$

For every $\sigma \in U_q^r$ (cf. Lemma 8.3), put

$$u'_{j\sigma} = \sum_{1\le\lambda\le r} u_{j\lambda}\overline{\sigma}_\lambda \in \mathbb{C},$$
$$\widehat{u}_\sigma = \sum_j u'_{j\sigma} t_j \in T \quad , \quad \widehat{e}_\sigma = \sum_\lambda \sigma_\lambda e_\lambda \in E.$$

Lemma 8.3 implies

$$\begin{aligned} q^{-r}\sum_{\sigma\in U_q^r} \Theta(\widehat{u}_\sigma\otimes\widehat{e}_\sigma, \widehat{u}_\sigma\otimes\widehat{e}_\sigma) &= q^{-r}\sum_{\sigma\in U_q^r} c_{jk\lambda\mu}u'_{j\sigma}\overline{u}'_{k\sigma}\sigma_\lambda\overline{\sigma}_\mu \\ &= \sum_{j,k,\lambda\neq\mu} c_{jk\lambda\mu}u_{j\lambda}\overline{u}_{k\mu} + \sum_{j,k,\lambda,\mu} c_{jk\lambda\lambda}u_{j\mu}\overline{u}_{k\mu}. \end{aligned}$$

The Griffiths positivity assumption shows that the left hand side is ≥ 0, hence

$$(\Theta + \mathrm{Tr}_E\,\Theta\otimes h)(u,u) \ge \sum_{j,k,\lambda} c_{jk\lambda\lambda}u_{j\lambda}\overline{u}_{k\lambda} \ge 0$$

with strict positivity if $\Theta >_{\mathrm{Grif}} 0$ and $u \neq 0$. □

(8.4) Example. Take $E = H$ over $\mathbb{P}^n = P(V)$. The exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{V} \longrightarrow H \longrightarrow 0$$

implies $\det \underline{V} = \det H \otimes \mathcal{O}(-1)$. Since $\det \underline{V}$ is a trivial bundle, we get (non canonical) isomorphisms

$$\det H \simeq \mathcal{O}(1),$$
$$T\mathbb{P}^n = H\otimes\mathcal{O}(1) \simeq H \otimes \det H.$$

We already know that $H \ge_{\mathrm{Grif}} 0$, hence $T\mathbb{P}^n \ge_{\mathrm{Nak}} 0$. A direct computation based on (6.9) shows that

$$\begin{aligned} \theta_{T\mathbb{P}^n}(u,u) &= (\theta_H + \mathrm{Tr}_H\,\theta_H\otimes h)(u,u) \\ &= \sum_{1\le j,k\le n} u_{jk}\overline{u}_{kj} + u_{jk}\overline{u}_{jk} = \frac{1}{2}\sum_{1\le j,k\le n} |u_{jk}+u_{kj}|^2. \end{aligned}$$

In addition, we have $T\mathbb{P}^n >_{\mathrm{Grif}} 0$. However, the Serre duality theorem gives

$$\begin{aligned} H^q(\mathbb{P}^n, K_{\mathbb{P}^n}\otimes T\mathbb{P}^n)^\star &\simeq H^{n-q}(\mathbb{P}^n, T^\star\mathbb{P}^n) \\ &= H^{1,n-q}(\mathbb{P}^n,\mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } q = n-1, \\ 0 & \text{if } q \neq n-1. \end{cases} \end{aligned}$$

For $n \ge 2$, Th. 7.3 implies that $T\mathbb{P}^n$ has no hermitian metric such that $\theta_{T\mathbb{P}^n} \ge_2 0$ on $\mathbb{P}^n$ and $\theta_{T\mathbb{P}^n} >_2 0$ in one point. This shows that the notion of 2-positivity is actually stronger than 1-positivity (i.e. Griffiths positivity).

(8.5) Remark. Since $\mathrm{Tr}_H\,\theta_H = \theta_{\mathcal{O}(1)}$ is positive and $\theta_{T\mathbb{P}^n}$ is not $>_{\mathrm{Nak}} 0$ when $n \geq 2$, we see that Prop. 8.2 is best possible in the sense that there cannot exist any constant $c < 1$ such that

$$\Theta >_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad \Theta + c\,\mathrm{Tr}_E\,\Theta \otimes h \geq_{\mathrm{Nak}} 0.$$

9. Applications to Griffiths Positive Bundles

We first need a preliminary result.

(9.1) Proposition. *Let T be a complex vector space and (E,h) a hermitian vector space of respective dimensions n,r with $r \geq 2$. Then for any hermitian form Θ on $T \otimes E$ and any integer $m \geq 1$*

$$\Theta >_{\mathrm{Grif}} 0 \quad\Longrightarrow\quad m\,\mathrm{Tr}_E\,\Theta \otimes h - \Theta >_m 0.$$

Proof. Let us distinguish two cases.

a) $m = 1$. Let $u \in T \otimes E$ be a tensor of rank 1. Then u can be written $u = \xi_1 \otimes e_1$ with $\xi_1 \in T$, $\xi_1 \neq 0$, and $e_1 \in E$, $|e_1| = 1$. Complete e_1 into an orthonormal basis $(e_1, \ldots, e_r)$ of E. One gets immediately

$$\begin{aligned}(\mathrm{Tr}_E\,\Theta \otimes h)(u,u) = \mathrm{Tr}_E\,\Theta(\xi_1,\xi_1) &= \sum_{1\leq\lambda\leq r} \Theta(\xi_1 \otimes e_\lambda, \xi_1 \otimes e_\lambda)\\ &> \Theta(\xi_1 \otimes e_1, \xi_1 \otimes e_1) = \Theta(u,u).\end{aligned}$$

b) $m \geq 2$. Every tensor $u \in T \otimes E$ of rank $\leq m$ can be written

$$u = \sum_{1\leq\lambda\leq q} \xi_\lambda \otimes e_\lambda \quad , \quad \xi_\lambda \in T,$$

with $q = \min(m,r)$ and $(e_\lambda)_{1\leq\lambda\leq r}$ an orthonormal basis of E. Let F be the vector subspace of E generated by $(e_1, \ldots, e_q)$ and Θ_F the restriction of Θ to $T \otimes F$. The first part shows that

$$\Theta' := \mathrm{Tr}_F\,\Theta_F \otimes h - \Theta_F >_{\mathrm{Grif}} 0.$$

Proposition 9.2 applied to Θ' on $T \otimes F$ yields

$$\Theta' + \mathrm{Tr}_F\,\Theta' \otimes h = q\,\mathrm{Tr}_F\,\Theta_F \otimes h - \Theta_F >_q 0.$$

Since $u \in T \otimes F$ is of rank $\leq q \leq m$, we get (for $u \neq 0$)

$$\begin{aligned}\Theta(u,u) = \Theta_F(u,u) &< q(\mathrm{Tr}_F\,\Theta_F \otimes h)(u,u)\\ &= q \sum_{1\le j,\lambda\le q} \Theta(\xi_j\otimes e_\lambda, \xi_j\otimes e_\lambda) \le m\,\mathrm{Tr}_E\,\Theta\otimes h(u,u)\square\end{aligned}$$

Proposition 9.1 is of course also true in the semi-positive case. From these facts, we deduce

(9.2) Theorem. *Let E be a Griffiths (semi-)positive bundle of rank $r \ge 2$. Then for any integer $m \ge 1$*

$$E^\star \otimes (\det E)^m >_m 0 \qquad (\textit{resp.}\quad \ge_m 0).$$

Proof. Apply Prop. 8.1 to $\Theta = -\theta_{E^\star} >_{\mathrm{Grif}} 0$ and observe that

$$\theta_{\det E} = -\theta_{\det E^\star} = \mathrm{Tr}_{E^\star}\,\Theta.$$

(9.3) Theorem. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of hermitian vector bundles. Then for any $m \ge 1$*

$$E >_m 0 \quad\Longrightarrow\quad S \otimes (\det Q)^m >_m 0.$$

Proof. Formulas (V-14.6) and (V-14.7) imply

$$\mathrm{i}\Theta(S) >_m \mathrm{i}\beta^\star\wedge\beta \quad , \quad \mathrm{i}\Theta(Q) >_m \mathrm{i}\beta\wedge\beta^\star,$$

$$\mathrm{i}\Theta(\det Q) = \mathrm{Tr}_Q(\mathrm{i}\Theta(Q)) > \mathrm{Tr}_Q(\mathrm{i}\beta\wedge\beta^\star).$$

If we write $\beta = \sum dz_j \otimes \beta_j$ as in the proof of Prop. 6.10, then

$$\begin{aligned}\mathrm{Tr}_Q(\mathrm{i}\beta\wedge\beta^\star) &= \sum i dz_j \wedge d\overline{z}_k\,\mathrm{Tr}_Q(\beta_j\beta_k^\star)\\ &= \sum i dz_j \wedge d\overline{z}_k\,\mathrm{Tr}_S(\beta_k^\star\beta_j) = \mathrm{Tr}_S(-\mathrm{i}\beta^\star\wedge\beta).\end{aligned}$$

Furthermore, it has been already proved that $-\mathrm{i}\beta^\star\wedge\beta \ge_{\mathrm{Nak}} 0$. By Prop. 8.1 applied to the corresponding hermitian form Θ on $TX\otimes S$, we get

$$m\,\mathrm{Tr}_S(-\mathrm{i}\beta^\star\wedge\beta)\otimes\mathrm{Id}_S + \mathrm{i}\beta^\star\wedge\beta \ge_m 0,$$

and Th. 9.3 follows.

(9.4) Corollary. *Let X be a compact n-dimensional complex manifold, E a vector bundle of rank $r \ge 2$ and $m \ge 1$ an integer. Then*

a) $E >_{\mathrm{Grif}} 0 \Longrightarrow H^{n,q}(X, E\otimes\det E) = 0$ *for* $q \ge 1$;

b) $E >_{\mathrm{Grif}} 0 \Longrightarrow H^{n,q}\big(X, E^\star\otimes(\det E)^m\big) = 0$ *for* $q \ge 1$ *and* $m \ge \min\{n-q+1, r\}$;

c) *Let* $0 \to S \to E \to Q \to 0$ *be an exact sequence of vector bundles and* $m = \min\{n - q + 1, \operatorname{rk} S\}$, $q \geq 1$. *If* $E >_m 0$ *and if* L *is a line bundle such that* $L \otimes (\det Q)^{-m} \geq 0$, *then*

$$H^{n,q}(X, S \otimes L) = 0.$$

Proof. Immediate consequence of Theorems 7.3, 8.1, 9.2 and 9.3. □

Note that under our hypotheses $\omega = \mathrm{i}\,\mathrm{Tr}_E\,\Theta(E) = \mathrm{i}\Theta(\Lambda^r E)$ is a Kähler metric on X. Corollary 2.5 shows that it is enough in a), b), c) to assume semi-positivity and strict positivity in one point (this is true a priori only if X is supposed in addition to be Kähler, but this hypothesis can be removed by means of Siu's result mentioned after (4.5).

a) is in fact a special case of a result of (Griffiths 1969), which we will prove in full generality in volume II (see the chapter on vanishing theorems for ample vector bundles); property b) will be also considerably strengthened there. Property c) is due to (Skoda 1978) for $q = 0$ and to (Demailly 1982c) in general. Let us take the tensor product of the exact sequence in c) with $(\det Q)^l$. The corresponding long cohomology exact sequence implies that the natural morphism

$$H^{n,q}\big(X, E \otimes (\det Q)^l\big) \longrightarrow H^{n,q}\big(X, Q \otimes (\det Q)^l\big)$$

is surjective for $q \geq 0$ and $l, m \geq \min\{n - q, \operatorname{rk} S\}$, bijective for $q \geq 1$ and $l, m \geq \min\{n - q + 1, \operatorname{rk} S\}$.

10. Cohomology Groups of $\mathcal{O}(k)$ over $\mathbb{P}^n$

As an illustration of the above results, we compute now the cohomology groups of all line bundles $\mathcal{O}(k) \to \mathbb{P}^n$. This precise evaluation will be needed in the proof of a general vanishing theorem for vector bundles, due to Le Potier (see volume II). As in §V-15, we consider a complex vector space V of dimension $n + 1$ and the exact sequence

$$(10.1)\quad 0 \longrightarrow \mathcal{O}(-1) \longrightarrow \underline{V} \longrightarrow H \longrightarrow 0$$

of vector bundles over $\mathbb{P}^n = P(V)$. We thus have $\det \underline{V} = \det H \otimes \mathcal{O}(-1)$, and as $TP(V) = H \otimes \mathcal{O}(1)$ by Th. V-15.7, we find

$$(10.2)\quad K_{P(V)} = \det T^\star P(V) = \det H^\star \otimes \mathcal{O}(-n) = \det \underline{V}^\star \otimes \mathcal{O}(-n-1)$$

where $\det \underline{V}$ is a trivial line bundle.

Before going further, we need some notations. For every integer $k \in \mathbb{N}$, we consider the homological complex $C^{\bullet,k}(V^\star)$ with differential γ such that

$$(10.3)\quad \begin{cases} C^{p,k}(V^\star) = \Lambda^p V^\star \otimes S^{k-p}V^\star, & 0 \le p \le k, \\ \qquad\qquad = 0 & \text{otherwise}, \\ \gamma \;:\; \Lambda^p V^\star \otimes S^{k-p}V^\star \longrightarrow \Lambda^{p-1}V^\star \otimes S^{k-p+1}V^\star, \end{cases}$$

where γ is the linear map obtained by contraction with the Euler vector field $\mathrm{Id}_V \in V \otimes V^\star$, through the obvious maps $V \otimes \Lambda^p V^\star \longrightarrow \Lambda^{p-1}V^\star$ and $V^\star \otimes S^{k-p}V^\star \longrightarrow S^{k-p+1}V^\star$. If $(z_0, \ldots, z_n)$ are coordinates on V, the module $C^{p,k}(V^\star)$ can be identified with the space of p-forms

$$\alpha(z) = \sum_{|I|=p} \alpha_I(z)\, dz_I$$

where the α_I 's are homogeneous polynomials of degree $k-p$. The differential γ is given by contraction with the Euler vector field $\xi = \sum_{0\le j\le n} z_j\, \partial/\partial z_j$.

Let us denote by $Z^{p,k}(V^\star)$ the space of p-cycles of $C^{\bullet,k}(V^\star)$, i.e. the space of forms $\alpha \in C^{p,k}(V^\star)$ such that $\xi \,\lrcorner\, \alpha = 0$. The exterior derivative d also acts on $C^{\bullet,k}(V^\star)$; we have

$$d \;:\; C^{p,k}(V^\star) \longrightarrow C^{p+1,k}(V^\star),$$

and a trivial computation shows that $d\gamma + \gamma d = k \cdot \mathrm{Id}_{C^{\bullet,k}(V^\star)}$.

(10.4) Theorem. *For $k \ne 0$, $C^{\bullet,k}(V^\star)$ is exact and there exist canonical isomorphisms*

$$C^{\bullet,k}(V^\star) = \Lambda^p V^\star \otimes S^{k-p}V^\star \simeq Z^{p,k}(V^\star) \oplus Z^{p-1,k}(V^\star).$$

Proof. The identity $d\gamma + \gamma d = k \cdot \mathrm{Id}$ implies the exactness. The isomorphism is given by $\frac{1}{k}\gamma d \oplus \gamma$ and its inverse by $\mathcal{P}_1 + \frac{1}{k} d \circ \mathcal{P}_2$. □

Let us consider now the canonical mappings

$$\pi \;:\; V \setminus \{0\} \longrightarrow P(V), \qquad \mu' \;:\; V \setminus \{0\} \longrightarrow \mathcal{O}(-1)$$

defined in §V-15. As $T_{[z]}P(V) \simeq V/\mathbb{C}\xi(z)$ for all $z \in V \setminus \{0\}$, every form $\alpha \in Z^{p,k}(V^\star)$ defines a holomorphic section of $\pi^\star\big(\Lambda^p T^\star P(V)\big)$, $\alpha(z)$ being homogeneous of degree k with respect to z. Hence $\alpha(z) \otimes \mu'(z)^{-k}$ is a holomorphic section of $\pi^\star\big(\Lambda^p T^\star P(V) \otimes \mathcal{O}(k)\big)$, and since its homogeneity degree is 0, it induces a holomorphic section of $\Lambda^p T^\star P(V) \otimes \mathcal{O}(k)$. We thus have an injective morphism

$$(10.5)\quad Z^{p,k}(V^\star) \longrightarrow H^{p,0}\big(P(V), \mathcal{O}(k)\big).$$

(10.6) Theorem. *The groups $H^{p,0}\big(P(V), \mathcal{O}(k)\big)$ are given by*

a) $H^{p,0}\big(P(V), \mathcal{O}(k)\big) \simeq Z^{p,k}(V^\star)$ *for* $k \ge p \ge 0$,

b) $H^{p,0}\big(P(V),\mathcal{O}(k)\big)=0$ *for* $k\le p$ *and* $(k,p)\ne(0,0)$.

Proof. Let s be a holomorphic section of $\Lambda^pT^\star P(V)\otimes\mathcal{O}(k)$. Set

$$\alpha(z)=(d\pi_z)^\star\big(s([z])\otimes\mu'(z)^k\big),\qquad z\in V\setminus\{0\}.$$

Then α is a holomorphic p-form on $V\setminus\{0\}$ such that $\xi\,\lrcorner\,\alpha=0$, and the coefficients of α are homogeneous of degree $k-p$ on $V\setminus\{0\}$ (recall that $d\pi_{\lambda z}=\lambda^{-1}d\pi_z$). It follows that $\alpha=0$ if $k<p$ and that $\alpha\in Z^{p,k}(V^\star)$ if $k\ge p$. The injective morphism (10.5) is therefore also surjective. Finally, $Z^{p,p}(V^\star)=0$ for $p=k\ne0$, because of the exactness of $C^{\bullet,k}(V^\star)$ when $k\ne0$. The proof is complete. □

(10.7) Theorem. *The cohomology groups $H^{p,q}\big(P(V),\mathcal{O}(k)\big)$ vanish in the following cases:*

a) $q\ne0,n,p$;

b) $q=0$, $k\le p$ *and* $(k,p)\ne(0,0)$;

c) $q=n$, $k\ge -n+p$ *and* $(k,p)\ne(0,n)$;

d) $q=p\ne0,n$, $k\ne0$.

The remaining non vanishing groups are:

$\overline{\mathrm{b}}$) $H^{p,0}\big(P(V),\mathcal{O}(k)\big)\simeq Z^{p,k}(V^\star)$ *for* $k>p$;

$\overline{\mathrm{c}}$) $H^{p,n}\big(P(V),\mathcal{O}(k)\big)\simeq Z^{n-p,-k}(V)$ *for* $k<-n+p$;

$\overline{\mathrm{d}}$) $H^{p,p}\big(P(V),\mathbb{C}\big)=\mathbb{C}$, $0\le p\le n$.

Proof. • $\overline{\mathrm{d}}$) is already known, and so is a) when $k=0$ (Th. VI-13.3).

• b) and $\overline{\mathrm{b}}$) follow from Th. 10.6, and c), $\overline{\mathrm{c}}$) are equivalent to b), $\overline{\mathrm{b}}$) via Serre duality:

$$H^{p,q}\big(P(V),\mathcal{O}(k)\big)^\star=H^{n-p,n-q}\big(P(V),\mathcal{O}(-k)\big),$$

thanks to the canonical isomorphism $\big(Z^{p,k}(V)\big)^\star=Z^{p,k}(V^\star)$.

• Let us prove now property a) when $k\ne0$ and property d). By Serre duality, we may assume $k>0$. Then

$$\Lambda^pT^\star P(V)\simeq K_{P(V)}\otimes\Lambda^{n-p}TP(V).$$

It is very easy to verify that $E\ge_{\rm Nak}0$ implies $\Lambda^sE\ge_{\rm Nak}0$ for every integer s. Since $TP(V)\ge_{\rm Nak}0$, we get therefore

$$F=\Lambda^{n-p}TP(V)\otimes\mathcal{O}(k)>_{\rm Nak}0\quad\text{ for }\ k>0,$$

and the Nakano vanishing theorem implies

$$\begin{aligned}H^{p,q}\big(P(V),\mathcal{O}(k)\big)&=H^q\big(P(V),\Lambda^pT^\star P(V)\otimes\mathcal{O}(k)\big)\\&=H^q\big(P(V),K_{P(V)}\otimes F\big)=0,\qquad q\ge1.\end{aligned}$$

□

11. Ample Vector Bundles

11.A. Globally Generated Vector Bundles

All definitions concerning ampleness are purely algebraic and do not involve differential geometry. We shall see however that ampleness is intimately connected with the differential geometric notion of positivity. For a general discussion of properties of ample vector bundles in arbitrary characteristic, we refer to (Hartshorne 1966).

(11.1) Definition. *Let $E \to X$ be a holomorphic vector bundle over an arbitrary complex manifold X.*

a) *E is said to be globally generated if for every $x \in X$ the evaluation map $H^0(X,E) \to E_x$ is onto.*

b) *E is said to be semi-ample if there exists an integer k_0 such that S^kE is globally generated for $k \ge k_0$.*

Any quotient of a trivial vector bundle is globally generated, for example the tautological quotient vector bundle Q over the Grassmannian $G_r(V)$ is globally generated. Conversely, every globally generated vector bundle E of rank r is isomorphic to the quotient of a trivial vector bundle of rank $\le n+r$, as shown by the following result.

(11.2) Proposition. *If a vector bundle E of rank r is globally generated, then there exists a finite dimensional subspace $V \subset H^0(X,E)$, $\dim V \le n+r$, such that V generates all fibers E_x, $x \in X$.*

Proof. Put an arbitrary hermitian metric on E and consider the Fréchet space $\mathcal{F} = \big(H^0(X,E)\big)^{n+r}$ of $(n+r)$-tuples of holomorphic sections of E, endowed with the topology of uniform convergence on compact subsets of X. For every compact set $K \subset X$, we set

$$A(K) = \{(s_1,\ldots,s_{n+r}) \in \mathcal{F} \text{ which do not generate } E \text{ on } K\}.$$

It is enough to prove that $A(K)$ is of first category in $\mathcal{F}$: indeed, Baire's theorem will imply that $A(X) = \bigcup A(K_\nu)$ is also of first category, if (K_ν) is an exhaustive sequence of compact subsets of X. It is clear that $A(K)$ is closed, because $A(K)$ is characterized by the closed condition

$$\min_K \sum_{i_1<\cdots<i_r} |s_{i_1} \wedge \cdots \wedge s_{i_r}| = 0.$$

It is therefore sufficient to prove that $A(K)$ has no interior point. By hypothesis, each fiber E_x, $x \in K$, is generated by r global sections $s'_1,\ldots,s'_r$. We have in fact $s'_1 \wedge \cdots \wedge s'_r \neq 0$ in a neighborhood U_x of x. By compactness of K, there exist finitely many sections $s'_1, \ldots, s'_N$ which generate E in a neighborhood Ω of the set K.

If T is a complex vector space of dimension r, define $R_k(T^p)$ as the set of p-tuples $(x_1, \ldots, x_p) \in T^p$ of rank k. Given $a \in R_k(T^p)$, we can reorder the p-tuple in such a way that $a_1 \wedge \cdots \wedge a_k \neq 0$. Complete these k vectors into a basis $(a_1, \ldots, a_k, b_1, \ldots, b_{r-k})$ of T. For every point $x \in T^p$ in a neighborhood of a, then $(x_1, \ldots, x_k, b_1, \ldots, b_{r-k})$ is again a basis of T. Therefore, we will have $x \in R_k(T^p)$ if and only if the coordinates of x_l, $k+1 \leq l \leq N$, relative to $b_1, \ldots, b_{r-k}$ vanish. It follows that $R_k(T^p)$ is a (non closed) submanifold of T^p of codimension $(r-k)(p-k)$.

Now, we have a surjective affine bundle-morphism

$$\begin{aligned} \Phi :\ \Omega \times \mathbb{C}^{N(n+r)} &\longrightarrow E^{n+r} \\ (x, \lambda) &\longmapsto \Big(s_j(x) + \sum_{1\leq k\leq N} \lambda_{jk} s'_k(x)\Big)_{1\leq j\leq n+r}. \end{aligned}$$

Therefore $\Phi^{-1}(R_k(E^{n+r}))$ is a locally trivial differentiable bundle over Ω, and the codimension of its fibers in $\mathbb{C}^{N(n+r)}$ is $(r-k)(n+r-k) \geq n+1$ if $k < r$; it follows that the dimension of the total space $\Phi^{-1}(R_k(E^{n+r}))$ is $\leq N(n+r) - 1$. By Sard's theorem

$$\bigcup_{k<r} \mathcal{P}_2\Big(\Phi^{-1}\big(R_k(E^{n+r})\big)\Big)$$

is of zero measure in $\mathbb{C}^{N(n+r)}$. This means that for almost every value of the parameter λ the vectors $s_j(x) + \sum_k \lambda_{jk} s'_k(x) \in E_x$, $1 \leq j \leq n+r$, are of maximum rank r at each point $x \in \Omega$. Therefore $A(K)$ has no interior point. □

Assume now that $V \subset H^0(X, E)$ generates E on X. Then there is an exact sequence

$$(11.3) \qquad 0 \longrightarrow S \longrightarrow \underline{V} \longrightarrow E \longrightarrow 0$$

of vector bundles over X, where $S_x = \{s \in V\ ;\ s(x) = 0\}$, $\operatorname{codim}_V S_x = r$. One obtains therefore a commutative diagram

$$(11.4) \qquad \begin{array}{ccc} E & \xrightarrow{\Psi_V} & Q \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi_V} & G_r(V) \end{array}$$

where ψ_V, Ψ_V are the holomorphic maps defined by

$$\begin{aligned} &\psi_V(x) = S_x, \quad x \in X, \\ &\Psi_V(u) = \{s \in V\ ;\ s(x) = u\} \in V/S_x, \quad u \in E_x. \end{aligned}$$

In particular, we see that every globally generated vector bundle E of rank r is the pull-back of the tautological quotient vector bundle Q of rank r over the Grassmannian by means of some holomorphic map $X \longrightarrow G_r(V)$. In the special case when $\operatorname{rk} E = r = 1$, the above diagram becomes

$$(11.4') \qquad \begin{array}{ccc} E & \xrightarrow{\Psi_V} & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi_V} & P(V^\star) \end{array}$$

(11.5) Corollary. *If E is globally generated, then E possesses a hermitian metric such that $E \geq_{\mathrm{Grif}} 0$ (and also $E^\star \leq_{\mathrm{Nak}} 0$).*

Proof. Apply Prop. 6.11 to the exact sequence (11.3), where $\underline{V}$ is endowed with an arbitrary hermitian metric. □

When E is of rank $r = 1$, then $S^k E = E^{\otimes k}$ and any hermitian metric of $E^{\otimes k}$ yields a metric on E after extracting k-th roots. Thus:

(11.6) Corollary. *If E is a semi-ample line bundle, then $E \geq 0$.* □

In the case of vector bundles ($r \geq 2$) the answer is unknown, mainly because there is no known procedure to get a Griffiths semipositive metric on E from one on $S^k E$.

11.B. Ampleness

We are now turning ourselves to the definition of ampleness. If $E \longrightarrow X$ is a holomorphic vector bundle, we define the bundle $J^k E$ of *k-jets of sections* of E by $(J^k E)_x = \mathcal{O}_x(E)/\big(\mathcal{M}_x^{k+1} \cdot \mathcal{O}_x(E)\big)$ for every $x \in X$, where $\mathcal{M}_x$ is the maximal ideal of $\mathcal{O}_x$. Let $(e_1, \ldots, e_r)$ be a holomorphic frame of E and $(z_1, \ldots, z_n)$ analytic coordinates on an open subset $\Omega \subset X$. The fiber $(J^k E)_x$ can be identified with the set of Taylor developments of order k :

$$\sum_{1 \leq \lambda \leq r, |\alpha| \leq k} c_{\lambda,\alpha} (z - x)^\alpha \, e_\lambda(z),$$

and the coefficients $c_{\lambda,\alpha}$ define coordinates along the fibers of $J^k E$. It is clear that the choice of another holomorphic frame (e_λ) would yield a linear change of coordinates $(c_{\lambda,\alpha})$ with holomorphic coefficients in x. Hence $J^k E$ is a holomorphic vector bundle of rank $r\binom{n+k}{n}$.

(11.7) Definition.

a) *E is said to be very ample if all evaluation maps $H^0(X, E) \to (J^1 E)_x$, $H^0(X, E) \to E_x \oplus E_y$, $x, y \in X$, $x \neq y$, are surjective.*

b) *E is said to be ample if there exists an integer k_0 such that $S^k E$ is very ample for $k \geq k_0$.*

(11.8) Example. $\mathcal{O}(1) \to \mathbb{P}^n$ is a very ample line bundle (immediate verification). Since the pull-back of a (very) ample vector bundle by an embedding is clearly also (very) ample, diagram (V-16.8) shows that $\Lambda^r Q \to G_r(V)$ is very ample. However, Q itself cannot be very ample if $r \geq 2$, because $\dim H^0(G_r(V), Q) = \dim V = d$, whereas

$$\operatorname{rank}(J^1 Q) = (\operatorname{rank} Q)\big(1 + \dim G_r(V)\big) = r\big(1 + r(d-r)\big) > d \quad \text{if} \quad r \geq 2.$$

(11.9) Proposition. *If E is very ample of rank r, there exists a subspace V of $H^0(X, E)$, $\dim V \leq \max\big(nr+n+r, 2(n+r)\big)$, such that all the evaluation maps $V \to E_x \oplus E_y$, $x \neq y$, and $V \to (J^1 E)_x$, $x \in X$, are surjective.*

Proof. The arguments are exactly the same as in the proof of Prop. 11.4, if we consider instead the bundles $J^1 E \longrightarrow X$ and $E \times E \longrightarrow X \times X \setminus \Delta_X$ of respective ranks $r(n+1)$ and $2r$, and sections $s'_1, \ldots, s'_N \in H^0(X, E)$ generating these bundles. □

(11.10) Proposition. *Let $E \to X$ be a holomorphic vector bundle.*

a) *If $V \subset H^0(X, E)$ generates $J^1 E \longrightarrow X$ and $E \times E \longrightarrow X \times X \setminus \Delta_X$, then ψ_V is an embedding.*

b) *Conversely, if $\operatorname{rank} E = 1$ and if there exists $V \subset H^0(X, E)$ generating E such that ψ_V is an embedding, then E is very ample.*

Proof. b) is immediate, because $E = \psi_V^\star(\mathcal{O}(1))$ and $\mathcal{O}(1)$ is very ample. Note that the result is false for $r \geq 2$ as shown by the example $E = Q$ over $X = G_r(V)$.

a) Under the assumption of a), it is clear since $S_x = \{s \in V \ ; \ s(x) = 0\}$ that $S_x = S_y$ implies $x = y$, hence ψ_V is injective. Therefore, it is enough to prove that the map $x \mapsto S_x$ has an injective differential. Let $x \in X$ and $W \subset V$ such that $S_x \oplus W = V$. Choose a coordinate system in a neighborhood of x in X and a small tangent vector $h \in T_x X$. The element $S_{x+h} \in G_r(V)$ is the graph of a small linear map $u = O(|h|) : S_x \to W$. Thus we have

$$S_{x+h} = \{s' = s + t \in V \ ; \ s \in S_x, \ t = u(s) \in W, \ s'(x+h) = 0\}.$$

Since $s(x) = 0$ and $|t| = O(|h|)$, we find

$$s'(x+h) = s'(x) + d_x s' \cdot h + O(|s'| \cdot |h|^2) = t(x) + d_x s \cdot h + O(|s| \cdot |h|^2),$$

thus $s'(x+h) = 0$ if and only if $t(x) = -d_x s \cdot h + O(|s| \cdot |h|^2)$. Thanks to the fiber isomorphism $\Psi_V : E_x \longrightarrow V/S_x \simeq W, \ t(x) \longmapsto t \bmod S_x$, we get:

$$u(s) = t = \Psi_V(t(x)) = -\Psi_V\big(d_x s \cdot h + O(|s| \cdot |h|^2)\big).$$

Recall that $T_y G_r(V) = \hom(S_y, Q_y) = \hom(y, V/y)$ (see V-16.5) and use these identifications at $y = S_x$. It follows that

$$(11.11) \quad (d_x\psi_V) \cdot h = u = \big(S_x \longrightarrow V/S_x,\ s \longmapsto -\Psi_V(d_x s \cdot h)\big),$$

Now hypothesis a) implies that $S_x \ni s \longmapsto d_x s \in \hom(T_x X, E_x)$ is onto, hence $d_x\psi_V$ is injective. □

(11.12) Corollary. *If E is an ample line bundle, then $E > 0$.*

Proof. If E is very ample, diagram (11.4′) shows that E is the pull-back of $\mathcal{O}(1)$ by the embedding ψ_V, hence $i\Theta(E) = \psi_V^\star\big(\mathrm{i}\Theta(\mathcal{O}(1))\big) > 0$ with the induced metric. The ample case follows by extracting roots. □

(11.13) Corollary. *If E is a very ample vector bundle, then E carries a hermitian metric such that $E^\star <_{\mathrm{Nak}} 0$, in particular $E >_{\mathrm{Grif}} 0$.*

Proof. Choose V as in Prop. 11.9 and select an arbitrary hermitian metric on V. Then diagram 11.4 yields $E = \psi_V^\star Q$, hence $\theta_E = \Psi_V^\star \theta_Q$. By formula (V-16.9) we have for every $\xi \in TG_r(V) = \hom(S, Q)$ and $t \in Q$:

$$\theta_Q(\xi \otimes t, \xi \otimes t) = \sum_{j,k,l} \xi_{jk}\overline{\xi}_{lk} t_l \overline{t}_j = \sum_k \Big|\sum_j \overline{t}_j \xi_{jk}\Big|^2 = |\langle \bullet, t\rangle \circ \xi|^2.$$

Let $h \in T_x X$, $t \in E_x$. Thanks to formula (11.11), we get

$$\begin{aligned}\theta_E(h \otimes t, h \otimes t) &= \theta_Q\big((d_x\psi_V \cdot h) \otimes \Psi_V(t), (d_x\psi_V \cdot h) \otimes \Psi_V(t)\big)\\ &= \big|\langle \bullet, \Psi_V(t)\rangle \circ (d_x\psi_V \cdot h)\big|^2 = \big|S_x \ni s \longmapsto \langle \Psi_V(d_x s \cdot h), \Psi_V(t)\rangle\big|^2\\ &= \big|S_x \ni s \longmapsto \langle d_x s \cdot h, t\rangle\big|^2 \geq 0.\end{aligned}$$

As $S_x \ni s \mapsto d_x s \in T^\star X \otimes E$ is surjective, it follows that $\theta_E(h \otimes t, h \otimes t) \neq 0$ when $h \neq 0$, $t \neq 0$. Now, $d_x s$ defines a linear form on $TX \otimes E^\star$ and the above formula for the curvature of E clearly yields

$$\theta_{E^\star}(u, u) = -|S_x \ni s \longmapsto d_x s \cdot u|^2 < 0 \quad \text{if } u \neq 0.$$ □

(11.14) Problem (Griffiths 1969). *If E is an ample vector bundle over a compact manifold X, then is $E >_{\mathrm{Grif}} 0$?*

Griffiths' problem has been solved in the affirmative when X is a curve (Umemura 1973), see also (Campana-Flenner 1990), but the general case is still unclear and seems very deep. The next sections will be concerned with the important result of Kodaira asserting the equivalence between positivity and ampleness for line bundles.

12. Blowing-up along a Submanifold

Here we generalize the blowing-up process already considered in Remark 4.5 to arbitrary manifolds. Let X be a complex n-dimensional manifold and Y a closed submanifold with $\operatorname{codim}_X Y = s$.

(12.1) Notations. *The normal bundle of Y in X is the vector bundle over Y defined as the quotient $NY = (TX)_{\restriction Y}/TY$. The fibers of NY are thus given by $N_yY = T_yX/T_yY$ at every point $y \in Y$. We also consider the projectivized normal bundle $P(NY) \longrightarrow Y$ whose fibers are the projective spaces $P(N_yY)$ associated to the fibers of NY.*

The *blow-up of X with center Y* (to be constructed later) is a complex n-dimensional manifold $\widetilde{X}$ together with a holomorphic map $\sigma : \widetilde{X} \longrightarrow X$ such that:

i) $E := \sigma^{-1}(Y)$ is a smooth *hypersurface* in $\widetilde{X}$, and the restriction $\sigma : E \to Y$ is a holomorphic fiber bundle isomorphic to the projectivized normal bundle $P(NY) \to Y$.

ii) $\sigma : \widetilde{X} \setminus E \longrightarrow X \setminus Y$ is a biholomorphism.

In order to construct $\widetilde{X}$ and σ, we first define the set-theoretic underlying objects as the disjoint sums

$$\begin{aligned} \widetilde{X} &= (X \setminus Y) \amalg E, \qquad \text{where } E := P(NY), \\ \sigma &= \mathrm{Id}_{X\setminus Y} \amalg \pi, \qquad \text{where } \pi : E \longrightarrow Y. \end{aligned}$$

This means intuitively that we have replaced each point $y \in Y$ by the projective space of all directions normal to Y. When Y is reduced to a single point, the geometric picture is given by Fig. 1 below. In general, the picture is obtained by slicing X transversally to Y near each point and by blowing-up each slice at the intersection point with Y.

It remains to construct the manifold structure on $\widetilde{X}$ and in particular to describe what are the holomorphic functions near a point of E. Let f, g be holomorphic functions on an open set $U \subset X$ such that $f = g = 0$ on $Y \cap U$. Then df and dg vanish on $TY_{\restriction Y\cap U}$, hence df and dg induce linear forms on $NY_{\restriction Y\cap U}$. The holomorphic function $h(z) = f(z)/g(z)$ on the open set

$$U_g := \big\{z \in U\ ;\ g(z) \neq 0\big\} \subset U \setminus Y$$

can be extended in a natural way to a function $\widetilde{h}$ on the set

$$\widetilde{U}_g = U_g \cup \big\{(z, [\xi]) \in P(NY)_{\restriction Y\cap U}\ ;\ dg_z(\xi) \neq 0\big\} \subset \widetilde{X}$$

by letting

$$\widetilde{h}(z, [\xi]) = \frac{df_z(\xi)}{dg_z(\xi)}, \qquad (z, [\xi]) \in P(NY)_{\restriction Y\cap U}.$$

Fig. 1 Blow-up of one point in X.

Using this observation, we now define the manifold structure on $\widetilde{X}$ by giving explicitly an atlas. Every coordinate chart of $X \setminus Y$ is taken to be also a coordinate chart of $\widetilde{X}$. Furthermore, for every point $y_0 \in Y$, there exists a neighborhood U of y_0 in X and a coordinate chart $\tau(z) = (z_1, \ldots, z_n) : U \to \mathbb{C}^n$ centered at y_0 such that $\tau(U) = B' \times B''$ for some balls $B' \subset \mathbb{C}^s$, $B'' \subset \mathbb{C}^{n-s}$, and such that $Y \cap U = \tau^{-1}(\{0\} \times B'') = \{z_1 = \ldots = z_s = 0\}$. It follows that $(z_{s+1}, \ldots, z_n)$ are local coordinates on $Y \cap U$ and that the vector fields $(\partial/\partial z_1, \ldots, \partial/\partial z_s)$ yield a holomorphic frame of $NY_{\restriction Y\cap U}$. Let us denote by $(\xi_1, \ldots, \xi_s)$ the corresponding coordinates along the fibers of NY. Then $(\xi_1, \ldots, \xi_s, z_{s+1}, \ldots, z_n)$ are coordinates on the total space NY. For every $j = 1, \ldots, s$, we set

$$\widetilde{U}_j = \widetilde{U}_{z_j} = \big\{z \in U \setminus Y \ ; \ z_j \neq 0\big\} \cup \big\{(z, [\xi]) \in P(NY)_{\restriction Y\cap U} \ ; \ \xi_j \neq 0\big\}.$$

Then $(\widetilde{U}_j)_{1 \le j \le s}$ is a covering of $\widetilde{U} = \sigma^{-1}(U)$ and for each j we define a coordinate chart $\widetilde{\tau}_j = (w_1, \ldots, w_n) : \widetilde{U}_j \longrightarrow \mathbb{C}^n$ by

$$w_k := \Big(\frac{z_k}{z_j}\Big)^{\sim} \quad \text{for } \ 1 \le k \le s, k \neq j\,; \qquad w_k := z_k \quad \text{for } \ k > s \ \text{ or } \ k = j.$$

For $z \in U \setminus Y$, resp. $(z, [\xi]) \in P(NY)_{\restriction Y \cap U}$, we get

$$\widetilde{\tau}_j(z) = (w_1, \ldots, w_n) = \Big(\frac{z_1}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, z_j, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_s}{z_j}, z_{s+1}, \ldots, z_n\Big),$$

$$\widetilde{\tau}_j(z, [\xi]) = (w_1, \ldots, w_n) = \Big(\frac{\xi_1}{\xi_j}, \ldots, \frac{\xi_{j-1}}{\xi_j}, 0\,, \frac{\xi_{j+1}}{\xi_j}, \ldots, \frac{\xi_s}{\xi_j}, \xi_{s+1}, \ldots, \xi_n\Big).$$

With respect to the coordinates (w_k) on $\widetilde{U}_j$ and (z_k) on U, the map σ is given by

$$\text{(12.2)}\qquad \begin{aligned} \widetilde{U}_j &\xrightarrow{\ \sigma\ } U \\ w &\xmapsto{\ \sigma_j\ } (w_1 w_j, \ldots, w_{j-1} w_j\,;\, w_j\,;\, w_{j+1} w_j, \ldots, w_s w_j\,;\, w_{s+1}, \ldots, w_n) \end{aligned}$$

where $\sigma_j = \tau \circ \sigma \circ \widetilde{\tau}_j^{-1}$, thus σ is holomorphic. The range of the coordinate chart $\widetilde{\tau}_j$ is $\widetilde{\tau}_j(\widetilde{U}_j) = \sigma_j^{-1}\big(\tau(U)\big)$, so it is actually open in $\mathbb{C}^n$. Furthermore $E \cap \widetilde{U}_j$ is defined by the single equation $w_j = 0$, thus E is a smooth hypersurface in $\widetilde{X}$. It remains only to verify that the coordinate changes $w \longmapsto w'$ associated to any coordinate change $z \longmapsto z'$ on X are holomorphic. For that purpose, it is sufficient to verify that $(f/g)^{\sim}$ is holomorphic in $(w_1, \ldots, w_n)$ on $\widetilde{U}_j \cap \widetilde{U}_g$. As g vanishes on $Y \cap U$, we can write $g(z) = \sum_{1 \le k \le s} z_k A_k(z)$ for some holomorphic functions A_k on U. Therefore

$$\frac{g(z)}{z_j} = A_j(\sigma_j(w)) + \sum_{k \ne j} w_k A_k(\sigma_j(w))$$

has an extension $(g/z_j)^{\sim}$ to $\widetilde{U}_j$ which is a holomorphic function of the variables $(w_1, \ldots, w_n)$. Since $(g/z_j)^{\sim}(z, [\xi]) = dg_z(\xi)/\xi_j$ on $E \cap \widetilde{U}_j$, it is clear that

$$\widetilde{U}_j \cap \widetilde{U}_g = \big\{w \in \widetilde{U}_j\ ;\ (g/z_j)^{\sim}(w) \ne 0\big\}.$$

Hence $\widetilde{U}_j \cap \widetilde{U}_g$ is open in $\widetilde{U}_g$ and $(f/g)^{\sim} = (f/z_j)^{\sim}/(g/z_j)^{\sim}$ is holomorphic on $\widetilde{U}_j \cap \widetilde{U}_g$.

(12.3) Definition. *The map $\sigma : \widetilde{X} \to X$ is called the blow-up of X with center Y and $E = \sigma^{-1}(Y) \simeq P(NY)$ is called the exceptional divisor of $\widetilde{X}$.*

According to (V-13.5), we denote by $\mathcal{O}(E)$ the line bundle on $\widetilde{X}$ associated to the divisor E and by $h \in H^0(\widetilde{X}, \mathcal{O}(E))$ the canonical section such that $\mathrm{div}(h) = [E]$. On the other hand, we denote by $\mathcal{O}_{P(NY)}(-1) \subset \pi^\star(NY)$ the tautological line subbundle over $E = P(NY)$ such that the fiber above the point $(z, [\xi])$ is $\mathbb{C}\xi \subset N_z Y$.

(12.4) Proposition. $\mathcal{O}(E)$ *enjoys the following properties:*

a) $\mathcal{O}(E)_{\restriction E}$ *is isomorphic to* $\mathcal{O}_{P(NY)}(-1)$.

b) *Assume that* X *is compact. For every positive line bundle* L *over* X, *the line bundle* $\mathcal{O}(-E)\otimes\sigma^\star(L^k)$ *over* $\widetilde{X}$ *is positive for* $k>0$ *large enough.*

Proof. a) The canonical section $h\in H^0(\widetilde{X},\mathcal{O}(E))$ vanishes at order 1 along E, hence the kernel of its differential

$$dh:(T\widetilde{X})_{\restriction E}\longrightarrow\mathcal{O}(E)_{\restriction E}$$

is TE. We get therefore an isomorphism $NE\simeq\mathcal{O}(E)_{\restriction E}$. Now, the map $\sigma:\widetilde{X}\to X$ satisfies $\sigma(E)\subset Y$, so its differential $d\sigma:T\widetilde{X}\longrightarrow\sigma^\star(TX)$ is such that $d\sigma(TE)\subset\sigma^\star(TY)$. Therefore $d\sigma$ induces a morphism

$$(12.5)\qquad NE\longrightarrow\sigma^\star(NY)=\pi^\star(NY)$$

of vector bundles over E. The vector field $\partial/\partial w_j$ yields a non vanishing section of NE on $\widetilde{U}_j$, and (12.2) implies

$$d\sigma_j\Big(\frac{\partial}{\partial w_j}\Big)=\frac{\partial}{\partial z_j}+\sum_{1\le k\le s,k\ne j}w_k\frac{\partial}{\partial z_k}\quad /\!/\quad\sum_{1\le k\le s}\xi_k\frac{\partial}{\partial z_k}$$

at every point $(z,[\xi])\in E$. This shows that (12.5) is an isomorphism of NE onto $\mathcal{O}_{P(NY)}(-1)\subset\pi^\star(NY)$, hence

$$(12.6)\qquad\mathcal{O}(E)_{\restriction E}\simeq NE\simeq\mathcal{O}_{P(NY)}(-1).$$

b) Select an arbitrary hermitian metric on TX and consider the induced metrics on NY and on $\mathcal{O}_{P(NY)}(1)\longrightarrow E=P(NY)$. The restriction of $\mathcal{O}_{P(NY)}(1)$ to each fiber $P(N_zY)$ is the standard line bundle $\mathcal{O}(1)$ over $\mathbb{P}^{s-1}$; thus by (V-15.10) this restriction has a positive definite curvature form. Extend now the metric of $\mathcal{O}_{P(NY)}(1)$ on E to a metric of $\mathcal{O}(-E)$ on X in an arbitrary way. If $F=\mathcal{O}(-E)\otimes\sigma^\star(L^k)$, then $\Theta(F)=\Theta(\mathcal{O}(-E))+k\,\sigma^\star\Theta(L)$, thus for every $t\in T\widetilde{X}$ we have

$$\theta_F(t,t)=\theta_{\mathcal{O}(-E)}(t,t)+k\,\theta_L\big(d\sigma(t),d\sigma(t)\big).$$

By the compactness of the unitary tangent bundle to $\widetilde{X}$ and the positivity of θ_L, it is sufficient to verify that $\theta_{\mathcal{O}(-E)}(t,t)>0$ for every unit vector $t\in T_z\widetilde{X}$ such that $d\sigma(t)=0$. However, from the computations of a), this can only happen when $z\in E$ and $t\in TE$, and in that case $d\sigma(t)=d\pi(t)=0$, so t is tangent to the fiber $P(N_zY)$. Therefore

$$\theta_{\mathcal{O}(-E)}(t,t)=\theta_{\mathcal{O}_{P(NY)}(1)}(t,t)>0.\qquad\square$$

(12.7) Proposition. *The canonical line bundle of* $\widetilde{X}$ *is given by*

$$K_{\widetilde{X}} = \mathcal{O}\big((s-1)E\big) \otimes \sigma^\star K_X, \qquad where \ \ s = \mathrm{codim}_X Y.$$

Proof. K_X is generated on U by the holomorphic n-form $dz_1 \wedge \ldots \wedge dz_n$. Using (12.2), we see that $\sigma^\star K_X$ is generated on $\widetilde{U}_j$ by

$$\sigma^\star(dz_1 \wedge \ldots \wedge dz_n) = w_j^{s-1}\, dw_1 \wedge \ldots \wedge dw_n.$$

Since the divisor of the section $h \in H^0(\widetilde{X}, \mathcal{O}(E))$ is the hypersurface E defined by the equation $w_j = 0$ in $\widetilde{U}_j$, we have a well defined line bundle isomorphism

$$\sigma^\star K_X \longrightarrow \mathcal{O}\big((1-s)E\big) \otimes K_{\widetilde{X}}, \qquad \alpha \longmapsto h^{1-s}\sigma^\star(\alpha). \qquad \square$$

13. Equivalence of Positivity and Ampleness for Line Bundles

We have seen in section 11 that every ample line bundle carries a hermitian metric of positive curvature. The converse will be a consequence of the following result.

(13.1) Theorem. *Let $L \longrightarrow X$ be a positive line bundle and L^k the k-th tensor power of L. For every N-tuple $(x_1, \ldots, x_N)$ of distinct points of X, there exists a constant $C > 0$ such that the evaluation maps*

$$H^0(X, L^k) \longrightarrow (J^m L^k)_{x_1} \oplus \cdots \oplus (J^m L^k)_{x_N}$$

are surjective for all integers $m \geq 0$, $k \geq C(m+1)$.

(13.2) Lemma. *Let $\sigma : \widetilde{X} \longrightarrow X$ be the blow-up of X with center the finite set $Y = \{x_1, \ldots, x_N\}$, and let $\mathcal{O}(E)$ be the line bundle associated to the exceptional divisor E. Then*

$$H^1(\widetilde{X}, \mathcal{O}(-mE) \otimes \sigma^\star L^k) = 0$$

for $m \geq 1$, $k \geq Cm$ and $C \geq 0$ large enough.

Proof. By Prop. 12.7 we get $K_{\widetilde{X}} = \mathcal{O}\big((n-1)E\big) \otimes \sigma^\star K_X$ and

$$H^1\big(\widetilde{X}, \mathcal{O}(-mE) \otimes \sigma^\star L^k\big) = H^{n,1}\big(\widetilde{X}, K_{\widetilde{X}}^{-1} \otimes \mathcal{O}(-mE) \otimes \sigma^\star L^k\big) = H^{n,1}\big(\widetilde{X}, F\big)$$

where $F = \mathcal{O}\big(-(m+n-1)E\big) \otimes \sigma^\star(K_X^{-1} \otimes L^k)$, so the conclusion will follow from the Kodaira-Nakano vanishing theorem if we can show that $F > 0$ when k is large enough. Fix an arbitrary hermitian metric on K_X. Then

$$\Theta(F) = (m+n-1)\Theta(\mathcal{O}(-E)) + \sigma^\star\big(k\Theta(L) - \Theta(K_X)\big).$$

There is $k_0 \geq 0$ such that $\mathrm{i}\big(k_0\Theta(L) - \Theta(K_X)\big) > 0$ on X, and Prop. 12.4 implies the existence of $C_0 > 0$ such that $\mathrm{i}\big(\Theta(\mathcal{O}(-E)) + C_0\sigma^\star\Theta(L)\big) > 0$ on $\widetilde{X}$. Thus $\mathrm{i}\Theta(F) > 0$ for $m \geq 2 - n$ and $k \geq k_0 + C_0(m + n - 1)$. □

Proof of Theorem 13.1. Let $v_j \in H^0(\Omega_j, L^k)$ be a holomorphic section of L^k in a neighborhood Ω_j of x_j having a prescribed m-jet at x_j. Set

$$v(x) = \sum_j \psi_j(x) v_j(x)$$

where $\psi_j = 1$ in a neighborhood of x_j and ψ_j has compact support in Ω_j. Then $d''v = \sum d''\psi_j \cdot v_j$ vanishes in a neighborhood of $x_1, \ldots, x_N$. Let h be the canonical section of $\mathcal{O}(E)^{-1}$ such that $\mathrm{div}(h) = [E]$. The $(0,1)$-form $\sigma^\star d''v$ vanishes in a neighborhood of $E = h^{-1}(0)$, hence

$$w = h^{-(m+1)}\sigma^\star d''v \in C^\infty_{0,1}\big(\widetilde{X}, \mathcal{O}(-(m+1)E) \otimes \sigma^\star L^k\big).$$

and w is a d''-closed form. By Lemma 13.2 there exists a smooth section $u \in C^\infty_{0,0}\big(\widetilde{X}, \mathcal{O}(-(m+1)E) \otimes \sigma^\star L^k\big)$ such that $d''u = w = h^{-(m+1)}\sigma^\star d''v$. This implies

$$\sigma^\star v - h^{m+1}u \in H^0(\widetilde{X}, \sigma^\star L^k),$$

and since $\sigma^\star L$ is trivial near E, there exists a section $g \in H^0(X, L^k)$ such that $\sigma^\star g = \sigma^\star v - h^{m+1}u$. As h vanishes at order 1 along E, the m-jet of g at x_j must be equal to that of v (or v_j). □

(13.3) Corollary. *For any holomorphic line bundle $L \longrightarrow X$, the following conditions are equivalent:*

a) *L is ample;*

b) *$L > 0$, i.e. L possesses a hermitian metric such that* $\mathrm{i}\Theta(L) > 0$.

Proof. a) $\Longrightarrow$ b) is given by Cor. 11.12, whereas b) $\Longrightarrow$ a) is a consequence of Th. 13.1 for $m = 1$. □

14. Kodaira's Projectivity Criterion

The following fundamental projectivity criterion is due to (Kodaira 1954).

(14.1) Theorem. *Let X be a compact complex manifold,* $\dim_{\mathbb{C}} X = n$. *The following conditions are equivalent.*

a) *X is projective algebraic, i.e. X can be embedded as an algebraic submanifold of the complex projective space $\mathbb{P}^N$ for N large.*

b) *X carries a positive line bundle L.*

c) *X carries a Hodge metric, i.e. a Kähler metric ω with rational cohomology class $\{\omega\} \in H^2(X, \mathbb{Q})$.*

Proof. a) $\Longrightarrow$ b). Take $L = \mathcal{O}(1)_{\restriction X}$.

b) $\Longrightarrow$ c). Take $\omega = \frac{\mathrm{i}}{2\pi}\Theta(L)$; then $\{\omega\}$ is the image of $c_1(L) \in H^2(X, \mathbb{Z})$.

c) $\Longrightarrow$ b). We can multiply $\{\omega\}$ by a common denominator of its coefficients and suppose that $\{\omega\}$ is in the image of $H^2(X, \mathbb{Z})$. Then Th. V-13.9 b) shows that there exists a hermitian line bundle L such that $\frac{\mathrm{i}}{2\pi}\Theta(L) = \omega > 0$.

b) $\Longrightarrow$ a). Corollary 13.3 shows that $F = L^k$ is very ample for some integer $k > 0$. Then Prop. 11.9 enables us to find a subspace V of $H^0(X, F)$, $\dim V \leq 2n + 2$, such that $\psi_V : X \longrightarrow G_1(V) = P(V^\star)$ is an embedding. Thus X can be embedded in $\mathbb{P}^{2n+1}$ and Chow's theorem II-7.10 shows that the image is an algebraic set in $\mathbb{P}^{2n+1}$. □

(14.2) Remark. The above proof shows in particular that every n-dimensional projective manifold X can be embedded in $\mathbb{P}^{2n+1}$. This can be shown directly by using generic projections $\mathbb{P}^N \to \mathbb{P}^{2n+1}$ and Whitney type arguments as in 11.2.

(14.3) Corollary. *Every compact Riemann surface X is isomorphic to an algebraic curve in $\mathbb{P}^3$.*

Proof. Any positive smooth form ω of type $(1,1)$ is Kähler, and ω is in fact a Hodge metric if we normalize its volume so that $\int_X \omega = 1$. □

This example can be somewhat generalized as follows.

(14.4) Corollary. *Every Kähler manifold (X, ω) such that $H^2(X, \mathcal{O}) = 0$ is projective.*

Proof. By hypothesis $H^{0,2}(X, \mathbb{C}) = 0 = H^{2,0}(X, \mathbb{C})$, hence

$$H^2(X, \mathbb{C}) = H^{1,1}(X, \mathbb{C})$$

admits a basis $\{\alpha_1\}, \ldots, \{\alpha_N\} \in H^2(X, \mathbb{Q})$ where $\alpha_1, \ldots, \alpha_N$ are harmonic real $(1,1)$-forms. Since $\{\omega\}$ is real, we have $\{\omega\} = \lambda_1\{\alpha_1\} + \ldots + \lambda_N\{\alpha_N\}$, $\lambda_j \in \mathbb{R}$, thus

$$\omega = \lambda_1\alpha_1 + \ldots + \lambda_N\alpha_N$$

because ω itself is harmonic. If $\mu_1, \ldots, \mu_N$ are rational numbers sufficiently close to $\lambda_1, \ldots, \lambda_N$, then $\widetilde{\omega} := \mu_1\alpha_1 + \cdots \mu_N\alpha_N$ is close to ω, so $\widetilde{\omega}$ is a positive definite d-closed $(1,1)$-form, and $\{\widetilde{\omega}\} \in H^2(X, \mathbb{Q})$. □

We obtain now as a consequence the celebrated Riemann criterion characterizing *abelian varieties* (= projective algebraic complex tori).

(14.5) Corollary. *A complex torus $X = \mathbb{C}^n/\Gamma$ (Γ a lattice of $\mathbb{C}^n$) is an abelian variety if and only if there exists a positive definite hermitian form h on $\mathbb{C}^n$ such that*

$$\operatorname{Im}\big(h(\gamma_1,\gamma_2)\big) \in \mathbb{Z} \qquad \text{for all} \quad \gamma_1, \gamma_2 \in \Gamma.$$

Proof (Sufficiency of the condition). Set $\omega = -\operatorname{Im} h$. Then ω defines a constant Kähler metric on $\mathbb{C}^n$, hence also on $X = \mathbb{C}^n/\Gamma$. Let $(a_1, \ldots, a_{2n})$ be an integral basis of the lattice Γ. We denote by T_j, T_{jk} the real 1- and 2-tori

$$T_j = (\mathbb{R}/\mathbb{Z})a_j, \quad 1 \le j \le n, \qquad T_{jk} = T_j \oplus T_k, \quad 1 \le j < k \le 2n.$$

Topologically we have $X \approx T_1 \times \ldots \times T_{2n}$, so the Künneth formula IV-15.7 yields

$$H^\bullet(X,\mathbb{Z}) \simeq \bigotimes_{1\le j\le 2n} \big(H^0(T_j,\mathbb{Z}) \oplus H^1(T_j,\mathbb{Z})\big),$$
$$H^2(X,\mathbb{Z}) \simeq \bigoplus_{1\le j<k\le 2n} H^1(T_j,\mathbb{Z}) \otimes H^1(T_k,\mathbb{Z}) \simeq \bigoplus_{1\le j<k\le 2n} H^2(T_{jk},\mathbb{Z})$$

where the projection $H^2(X,\mathbb{Z}) \longrightarrow H^2(T_{jk},\mathbb{Z})$ is induced by the injection $T_{jk} \subset X$. In the identification $H^2(T_{jk},\mathbb{R}) \simeq \mathbb{R}$, we get

$$\text{(14.6)} \quad \{\omega\}_{\restriction T_{jk}} = \int_{T_{jk}} \omega = \omega(a_j, a_k) = -\operatorname{Im} h(a_j, a_k).$$

The assumption on h implies $\{\omega\}_{\restriction T_{jk}} \in H^2(T_{jk},\mathbb{Z})$ for all j,k, therefore $\{\omega\} \in H^2(X,\mathbb{Z})$ and X is projective by Th. (14.1).

Proof (Necessity of the condition). If X is projective, then X admits a Kähler metric ω such that $\{\omega\}$ is in the image of $H^2(X,\mathbb{Z})$. In general, ω is not invariant under the translations $\tau_x(y) = y - x$ of X. Therefore, we replace ω by its "mean value":

$$\widetilde{\omega} = \frac{1}{\operatorname{Vol}(X)} \int_{x\in X} (\tau_x^\star \omega)\, dx,$$

which has the same cohomology class as ω (τ_x is homotopic to the identity). Now $\widetilde{\omega}$ is the imaginary part of a constant positive definite hermitian form h on $\mathbb{C}^n$, and formula (14.6) shows that $\operatorname{Im} h(a_j, a_k) \in \mathbb{Z}$. □

(14.7) Example. Let X be a projective manifold. We shall prove that the Jacobian $\operatorname{Jac}(X)$ and the Albanese variety $\operatorname{Alb}(X)$ (cf. § VI-13 for definitions) are abelian varieties.

In fact, let ω be a Kähler metric on X such that $\{\omega\}$ is in the image of $H^2(X,\mathbb{Z})$ and let h be the hermitian metric on $H^1(X,\mathcal{O}) \simeq H^{0,1}(X,\mathbb{C})$ defined by

$$h(u,v) = \int_X -2\mathrm{i}\, u \wedge \overline{v} \wedge \omega^{n-1}$$

for all closed $(0,1)$-forms u, v. As

$$-2\mathrm{i}\, u \wedge \overline{v} \wedge \omega^{n-1} = \frac{2}{n}\, |u|^2\, \omega^n,$$

we see that h is a positive definite hermitian form on $H^{0,1}(X,\mathbb{C})$. Consider elements $\gamma_j \in H^1(X,\mathbb{Z})$, $j = 1,2$. If we write $\gamma_j = \gamma_j' + \gamma_j''$ in the decomposition $H^1(X,\mathbb{C}) = H^{1,0}(X,\mathbb{C}) \oplus H^{0,1}(X,\mathbb{C})$, we get

$$h(\gamma_1'',\gamma_2'') = \int_X -2\mathrm{i}\,\gamma_1'' \wedge \gamma_2' \wedge \omega^{n-1},$$

$$\operatorname{Im} h(\gamma_1'',\gamma_2'') = \int_X (\gamma_1' \wedge \gamma_2'' + \gamma_1'' \wedge \gamma_2') \wedge \omega^{n-1} = \int_X \gamma_1 \wedge \gamma_2 \wedge \omega^{n-1} \in \mathbb{Z}.$$

Therefore $\mathrm{Jac}(X)$ is an abelian variety.

Now, we observe that $H^{n-1,n}(X,\mathbb{C})$ is the anti-dual of $H^{0,1}(X,\mathbb{C})$ by Serre duality. We select on $H^{n-1,n}(X,\mathbb{C})$ the dual hermitian metric $h^\star$. Since the Poincaré bilinear pairing yields a unimodular bilinear map

$$H^1(X,\mathbb{Z}) \times H^{2n-1}(X,\mathbb{Z}) \longrightarrow \mathbb{Z},$$

we easily conclude that $\operatorname{Im} h^\star(\gamma_1'',\gamma_2'') \in \mathbb{Q}$ for all $\gamma_1, \gamma_2 \in H^{2n-1}(X,\mathbb{Z})$. Therefore $\mathrm{Alb}(X)$ is also an abelian variety.

Chapter VIII
L^2 Estimates on Pseudoconvex Manifolds

The main goal of this chapter is to show that the differential geometric technique that has been used in order to prove vanishing theorems also yields very precise L^2 estimates for the solutions of equations $d''u = v$ on pseudoconvex manifolds. The central idea, due to (Hörmander 1965), is to introduce weights of the type $e^{-\varphi}$ where φ is a function satisfying suitable convexity conditions. This method leads to generalizations of many standard vanishing theorems to weakly pseudoconvex manifolds. As a special case, we obtain the original Hörmander estimates for pseudoconvex domains of $\mathbb{C}^n$, and give some applications to algebraic geometry (Hörmander-Bombieri-Skoda theorem, properties of zero sets of polynomials in $\mathbb{C}^n$). We also derive the Ohsawa-Takegoshi extension theorem for L^2 holomorphic functions and Skoda's L^2 estimates for surjective bundle morphisms (Skoda 1972a, 1978, Demailly 1982c). Skoda's estimates can be used to obtain a quick solution of the Levi problem, and have important applications to local algebra and Nullstellensatz theorems. Finally, L^2 estimates are used to prove the Newlander-Nirenberg theorem on the analyticity of almost complex structures. We apply it to establish Kuranishi's theorem on deformation theory of compact complex manifolds.

1. Non Bounded Operators on Hilbert Spaces

A few preliminaries of functional analysis will be needed here. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be complex Hilbert spaces. We consider a linear operator T defined on a subspace $\mathrm{Dom}\,T \subset \mathcal{H}_1$ (called the domain of T) into $\mathcal{H}_2$. The operator T is said to be *densely defined* if $\mathrm{Dom}\,T$ is dense in $\mathcal{H}_1$, and *closed* if its graph

$$\mathrm{Gr}\,T = \big\{(x, Tx)\ ;\ x \in \mathrm{Dom}\,T\big\}$$

is closed in $\mathcal{H}_1 \times \mathcal{H}_2$.

Assume now that T is closed and densely defined. The adjoint $T^\star$ of T (in Von Neumann's sense) is constructed as follows: $\mathrm{Dom}\,T^\star$ is the set of $y \in \mathcal{H}_2$ such that the linear form

$$\mathrm{Dom}\,T \ni x \longmapsto \langle Tx, y\rangle_2$$

is bounded in $\mathcal{H}_1$-norm. Since $\mathrm{Dom}\,T$ is dense, there exists for every y in $\mathrm{Dom}\,T^\star$ a unique element $T^\star y \in \mathcal{H}_1$ such that $\langle Tx, y\rangle_2 = \langle x, T^\star y\rangle_1$ for all $x \in \mathrm{Dom}\,T^\star$. It is immediate to verify that $\mathrm{Gr}\,T^\star = \big(\mathrm{Gr}(-T)\big)^\perp$ in $\mathcal{H}_1 \times \mathcal{H}_2$.

It follows that $T^\star$ is closed and that every pair $(u,v) \in \mathcal{H}_1 \times \mathcal{H}_2$ can be written

$$(u,v) = (x, -Tx) + (T^\star y, y), \qquad x \in \mathrm{Dom}\, T, \quad y \in \mathrm{Dom}\, T^\star.$$

Take in particular $u = 0$. Then

$$x + T^\star y = 0, \qquad v = y - Tx = y + TT^\star y, \qquad \langle v, y\rangle_2 = \|y\|_2^2 + \|T^\star y\|_1^2.$$

If $v \in (\mathrm{Dom}\, T^\star)^\perp$ we get $\langle v, y\rangle_2 = 0$, thus $y = 0$ and $v = 0$. Therefore $T^\star$ is densely defined and our discussion implies:

(1.1) Theorem (Von Neumann 19??). *If $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a closed and densely defined operator, then its adjoint $T^\star$ is also closed and densely defined and $(T^\star)^\star = T$. Furthermore, we have the relation $\mathrm{Ker}\, T^\star = (\mathrm{Im}\, T)^\perp$ and its dual $(\mathrm{Ker}\, T)^\perp = \overline{\mathrm{Im}\, T^\star}$.* □

Consider now two closed and densely defined operators T, S :

$$\mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2 \xrightarrow{S} \mathcal{H}_3$$

such that $S \circ T = 0$. By this, we mean that the range $T(\mathrm{Dom}\, T)$ is contained in $\mathrm{Ker}\, S \subset \mathrm{Dom}\, S$, in such a way that there is no problem for defining the composition $S \circ T$. The starting point of all L^2 estimates is the following abstract existence theorem.

(1.2) Theorem. *There are orthogonal decompositions*

$$\begin{aligned} \mathcal{H}_2 &= (\mathrm{Ker}\, S \cap \mathrm{Ker}\, T^\star) \oplus \overline{\mathrm{Im}\, T} \oplus \overline{\mathrm{Im}\, S^\star}, \\ \mathrm{Ker}\, S &= (\mathrm{Ker}\, S \cap \mathrm{Ker}\, T^\star) \oplus \overline{\mathrm{Im}\, T}. \end{aligned}$$

In order that $\mathrm{Im}\, T = \mathrm{Ker}\, S$, it suffices that

$$(1.3) \qquad \|T^\star x\|_1^2 + \|Sx\|_3^2 \geq C\|x\|_2^2, \qquad \forall x \in \mathrm{Dom}\, S \cap \mathrm{Dom}\, T^\star$$

for some constant $C > 0$. In that case, for every $v \in \mathcal{H}_2$ such that $Sv = 0$, there exists $u \in \mathcal{H}_1$ such that $Tu = v$ and

$$\|u\|_1^2 \leq \frac{1}{C}\|v\|_2^2.$$

In particular

$$\overline{\mathrm{Im}\, T} = \mathrm{Im}\, T = \mathrm{Ker}\, S, \qquad \overline{\mathrm{Im}\, S^\star} = \mathrm{Im}\, S^\star = \mathrm{Ker}\, T^\star.$$

Proof. Since S is closed, the kernel $\mathrm{Ker}\, S$ is closed in $\mathcal{H}_2$. The relation $(\mathrm{Ker}\, S)^\perp = \overline{\mathrm{Im}\, S^\star}$ implies

$$(1.4) \qquad \mathcal{H}_2 = \mathrm{Ker}\, S \oplus \overline{\mathrm{Im}\, S^\star}$$

and similarly $\mathcal{H}_2 = \operatorname{Ker} T^\star \oplus \overline{\operatorname{Im} T}$. However, the assumption $S \circ T = 0$ shows that $\overline{\operatorname{Im} T} \subset \operatorname{Ker} S$, therefore

$$\operatorname{Ker} S = (\operatorname{Ker} S \cap \operatorname{Ker} T^\star) \oplus \overline{\operatorname{Im} T}. \tag{1.5}$$

The first two equalities in Th. 1.2 are then equivalent to the conjunction of (1.4) and (1.5).

Now, under assumption (1.3), we are going to show that the equation $Tu = v$ is always solvable if $Sv = 0$. Let $x \in \operatorname{Dom} T^\star$. One can write

$$x = x' + x'' \quad \text{where} \quad x' \in \operatorname{Ker} S \quad \text{and} \quad x'' \in (\operatorname{Ker} S)^\perp \subset (\operatorname{Im} T)^\perp = \operatorname{Ker} T^\star.$$

Since $x, x'' \in \operatorname{Dom} T^\star$, we have also $x' \in \operatorname{Dom} T^\star$. We get

$$\langle v, x\rangle_2 = \langle v, x'\rangle_2 + \langle v, x''\rangle_2 = \langle v, x'\rangle_2$$

because $v \in \operatorname{Ker} S$ and $x'' \in (\operatorname{Ker} S)^\perp$. As $Sx' = 0$ and $T^\star x'' = 0$, the Cauchy-Schwarz inequality combined with (1.3) implies

$$|\langle v, x\rangle_2|^2 \le \|v\|_2^2\ \|x'\|_2^2 \le \frac{1}{C}\|v\|_2^2\ \|T^\star x'\|_1^2 = \frac{1}{C}\|v\|_2^2\ \|T^\star x\|_1^2.$$

This shows that the linear form $T^\star_X \ni x \longmapsto \langle x, v\rangle_2$ is continuous on $\operatorname{Im} T^\star \subset \mathcal{H}_1$ with norm $\le C^{-1/2}\|v\|_2$. By the Hahn-Banach theorem, this form can be extended to a continuous linear form on $\mathcal{H}_1$ of norm $\le C^{-1/2}\|v\|_2$, i.e. we can find $u \in \mathcal{H}_1$ such that $\|u\|_1 \le C^{-1/2}\|v\|_2$ and

$$\langle x, v\rangle_2 = \langle T^\star x, u\rangle_1, \qquad \forall x \in \operatorname{Dom} T^\star.$$

This means that $u \in \operatorname{Dom}(T^\star)^\star = \operatorname{Dom} T$ and $v = Tu$. We have thus shown that $\operatorname{Im} T = \operatorname{Ker} S$, in particular $\operatorname{Im} T$ is closed. The dual equality $\operatorname{Im} S^\star = \operatorname{Ker} T^\star$ follows by considering the dual pair $(S^\star, T^\star)$. □

2. Complete Riemannian Manifolds

Let (M, g) be a riemannian manifold of dimension m, with metric

$$g(x) = \sum g_{jk}(x)\, dx_j \otimes dx_k, \qquad 1 \le j, k \le m.$$

The length of a path $\gamma : [a, b] \longrightarrow M$ is by definition

$$\ell(\gamma) = \int_a^b |\gamma'(t)|_g dt = \int_a^b \Big(\sum_{j,k} g_{jk}\big(\gamma(t)\big)\, \gamma'_j(t)\gamma'_k(t)\Big)^{1/2} dt.$$

The geodesic distance of two points $x, y \in M$ is

$$\delta(x, y) = \inf_\gamma \ell(\gamma) \qquad \text{over paths } \gamma \text{ with } \ \gamma(a) = x, \quad \gamma(b) = y,$$

if x, y are in the same connected component of M, $\delta(x,y) = +\infty$ otherwise. It is easy to check that δ satisfies the usual axioms of distances: for the separation axiom, use the fact that if y is outside some closed coordinate ball $\overline{B}$ of radius r centered at x and if $g \geq c|dx|^2$ on $\overline{B}$, then $\delta(x,y) \geq c^{1/2}r$. In addition, δ satisfies the axiom:

$$\text{(2.1)} \quad \text{for every } x, y \in M, \qquad \inf_{z\in M} \max\{\delta(x,z), \delta(y,z)\} = \frac{1}{2}\delta(x,y).$$

In fact for every $\varepsilon > 0$ there is a path γ such that $\gamma(a) = x$, $\gamma(b) = y$, $\ell(\gamma) < \delta(x,y) + \varepsilon$ and we can take z to be at mid-distance between x and y along γ. A metric space E with a distance δ satisfying the additional axiom (2.1) will be called a *geodesic* metric space. It is then easy to see by dichotomy that any two points $x, y \in E$ can be joined by a chain of points $x = x_0,\ x_1, \ldots, x_N = y$ such that $\delta(x_j, x_{j+1}) < \varepsilon$ and $\sum \delta(x_j, x_{j+1}) < \delta(x,y) + \varepsilon$.

(2.2) Lemma (Hopf-Rinow)**.** *Let (E,δ) be a geodesic metric space. Then the following properties are equivalent:*

a) *E is locally compact and complete;*

b) *all closed geodesic balls $\overline{B}(x_0,r)$ are compact.*

Proof. Since any Cauchy sequence is bounded, it is immediate that b) implies a). We now check that a) $\Longrightarrow$ b). Fix x_0 and define R to be the supremum of all $r > 0$ such that $\overline{B}(x_0,r)$ is compact. Since E is locally compact, we have $R > 0$. Suppose that $R < +\infty$. Then $\overline{B}(x_0,r)$ is compact for every $r < R$. Let y_ν be a sequence of points in $\overline{B}(x_0,R)$. Fix an integer p. As $\delta(x_0,y_\nu) \leq R$, axiom (2.1) shows that we can find points $z_\nu \in M$ such that $\delta(x_0,z_\nu) \leq (1-2^{-p})R$ and $\delta(z_\nu,y_\nu) \leq 2^{1-p}R$. Since $\overline{B}(x_0,(1-2^{-p})R)$ is compact, there is a subsequence $(z_{\nu(p,q)})_{q\in\mathbb{N}}$ converging to a limit point w_p with $\delta(z_{\nu(p,q)}, w_p) \leq 2^{-q}$. We proceed by induction on p and take $\nu(p+1,q)$ to be a subsequence of $\nu(p,q)$. Then

$$\delta(y_{\nu(p,q)}, w_p) \leq \delta(y_{\nu(p,q)}, z_{\nu(p,q)}) + \delta(z_{\nu(p,q)}, w_p) \leq 2^{1-p}R + 2^{-q}.$$

Since $(y_{\nu(p+1,q)})$ is a subsequence of $(y_{\nu(p,q)})$, we infer from this that $\delta(w_p, w_{p+1}) \leq 3\,2^{-p}R$ by letting q tend to $+\infty$. By the completeness hypothesis, the Cauchy sequence (w_p) converges to a limit point $w \in M$, and the above inequalities show that $(y_{\nu(p,p)})$ converges to $w \in \overline{B}(x_0,R)$. Therefore $\overline{B}(x_0,R)$ is compact. Now, each point $y \in \overline{B}(x_0,R)$ can be covered by a compact ball $\overline{B}(y,\varepsilon_y)$, and the compact set $\overline{B}(x_0,R)$ admits a finite covering by concentric balls $B(y_j,\varepsilon_{y_j}/2)$. Set $\varepsilon = \min \varepsilon_{y_j}$. Every point $z \in \overline{B}(x_0, R+\varepsilon/2)$ is at distance $\leq \varepsilon/2$ of some point $y \in \overline{B}(x_0,R)$, hence at distance $\leq \varepsilon/2 + \varepsilon_{y_j}/2$ of some point y_j, in particular $\overline{B}(x_0, R+\varepsilon/2) \subset \bigcup \overline{B}(y_j,\varepsilon_{y_j})$ is compact. This is a contradiction, so $R = +\infty$. $\square$

The following standard definitions and properties will be useful in order to deal with the completeness of the metric.

(2.3) Definitions.

a) *A riemannian manifold* (M, g) *is said to be complete if* (M, δ) *is complete as a metric space.*

b) *A continuous function* $\psi : M \to \mathbb{R}$ *is said to be exhaustive if for every* $c \in \mathbb{R}$ *the sublevel set* $M_c = \{x \in M \;;\; \psi(x) < c\}$ *is relatively compact in* M*.*

c) *A sequence* $(K_\nu)_{\nu \in \mathbb{N}}$ *of compact subsets of* M *is said to be exhaustive if* $M = \bigcup K_\nu$ *and if* K_ν *is contained in the interior of* $K_{\nu+1}$ *for all* ν *(so that every compact subset of* M *is contained in some* K_ν*).*

(2.4) Lemma. *The following properties are equivalent:*

a) (M, g) *is complete;*

b) *there exists an exhaustive function* $\psi \in C^\infty(M, \mathbb{R})$ *such that* $|d\psi|_g \le 1$*;*

c) *there exists an exhaustive sequence* $(K_\nu)_{\nu \in \mathbb{N}}$ *of compact subsets of* M *and functions* $\psi_\nu \in C^\infty(M, \mathbb{R})$ *such that*

$$\begin{aligned} &\psi_\nu = 1 \quad \text{in a neighborhood of } K_\nu, \qquad \operatorname{Supp} \psi_\nu \subset K^\circ_{\nu+1}, \\ &0 \le \psi_\nu \le 1 \quad \text{and} \quad |d\psi_\nu|_g \le 2^{-\nu}. \end{aligned}$$

Proof. a) $\Longrightarrow$ b). Without loss of generality, we may assume that M is connected. Select a point $x_0 \in M$ and set $\psi_0(x) = \frac{1}{2}\delta(x_0, x)$. Then ψ_0 is a Lipschitz function with constant $\frac{1}{2}$, thus ψ_0 is differentiable almost everywhere on M and $|d\psi_0|_g \le \frac{1}{2}$. We can find a smoothing ψ of ψ_0 such that $|d\psi|_g \le 1$ and $|\psi - \psi_0| \le 1$. Then ψ is an exhaustion function of M.

b) $\Longrightarrow$ c). Choose ψ as in a) and a function $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\rho = 1$ on $]-\infty, 1.1]$, $\rho = 0$ on $[1.9, +\infty[$ and $0 \le \rho' \le 2$ on $[1, 2]$. Then

$$K_\nu = \{x \in M \;;\; \psi(x) \le 2^{\nu+1}\}, \qquad \psi_\nu(x) = \rho\big(2^{-\nu-1}\psi(x)\big)$$

satisfy our requirements.

c) $\Longrightarrow$ b). Set $\psi = \sum 2^\nu(1 - \psi_\nu)$.

b) $\Longrightarrow$ a). The inequality $|d\psi|_g \le 1$ implies $|\psi(x) - \psi(y)| \le \delta(x, y)$ for all $x, y \in M$, so all δ-balls must be relatively compact in M. $\square$

3. L^2 Hodge Theory on Complete Riemannian Manifolds

Let (M,g) be a riemannian manifold and let F_1, F_2 be hermitian C^∞ vector bundles over M. If $P : C^\infty(M,F_1) \longrightarrow C^\infty(M,F_2)$ is a differential operator with smooth coefficients, then P induces a non bounded operator

$$\widetilde{P} : L^2(M,F_1) \longrightarrow L^2(M,F_2)$$

as follows: if $u \in L^2(M,F_1)$, we compute $\widetilde{P}u$ in the sense of distribution theory and we say that $u \in \operatorname{Dom}\widetilde{P}$ if $\widetilde{P}u \in L^2(M,F_2)$. It follows that $\widetilde{P}$ is densely defined, since $\operatorname{Dom}P$ contains the set $\mathcal{D}(M,F_1)$ of compactly supported sections of $C^\infty(M,F_1)$, which is dense in $L^2(M,F_1)$. Furthermore $\operatorname{Gr}\widetilde{P}$ is closed: if $u_\nu \to u$ in $L^2(M,F_1)$ and $\widetilde{P}u_\nu \to v$ in $L^2(M,F_2)$ then $\widetilde{P}u_\nu \to \widetilde{P}u$ in the weak topology of distributions, thus we must have $\widetilde{P}u = v$ and $(u,v) \in \operatorname{Gr}\widetilde{P}$. By the general results of § 1, we see that $\widetilde{P}$ has a closed and densely defined Von Neumann adjoint $\big(\widetilde{P}\big)^\star$. We want to stress, however, that $\big(\widetilde{P}\big)^\star$ *does not always* coincide with the extension $(P^\star)^\sim$ of the formal adjoint $P^\star : C^\infty(M,F_2) \longrightarrow C^\infty(M,F_1)$, computed in the sense of distribution theory. In fact $u \in \operatorname{Dom}(\widetilde{P})^\star$, resp. $u \in \operatorname{Dom}(P^\star)^\sim$, if and only if there is an element $v \in L^2(M,F_1)$ such that $\langle u, \widetilde{P}f\rangle = \langle v, f\rangle$ for all $f \in \operatorname{Dom}\widetilde{P}$, resp. for all $f \in \mathcal{D}(M,F_1)$. Therefore we always have $\operatorname{Dom}(\widetilde{P})^\star \subset \operatorname{Dom}(P^\star)^\sim$ and the inclusion may be strict because the integration by parts to perform may involve boundary integrals for $(\widetilde{P})^\star$.

(3.1) Example. Consider

$$P = \frac{d}{dx} : L^2\big(]0,1[\big) \longrightarrow L^2\big(]0,1[\big)$$

where the L^2 space is taken with respect to the Lebesgue measure dx. Then $\operatorname{Dom}\widetilde{P}$ consists of all L^2 functions with L^2 derivatives on $]0,1[$. Such functions have a continuous extension to the interval $[0,1]$. An integration by parts shows that

$$\int_0^1 u\overline{\frac{df}{dx}}\,dx = \int_0^1 -\frac{du}{dx}\overline{f}\,dx$$

for all $f \in \mathcal{D}(]0,1[)$, thus $P^\star = -d/dx = -P$. However for $f \in \operatorname{Dom}\widetilde{P}$ the integration by parts involves the extra term $u(1)\overline{f}(1) - u(0)\overline{f}(0)$ in the right hand side, which is thus continuous in f with respect to the L^2 topology if and only if $du/dx \in L^2$ and $u(0) = u(1) = 0$. Therefore $\operatorname{Dom}(\widetilde{P})^\star$ consists of all $u \in \operatorname{Dom}(P^\star)^\sim = \operatorname{Dom}\widetilde{P}$ satisfying the additional boundary condition $u(0) = u(1) = 0$. □

Let $E \to M$ be a differentiable hermitian bundle. In what follows, we still denote by $D, \delta, \varDelta$ the differential operators of § VI-2 extended in the sense of distribution theory (as explained above). These operators are thus closed and densely defined operators on $L^2_\bullet(M,E)=\bigoplus_p L^2_p(M,E)$. We also introduce the space $\mathcal{D}_p(M,E)$ of compactly supported forms in $C^\infty_p(M,E)$. The theory relies heavily on the following important result.

(3.2) Theorem. *Assume that (M,g) is complete. Then*

a) *$\mathcal{D}_\bullet(M,E)$ is dense in $\mathrm{Dom}\,D$, $\mathrm{Dom}\,\delta$ and $\mathrm{Dom}\,D\cap\mathrm{Dom}\,\delta$ respectively for the graph norms*

$$u\mapsto \|u\|+\|Du\|,\qquad u\mapsto\|u\|+\|\delta u\|,\qquad u\mapsto\|u\|+\|Du\|+\|\delta u\|.$$

b) *$D^\star=\delta$, $\delta^\star=D$ as adjoint operators in Von Neumann's sense.*

c) *One has $\langle u,\Delta u\rangle=\|Du\|^2+\|\delta u\|^2$ for every $u\in\mathrm{Dom}\,\Delta$. In particular*

$$\mathrm{Dom}\,\Delta\subset\mathrm{Dom}\,D\cap\mathrm{Dom}\,\delta,\qquad \mathrm{Ker}\,\Delta=\mathrm{Ker}\,D\cap\mathrm{Ker}\,\delta,$$

and Δ is self-adjoint.

d) *If $D^2=0$, there are orthogonal decompositions*

$$\begin{aligned}L^2_\bullet(M,E)&=\mathcal{H}^\bullet(M,E)\oplus\overline{\mathrm{Im}\,D}\oplus\overline{\mathrm{Im}\,\delta},\\ \mathrm{Ker}\,D&=\mathcal{H}^\bullet(M,E)\oplus\overline{\mathrm{Im}\,D},\end{aligned}$$

where $\mathcal{H}^\bullet(M,E)=\big\{u\in L^2_\bullet(M,E)\,;\ \Delta u=0\big\}\subset C^\infty_\bullet(M,E)$ is the space of L^2 harmonic forms.

Proof. a) We show that every element $u\in\mathrm{Dom}\,D$ can be approximated in the graph norm of D by smooth and compactly supported forms. By hypothesis, u and Du belong to $L^2_\bullet(M,E)$. Let (ψ_ν) be a sequence of functions as in Lemma 2.4 c). Then $\psi_\nu u\to u$ in $L^2_\bullet(M,E)$ and $D(\psi_\nu u)=\psi_\nu Du+d\psi_\nu\wedge u$ where

$$|d\psi_\nu\wedge u|\le|d\psi_\nu|\;|u|\le 2^{-\nu}|u|.$$

Therefore $d\psi_\nu\wedge u\to0$ and $D(\psi_\nu u)\to Du$. After replacing u by $\psi_\nu u$, we may assume that u has compact support, and by using a finite partition of unity on a neighborhood of $\mathrm{Supp}\,u$ we may also assume that $\mathrm{Supp}\,u$ is contained in a coordinate chart of M on which E is trivial. Let A be the connection form of D on this chart and (ρ_ε) a family of smoothing kernels. Then $u\star\rho_\varepsilon\in\mathcal{D}_\bullet(M,E)$ converges to u in $L^2(M,E)$ and

$$D(u\star\rho_\varepsilon)-(Du)\star\rho_\varepsilon=A\wedge(u\star\rho_\varepsilon)-(A\wedge u)\star\rho_\varepsilon$$

because d commutes with convolution (as any differential operator with constant coefficients). Moreover $(Du)\star\rho_\varepsilon$ converges to Du in $L^2(M,E)$ and $A\wedge(u\star\rho_\varepsilon)$, $(A\wedge u)\star\rho_\varepsilon$ both converge to $A\wedge u$ since $A\wedge\bullet$ acts continuously on L^2. Thus $D(u\star\rho_\varepsilon)$ converges to Du and the density of $\mathcal{D}_\bullet(M,E)$ in $\mathrm{Dom}\,D$ follows. The proof for $\mathrm{Dom}\,\delta$ and $\mathrm{Dom}\,D\cap\mathrm{Dom}\,\delta$ is similar, except that the principal part of δ no longer has constant coefficients in general. The convolution technique requires in this case the following lemma due to K.O. Friedrichs (see e.g. Hörmander 1963).

(3.3) Lemma. *Let $Pf = \sum a_k \, \partial f / \partial x_k + bf$ be a differential operator of order 1 on an open set $\Omega \subset \mathbb{R}^n$, with coefficients $a_k \in C^1(\Omega)$, $b \in C^0(\Omega)$. Then for any $v \in L^2(\mathbb{R}^n)$ with compact support in Ω we have*

$$\lim_{\varepsilon \to 0} ||P(v \star \rho_\varepsilon) - (Pv) \star \rho_\varepsilon||_{L^2} = 0.$$

Proof. It is enough to consider the case when $P = a\partial/\partial x_k$. As the result is obvious if $v \in C^1$, we only have to show that

$$||P(v \star \rho_\varepsilon) - (Pv) \star \rho_\varepsilon||_{L^2} \le C||v||_{L^2}$$

and to use a density argument. A computation of $w_\varepsilon = P(v \star \rho_\varepsilon) - (Pv) \star \rho_\varepsilon$ by means of an integration by parts gives

$$\begin{aligned} w_\varepsilon(x) &= \int_{\mathbb{R}^n} \Big(a(x) \frac{\partial v}{\partial x_k}(x - \varepsilon y)\rho(y) - a(x - \varepsilon y)\frac{\partial v}{\partial x_k}(x - \varepsilon y)\rho(y)\Big) dy \\ &= \int_{\mathbb{R}^n} \Big(\big(a(x) - a(x - \varepsilon y)\big) v(x - \varepsilon y) \frac{1}{\varepsilon} \partial_k \rho(y) \\ &\qquad\qquad + \partial_k a(x - \varepsilon y) v(x - \varepsilon y)\rho(y)\Big) dy. \end{aligned}$$

If C is a bound for $|da|$ in a neighborhood of Supp v, we get

$$|w_\varepsilon(x)| \le C \int_{\mathbb{R}^n} |v(x - \varepsilon y)| \big(|y| \, |\partial_k \rho(y)| + |\rho(y)|\big) dy,$$

so Minkowski's inequality $||f \star g||_{L^p} \le ||f||_{L^1} ||g||_{L^p}$ gives

$$||w_\varepsilon||_{L^2} \le C \Big(\int_{\mathbb{R}^n} \big(|y| \, |\partial_k \rho(y)| + |\rho(y)|\big) dy \Big) ||v||_{L^2}. \qquad \square$$

Proof (end). b) is equivalent to the fact that

$$\langle\!\langle Du, v \rangle\!\rangle = \langle\!\langle u, \delta v \rangle\!\rangle, \qquad \forall u \in \operatorname{Dom} D, \quad \forall v \in \operatorname{Dom} \delta.$$

By a), we can find $u_\nu, \ v_\nu \in \mathcal{D}_\bullet(M, E)$ such that

$$u_\nu \to u, \qquad v_\nu \to v, \qquad Du_\nu \to Du \quad \text{ and } \quad \delta v_\nu \to \delta v \quad \text{ in } \; L^2_\bullet(M, E),$$

and the required equality is the limit of the equalities $\langle\!\langle Du_\nu, v_\nu \rangle\!\rangle = \langle\!\langle u_\nu, \delta v_\nu \rangle\!\rangle$.

c) Let $u \in \operatorname{Dom} \Delta$. As Δ is an elliptic operator of order 2, u must be in $W^2_\bullet(M, E, \mathrm{loc})$ by Gårding's inequality. In particular $Du, \ \delta u \in L^2(M, E, \mathrm{loc})$

and we can perform all integrations by parts that we want if the forms are multiplied by compactly supported functions ψ_ν. Let us compute

$$\begin{aligned}
&\|\psi_\nu Du\|^2 + \|\psi_\nu \delta u\|^2 = \\
&= \langle\!\langle \psi_\nu^2 Du, Du\rangle\!\rangle + \langle\!\langle u, D(\psi_\nu^2 \delta u)\rangle\!\rangle \\
&= \langle\!\langle D(\psi_\nu^2 u), Du\rangle\!\rangle + \langle\!\langle u, \psi_\nu^2 D\delta u\rangle\!\rangle - 2\langle\!\langle \psi_\nu d\psi_\nu \wedge u, Du\rangle\!\rangle + 2\langle\!\langle u, \psi_\nu d\psi_\nu \wedge \delta u\rangle\!\rangle \\
&= \langle\!\langle \psi_\nu^2 u, \Delta u\rangle\!\rangle - 2\langle\!\langle d\psi_\nu \wedge u, \psi_\nu Du\rangle\!\rangle + 2\langle\!\langle u, d\psi_\nu \wedge (\psi_\nu \delta u)\rangle\!\rangle \\
&\le \langle\!\langle \psi_\nu^2 u, \Delta u\rangle\!\rangle + 2^{-\nu}\big(2\|\psi_\nu Du\|\,\|u\| + 2\|\psi_\nu \delta u\|\,\|u\|\big) \\
&\le \langle\!\langle \psi_\nu^2 u, \Delta u\rangle\!\rangle + 2^{-\nu}\big(\|\psi_\nu Du\|^2 + \|\psi_\nu \delta u\|^2 + 2\|u\|^2\big).
\end{aligned}$$

We get therefore

$$\|\psi_\nu Du\|^2 + \|\psi_\nu \delta u\|^2 \le \frac{1}{1-2^{-\nu}}\big(\langle\!\langle \psi_\nu^2 u, \Delta u\rangle\!\rangle + 2^{1-\nu}\|u\|^2\big).$$

By letting ν tend to $+\infty$, we obtain $\|Du\|^2 + \|\delta u\|^2 \le \langle\!\langle u, \Delta u\rangle\!\rangle$, in particular Du, δu are in $L^2_\bullet(M,E)$. This implies

$$\langle\!\langle u, \Delta v\rangle\!\rangle = \langle\!\langle Du, Dv\rangle\!\rangle + \langle\!\langle \delta u, \delta v\rangle\!\rangle, \qquad \forall u, v \in \operatorname{Dom}\Delta,$$

because the equality holds for $\psi_\nu u$ and v, and because we have $\psi_\nu u \to u$, $D(\psi_\nu u) \to Du$ and $\delta(\psi_\nu u) \to \delta u$ in L^2. Therefore Δ is self-adjoint.

d) is an immediate consequence of b), c) and Th. 1.2. □

On a complete hermitian manifold (X,ω), there are of course similar results for the operators $D', D'', \delta', \delta'', \Delta', \Delta''$ attached to a hermitian vector bundle E.

4. General Estimate for d'' on Hermitian Manifolds

Let (X,ω) be a *complete* hermitian manifold and E a hermitian holomorphic vector bundle of rank r over X. Assume that the hermitian operator

(4.1) $\quad A_{E,\omega} = [\mathrm{i}\Theta(E), \Lambda] + T_\omega$

is *semi-positive* on $\Lambda^{p,q}T_X^\star \otimes E$. Then for every form $u \in \operatorname{Dom} D'' \cap \operatorname{Dom}\delta''$ of bidegree (p,q) we have

(4.2) $\quad \|D''u\|^2 + \|\delta''u\|^2 \ge \displaystyle\int_X \langle A_{E,\omega}u, u\rangle\, dV.$

In fact (4.2) is true for all $u \in \mathcal{D}_{p,q}(X,E)$ in view of the Bochner-Kodaira-Nakano identity VII-2.3, and this result is easily extended to every u in $\operatorname{Dom} D'' \cap \operatorname{Dom}\delta''$ by density of $\mathcal{D}_{p,q}(X,E)$ (Th. 3.2 a)).

Assume now that a form $g \in L^2_{p,q}(X,E)$ is given such that

(4.3) $\quad D''g = 0,$

and that for almost every $x \in X$ there exists $\alpha \in [0, +\infty[$ such that

$$|\langle g(x), u\rangle|^2 \le \alpha \langle A_{E,\omega} u, u\rangle$$

for every $u \in (\Lambda^{p,q} T_X^\star \otimes E)_x$. If the operator $A_{E,\omega}$ is invertible, the minimal such number α is $|A_{E,\omega}^{-1/2} g(x)|^2 = \langle A_{E,\omega}^{-1} g(x), g(x)\rangle$, so we shall always denote it in this way even when $A_{E,\omega}$ is no longer invertible. Assume furthermore that

(4.4) $$\int_X \langle A_{E,\omega}^{-1} g, g\rangle \, dV < +\infty.$$

The basic result of L^2 theory can be stated as follows.

(4.5) Theorem. *If (X, ω) is complete and $A_{E,\omega} \ge 0$ in bidegree (p, q), then for any $g \in L^2_{p,q}(X, E)$ satisfying (4.4) such that $D''g = 0$ there exists $f \in L^2_{p,q-1}(X, E)$ such that $D''f = g$ and*

$$\|f\|^2 \le \int_X \langle A_{E,\omega}^{-1} g, g\rangle \, dV.$$

Proof. For every $u \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$ we have

$$\begin{aligned} \big|\langle\!\langle u, g\rangle\!\rangle\big|^2 = \Big|\int_X \langle u, g\rangle \, dV\Big|^2 &\le \Big(\int_X \langle A_{E,\omega} u, u\rangle^{1/2} \langle A_{E,\omega}^{-1} g, g\rangle^{1/2} \, dV\Big)^2 \\ &\le \int_X \langle A_{E,\omega}^{-1} g, g\rangle \, dV \cdot \int_X \langle A_{E,\omega} u, u\rangle \, dV \end{aligned}$$

by means of the Cauchy-Schwarz inequality. The a priori estimate (4.2) implies

$$\big|\langle\!\langle u, g\rangle\!\rangle\big|^2 \le C\big(\|D''u\|^2 + \|\delta''u\|^2\big), \qquad \forall u \in \operatorname{Dom} D'' \cap \operatorname{Dom} \delta''$$

where C is the integral (4.4). Now we just have to repeat the proof of the existence part of Th. 1.2. For any $u \in \operatorname{Dom} \delta''$, let us write

$$u = u_1 + u_2, \qquad u_1 \in \operatorname{Ker} D'', \qquad u_2 \in (\operatorname{Ker} D'')^\perp = \overline{\operatorname{Im} \delta''}.$$

Then $D''u_1 = 0$ and $\delta''u_2 = 0$. Since $g \in \operatorname{Ker} D''$, we get

$$\big|\langle\!\langle u, g\rangle\!\rangle\big|^2 = \big|\langle\!\langle u_1, g\rangle\!\rangle\big|^2 \le C\|\delta''u_1\|^2 = C\|\delta''u\|^2.$$

The Hahn-Banach theorem shows that the continuous linear form

$$L^2_{p,q-1}(X, E) \ni \delta''u \longmapsto \langle\!\langle u, g\rangle\!\rangle$$

can be extended to a linear form $v \longmapsto \langle\!\langle v, f\rangle\!\rangle$, $f \in L^2_{p,q-1}(X, E)$, of norm $\|f\| \le C^{1/2}$. This means that

$$\langle\!\langle u, g\rangle\!\rangle = \langle\!\langle \delta'' u, f\rangle\!\rangle, \qquad \forall u \in \operatorname{Dom}\delta'',$$

i.e. that $D''f = g$. The theorem is proved. □

(4.6) Remark. One can always find a solution $f \in (\operatorname{Ker} D'')^\perp$: otherwise replace f by its orthogonal projection on $(\operatorname{Ker} D'')^\perp$. This solution is clearly unique and is precisely the solution of minimal L^2 norm of the equation $D''f = g$. We have $f \in \overline{\operatorname{Im}\delta''}$, thus f satifies the additional equation

$$(4.7) \quad \delta'' f = 0.$$

Consequently $\Delta'' f = \delta'' D'' f = \delta'' g$. If $g \in C^\infty_{p,q}(X, E)$, the ellipticity of Δ'' shows that $f \in C^\infty_{p,q-1}(X, E)$.

(4.8) Remark. If $A_{E,\omega}$ is positive definite, let $\lambda(x) > 0$ be the smallest eigenvalue of this operator at $x \in X$. Then λ is continuous on X and we have

$$\int_X \langle A^{-1}_{E,\omega} g, g\rangle \, dV \le \int_X \lambda(x)^{-1} |g(x)|^2 \, dV.$$

The above situation occurs for example if ω is complete Kähler, $E >_m 0$ and $p = n$, $q \ge 1$, $m \ge \min\{n - q + 1, r\}$ (apply Lemma VII-7.2).

5. Estimates on Weakly Pseudoconvex Manifolds

We first introduce a large class of complex manifolds on which the L^2 estimates will be easily tractable.

(5.1) Definition. *A complex manifold X is said to be weakly pseudoconvex if there exists an exhaustion function $\psi \in C^\infty(X, \mathbb{R})$ such that $\mathrm{i}d'd''\psi \ge 0$ on X, i.e. ψ is plurisubharmonic.*

For domains $\Omega \subset \mathbb{C}^n$, the above weak pseudoconvexity notion is equivalent to pseudoconvexity (cf. Th. I-4.14). Note that every compact manifold is also weakly pseudoconvex (take $\psi \equiv 0$). Other examples that will appear later are Stein manifolds, or the total space of a Griffiths semi-negative vector bundle over a compact manifold (cf. Prop. IX-?.?).

(5.2) Theorem. *Every weakly pseudoconvex Kähler manifold (X, ω) carries a complete Kähler metric $\widehat{\omega}$.*

Proof. Let $\psi \in C^\infty(X,\mathbb{R})$ be an exhaustive plurisubharmonic function on X. After addition of a constant to ψ, we can assume $\psi \geq 0$. Then $\widehat{\omega} = \omega + \mathrm{i}d'd''(\psi^2)$ is Kähler and

$$\widehat{\omega} = \omega + 2i\psi d'd''\psi + 2\mathrm{i}d'\psi\wedge d''\psi \geq \omega + 2\mathrm{i}d'\psi \wedge d''\psi.$$

Since $d\psi = d'\psi + d''\psi$, we get $|d\psi|_{\widehat{\omega}} = \sqrt{2}|d'\psi|_{\widehat{\omega}} \leq 1$ and Lemma 2.4 shows that $\widehat{\omega}$ is complete. □

Observe that we could have set more generally $\widehat{\omega} = \omega + \mathrm{i}d'd''(\chi\circ\psi)$ where χ is a convex increasing function. Then

$$\begin{aligned} \widehat{\omega} &= \omega + i(\chi'\circ\psi)d'd''\psi + i(\chi''\circ\psi)d'\psi\wedge d''\psi \\ &\geq \omega + \mathrm{i}d'(\rho\circ\psi)\wedge d''(\rho\circ\psi) \end{aligned} \tag{5.3}$$

where $\rho(t) = \int_0^t \sqrt{\chi''(u)}\,du$. We thus have $|d'(\rho\circ\psi)|_{\widehat{\omega}} \leq 1$ and $\widehat{\omega}$ will be complete as soon as $\lim_{t\to+\infty}\rho(t) = +\infty$, i.e.

$$\int_0^{+\infty} \sqrt{\chi''(u)}\,du = +\infty. \tag{5.4}$$

One can take for example $\chi(t) = t - \log(t)$ for $t \geq 1$.

It follows from the above considerations that almost all vanishing theorems for positive vector bundles over compact manifolds are also valid on weakly pseudoconvex manifolds. Let us mention here the analogues of some results proved in Chapter 7.

(5.5) Theorem. *For any m-positive vector bundle of rank r over a weakly pseudoconvex manifold X, we have $H^{n,q}(X,E) = 0$ for all $q \geq 1$ and $m \geq \min\{n-q+1, r\}$.*

Proof. The curvature form $\mathrm{i}\Theta(\det E)$ is a Kähler metric on X, hence X possesses a complete Kähler metric ω. Let $\psi \in C^\infty(X,\mathbb{R})$ be an exhaustive plurisubharmonic function. For any convex increasing function $\chi \in C^\infty(\mathbb{R},\mathbb{R})$, we denote by E_χ the holomorphic vector bundle E together with the modified metric $|u|_\chi^2 = |u|^2 \exp\big(-\chi\circ\psi(x)\big)$, $u \in E_x$. We get

$$\mathrm{i}\Theta(E_\chi) = \mathrm{i}\Theta(E) + \mathrm{i}d'd''(\chi\circ\psi)\otimes \mathrm{Id}_E \geq_m \mathrm{i}\Theta(E),$$

thus $A_{E_\chi,\omega} \geq A_{E,\omega} > 0$ in bidegree (n,q). Let g be a given form of bidegree (n,q) with L^2_{loc} coefficients, such that $D''g = 0$. The integrals

$$\int_X \langle A_{E_\chi,\omega}^{-1} g, g\rangle_\chi\, dV \leq \int_X \langle A_{E,\omega}^{-1} g, g\rangle\, e^{-\chi\circ\psi}\, dV, \qquad \int_X |g|^2\, e^{-\chi\circ\psi}\, dV$$

become convergent if χ grows fast enough. We can thus apply Th. 4.5 to (X, E_χ, ω) and find a $(n,q-1)$ form f such that $D''f = g$. If g is smooth, Remark 4.6 shows that f can also be chosen smooth. □

(5.6) Theorem. *If E is a positive line bundle over a weakly pseudoconvex manifold X, then $H^{p,q}(X,E) = 0$ for $p+q \ge n+1$.*

Proof. The proof is similar to that of Th. 5.5, except that we use here the Kähler metric

$$\omega_\chi = \mathrm{i}\Theta(E_\chi) = \omega + \mathrm{i}d'd''(\chi\circ\psi), \qquad \omega = \mathrm{i}\Theta(E),$$

which depends on χ. By (5.4) ω_χ is complete as soon as χ is a convex increasing function that grows fast enough. Apply now Th. 4.5 to (X, E_χ, ω_χ) and observe that $A_{E_\chi,\omega_\chi} = [\mathrm{i}\Theta(E_\chi), \Lambda_\chi] = (p+q-n)\ \mathrm{Id}$ in bidegree (p,q) in virtue of Cor. VI-8.4 It remains to show that for every form $g \in C^\infty_{p,q}(X,E)$ there exists a choice of χ such that $g \in L^2_{p,q}(X, E_\chi, \omega_\chi)$. By (5.3) the norm of a scalar form with respect to ω_χ is less than its norm with respect to ω, hence $|g|^2_\chi \le |g|^2 \exp(-\chi\circ\psi)$. On the other hand

$$dV_\chi \le C\big(1 + \chi'\circ\psi + \chi''\circ\psi\big)^n\, dV$$

where C is a positive continuous function on X. The following lemma implies that we can always choose χ in order that the integral of $|g|^2_\chi\, dV_\chi$ converges on X.

(5.7) Lemma. *For any positive function $\lambda \in C^\infty\big([0,+\infty[,\mathbb{R}\big)$, there exists a smooth convex function $\chi \in C^\infty\big([0,+\infty[,\mathbb{R}\big)$ such that $\chi, \chi', \chi'' \ge \lambda$ and $(1+\chi'+\chi'')^n e^{-\chi} \le 1/\lambda$.*

Proof. We shall construct χ such that $\chi'' \ge \chi' \ge \chi \ge \lambda$ and $\chi''/\chi^2 \le C$ for some constant C. Then χ satisties the conclusion of the lemma after addition of a constant. Without loss of generality, we may assume that λ is increasing and $\lambda \ge 1$. We define χ as a power series

$$\chi(t) = \sum_{k=0}^{+\infty} a_0 a_1 \ldots a_k\, t^k,$$

where $a_k > 0$ is a decreasing sequence converging to 0 very slowly. Then χ is real analytic on $\mathbb{R}$ and the inequalities $\chi'' \ge \chi' \ge \chi$ are realized if we choose $a_k \ge 1/k$, $k \ge 1$. Select a strictly increasing sequence of integers $(N_p)_{p\ge1}$ so large that $\frac{1}{p}\lambda(p+1)^{1/N_p} \in [1/p, 1/(p-1)[$. We set

$$a_0 = \ldots = a_{N_1-1} = e\,\lambda(2),$$
$$a_k = \frac{1}{p}\,\lambda(p+1)^{1/N_p}\, e^{1/\sqrt{k}}, \qquad N_p \le k < N_{p+1}.$$

Then (a_k) is decreasing. For $t \in [0,1]$ we have $\chi(t) \ge a_0 \ge \lambda(t)$ and for $t \in [1,+\infty[$ the choice $k = N_p$ where $p = [t]$ is the integer part of t gives

$$\chi(t) \ge \chi(p) \ge (a_0 a_1 \ldots a_k)p^k \ge (a_k p)^k \ge \lambda(p+1) \ge \lambda(t).$$

Furthermore, we have

$$\chi(t)^2 \geq \sum_{k\geq 0} (a_0a_1\ldots a_k)^2\, t^{2k},$$
$$\chi''(t) = \sum_{k\geq 0} (k+1)(k+2)\, a_0a_1\ldots a_{k+2}\, t^k,$$

thus we will get $\chi''(t) \leq C\chi(t)^2$ if we can prove that

$$m^2\, a_0a_1\ldots a_{2m} \leq C'(a_0a_1\ldots a_m)^2, \qquad m \geq 0.$$

However, as $\frac{1}{p}\lambda(p+1)^{1/N_p}$ is decreasing, we find

$$\begin{aligned}\frac{a_0a_1\ldots a_{2m}}{(a_0a_1\ldots a_m)^2} &= \frac{a_{m+1}\ldots a_{2m}}{a_0a_1\ldots a_m}\\ &\leq \exp\Big(\frac{1}{\sqrt{m+1}}+\cdots+\frac{1}{\sqrt{2m}}-\frac{1}{\sqrt{1}}-\cdots-\frac{1}{\sqrt{m}}+O(1)\Big)\\ &\leq \exp\big(2\sqrt{2m}-4\sqrt{m}+O(1)\big) \leq C'm^{-2}. \qquad \square\end{aligned}$$

As a last application, we generalize the Girbau vanishing theorem in the case of weakly pseudoconvex manifolds. This result is due to (Abdelkader 1980) and (Ohsawa 1981). We present here a simplified proof which appeared in (Demailly 1985).

(5.8) Theorem. *Let (X,ω) be a weakly pseudoconvex Kähler manifold. If E is a semi-positive line bundle such that $\mathrm{i}\Theta(E)$ has at least $n-s+1$ positive eigenvalues at every point, then*

$$H^{p,q}(X,E) = 0 \qquad \text{for} \quad p+q \geq n+s.$$

Proof. Let $\chi, \rho \in C^\infty(\mathbb{R},\mathbb{R})$ be convex increasing functions to be specified later. We use here the *hermitian* metric

$$\begin{aligned}\alpha &= \mathrm{i}\Theta(E_\chi) + \exp(-\rho\circ\psi)\,\omega\\ &= \mathrm{i}\Theta(E) + \mathrm{i}d'd''(\chi\circ\psi) + \exp(-\rho\circ\psi)\,\omega.\end{aligned}$$

Although ω is Kähler, the metric α is not so. Denote by $\gamma_j^{\chi,\omega}$ (resp. $\gamma_j^{\chi,\alpha}$), $1 \leq j \leq n$, the eigenvalues of $\mathrm{i}\Theta(E_\chi)$ with respect to ω (resp. α), rearranged in increasing order. The minimax principle implies $\gamma_j^{\chi,\omega} \geq \gamma_j^{0,\omega}$, and the hypothesis yields $0 < \gamma_s^{0,\omega} \leq \gamma_{s+1}^{0,\omega} \leq \ldots \leq \gamma_n^{0,\omega}$ on X. By means of a diagonalization of $\mathrm{i}\Theta(E_\chi)$ with respect to ω, we find

$$1 \geq \gamma_j^{\chi,\alpha} = \frac{\gamma_j^{\chi,\omega}}{\gamma_j^{\chi,\omega}+\exp(-\rho\circ\psi)} \geq \frac{\gamma_j^{0,\omega}}{\gamma_j^{0,\omega}+\exp(-\rho\circ\psi)}.$$

Let $\varepsilon > 0$ be small. Select ρ such that $\exp(-\rho\circ\psi(x)) \le \varepsilon\gamma_s^{0,\omega}(x)$ at every point. Then for $j \ge s$ we get

$$\gamma_j^{\chi,\alpha} \ge \frac{\gamma_j^{0,\omega}}{\gamma_j^{0,\omega}+\varepsilon\gamma_j^{0,\omega}} = \frac{1}{1+\varepsilon} \ge 1-\varepsilon,$$

and Th. VI-8.3 implies

$$\begin{aligned}\langle[\mathrm{i}\Theta(E_\chi),\Lambda_\alpha]u,u\rangle_\alpha &\ge \big(\gamma_1^{\chi,\alpha}+\cdots+\gamma_p^{\chi,\alpha}-\gamma_{q+1}^{\chi,\alpha}-\ldots-\gamma_n^{\chi,\alpha}\big)|u|^2\\ &\ge \big((p-s+1)(1-\varepsilon)-(n-q)\big)|u|^2\\ &\ge \big(1-(p-s+1)\varepsilon\big)|u|^2.\end{aligned}$$

It remains however to control the torsion term T_α. As ω is Kähler, trivial computations yield

$$\begin{aligned}d'\alpha &= -\rho'\circ\psi\ \exp(-\rho\circ\psi)\ d'\psi\wedge\omega,\\ d'd''\alpha &= \exp(-\rho\circ\psi)\ \big[\big((\rho'\circ\psi)^2-\rho''\circ\psi\big)d'\psi\wedge d''\psi-\rho'\circ\psi\ d'd''\psi\big]\wedge\omega.\end{aligned}$$

Since

$$\alpha \ge i(\chi'\circ\psi\ d'd''\psi+\chi''\circ\psi\ d'\psi\wedge d''\psi)+\exp(-\rho\circ\psi)\omega,$$

we get the upper bounds

$$\begin{aligned}|d'\alpha|_\alpha &\le \rho'\circ\psi\ |d'\psi|_\alpha\ |\exp(-\rho\circ\psi)\omega|_\alpha \le \rho'\circ\psi\ (\chi''\circ\psi)^{-\frac12}\\ |d'd''\alpha|_\alpha &\le \frac{(\rho'\circ\psi)^2+\rho''\circ\psi}{\chi''\circ\psi}+\frac{\rho'\circ\psi}{\chi'\circ\psi}.\end{aligned}$$

It is then clear that we can choose χ growing sufficiently fast in order that $|T_\alpha|_\alpha \le \varepsilon$. If ε is chosen sufficiently small, we get $A_{E_\chi,\alpha} \ge \frac12\,\mathrm{Id}$, and the conclusion is obtained in the same way as for Th. 5.6. □

6. Hörmander's Estimates for non Complete Kähler Metrics

Our aim here is to derive also estimates for a non complete Kähler metric, for example the standard metric of $\mathbb{C}^n$ on a bounded domain $\Omega \subset\subset \mathbb{C}^n$. A result of this type can be obtained in the situation described at the end of Remark 4.8. The underlying idea is due to (Hörmander 1966), although we do not apply his so called "three weights" technique, but use instead an approximation of the given metric ω by complete Kähler metrics.

(6.1) Theorem. *Let $(X,\widehat\omega)$ be a complete Kähler manifold, ω another Kähler metric, possibly non complete, and $E \longrightarrow X$ a m-semi-positive vector bundle. Let $g \in L^2_{n,q}(X,E)$ be such that $D''g = 0$ and*

$$\int_X \langle A_q^{-1} g, g\rangle\, dV < +\infty$$

with respect to ω, *where* A_q *stands for the operator* $\mathrm{i}\Theta(E) \wedge \Lambda$ *in bidegree* (n,q) *and* $q \geq 1$, $m \geq \min\{n-q+1, r\}$. *Then there exists* $f \in L^2_{n,q-1}(X,E)$ *such that* $D''f = g$ *and*

$$\|f\|^2 \leq \int_X \langle A_q^{-1} g, g\rangle\, dV.$$

Proof. For every $\varepsilon > 0$, the Kähler metric

$$\omega_\varepsilon = \omega + \varepsilon\widehat{\omega}$$

is complete. The idea of the proof is to apply the L^2 estimates to ω_ε and to let ε tend to zero. Let us put an index ε to all objects depending on ω_ε. It follows from Lemma 6.3 below that

$$(6.2)\quad |u|^2_\varepsilon\, dV_\varepsilon \leq |u|^2\, dV, \qquad \langle A_{q,\varepsilon}^{-1} u, u\rangle_\varepsilon\, dV_\varepsilon \leq \langle A_q^{-1} u, u\rangle\, dV$$

for every $u \in \Lambda^{n,q} T^\star_X \otimes E$. If these estimates are taken for granted, Th. 4.5 applied to ω_ε yields a section $f_\varepsilon \in L^2_{n,q-1}(X,E)$ such that $D'' f_\varepsilon = g$ and

$$\int_X |f_\varepsilon|^2_\varepsilon\, dV_\varepsilon \leq \int_X \langle A_{q,\varepsilon}^{-1} g, g\rangle_\varepsilon\, dV_\varepsilon \leq \int_X \langle A_q^{-1} g, g\rangle\, dV.$$

This implies that the family (f_ε) is bounded in L^2 norm on every compact subset of X. We can thus find a weakly convergent subsequence (f_{ε_ν}) in L^2_{loc}. The weak limit f is the solution we are looking for. □

(6.3) Lemma. *Let* ω, γ *be hermitian metrics on* X *such that* $\gamma \geq \omega$. *For every* $u \in \Lambda^{n,q} T^\star_X \otimes E$, $q \geq 1$, *we have*

$$|u|^2_\gamma\, dV_\gamma \leq |u|^2\, dV, \qquad \langle A_{q,\gamma}^{-1} u, u\rangle_\gamma\, dV_\gamma \leq \langle A_q^{-1} u, u\rangle\, dV$$

where an index γ *means that the corresponding term is computed in terms of* γ *instead of* ω.

Proof. Let $x_0 \in X$ be a given point and $(z_1, \ldots, z_n)$ coordinates such that

$$\omega = \mathrm{i} \sum_{1 \leq j \leq n} dz_j \wedge d\overline{z}_j, \qquad \gamma = \mathrm{i} \sum_{1 \leq j \leq n} \gamma_j\, dz_j \wedge d\overline{z}_j \quad \text{at} \quad x_0,$$

where $\gamma_1 \leq \ldots \leq \gamma_n$ are the eigenvalues of γ with respect to ω (thus $\gamma_j \geq 1$). We have $|dz_j|^2_\gamma = \gamma_j^{-1}$ and $|dz_K|^2_\gamma = \gamma_K^{-1}$ for any multi-index K, with the notation $\gamma_K = \prod_{j \in K} \gamma_j$. For every $u = \sum u_{K,\lambda} dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_K \otimes e_\lambda$, $|K| = q$, $1 \leq \lambda \leq r$, the computations of § VII-7 yield

$$
\begin{aligned}
|u|_\gamma^2 &= \sum_{K,\lambda} (\gamma_1 \ldots \gamma_n)^{-1} \gamma_K^{-1} \, |u_{K,\lambda}|^2, \qquad dV_\gamma = \gamma_1 \ldots \gamma_n \, dV, \\
|u|_\gamma^2 \, dV_\gamma &= \sum_{K,\lambda} \gamma_K^{-1} \, |u_{K,\lambda}|^2 \, dV \le |u|^2 \, dV, \\
\Lambda_\gamma u &= \sum_{|I|=q-1} \sum_{j,\lambda} \mathrm{i}(-1)^{n+j-1} \gamma_j^{-1} \, u_{jI,\lambda} \, (\widehat{dz_j}) \wedge d\overline{z}_I \otimes e_\lambda,
\end{aligned}
$$

where $(\widehat{dz_j})$ means $dz_1 \wedge \ldots \widehat{dz_j} \ldots \wedge dz_n$,

$$
\begin{aligned}
A_{q,\gamma} u &= \sum_{|I|=q-1} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} \, c_{jk\lambda\mu} \, u_{jI,\lambda} \, dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_{kI} \otimes e_\mu, \\
\langle A_{q,\gamma} u, u \rangle_\gamma &= (\gamma_1 \ldots \gamma_n)^{-1} \sum_{|I|=q-1} \gamma_I^{-1} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} \gamma_k^{-1} c_{jk\lambda\mu} \, u_{jI,\lambda} \overline{u}_{kI,\mu} \\
&\ge (\gamma_1 \ldots \gamma_n)^{-1} \sum_{|I|=q-1} \gamma_I^{-2} \sum_{j,k,\lambda,\mu} \gamma_j^{-1} \gamma_k^{-1} c_{jk\lambda\mu} \, u_{jI,\lambda} \overline{u}_{kI,\mu} \\
&= \gamma_1 \ldots \gamma_n \, \langle A_q S_\gamma u, S_\gamma u \rangle
\end{aligned}
$$

where S_γ is the operator defined by

$$
S_\gamma u = \sum_K (\gamma_1 \ldots \gamma_n \gamma_K)^{-1} \, u_{K,\lambda} \, dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_K \otimes e_\lambda.
$$

We get therefore

$$
\begin{aligned}
|\langle u, v \rangle_\gamma|^2 = |\langle u, S_\gamma v \rangle|^2 &\le \langle A_q^{-1} u, u \rangle \langle A_q S_\gamma v, S_\gamma v \rangle \\
&\le (\gamma_1 \ldots \gamma_n)^{-1} \langle A_q^{-1} u, u \rangle \langle A_{q,\gamma} v, v \rangle_\gamma,
\end{aligned}
$$

and the choice $v = A_{q,\gamma}^{-1} u$ implies

$$
\langle A_{q,\gamma}^{-1} u, u \rangle_\gamma \le (\gamma_1 \ldots \gamma_n)^{-1} \, \langle A_q^{-1} u, u \rangle \, ;
$$

this relation is equivalent to the last one in the lemma. □

An important special case is that of a semi-positive line bundle E. If we let $0 \le \lambda_1(x) \le \ldots \le \lambda_n(x)$ be the eigenvalues of $\mathrm{i}\Theta(E)_x$ with respect to ω_x for all $x \in X$, formula VI-8.3 implies

$$
\begin{aligned}
&\langle A_q u, u \rangle \ge (\lambda_1 + \cdots + \lambda_q) |u|^2, \\
(6.4) \qquad &\int_X \langle A_q^{-1} g, g \rangle \, dV \le \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} \, |g|^2 \, dV.
\end{aligned}
$$

A typical situation where these estimates can be applied is the case when E is the trivial line bundle $X \times \mathbb{C}$ with metric given by a weight $e^{-\varphi}$. One can assume for example that φ is plurisubharmonic and that $\mathrm{i}d'd''\varphi$ has at least $n - q + 1$ positive eigenvalues at every point, i.e. $\lambda_q > 0$ on X. This situation leads to very important L^2 estimates, which are precisely those given by (Hörmander 1965, 1966). We state here a slightly more general result.

(6.5) Theorem. *Let (X,ω) be a weakly pseudoconvex Kähler manifold, E a hermitian line bundle on X, $\varphi \in C^\infty(X,\mathbb{R})$ a weight function such that the eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ of $\mathrm{i}\Theta(E) + \mathrm{i}d'd''\varphi$ are ≥ 0. Then for every form g of type (n,q), $q \ge 1$, with L^2_{loc} (resp. C^∞) coefficients such that $D''g = 0$ and*

$$\int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |g|^2 e^{-\varphi}\, dV < +\infty,$$

we can find a L^2_{loc} (resp. C^∞) form f of type $(n,q-1)$ such that $D''f = g$ and

$$\int_X |f|^2 e^{-\varphi}\, dV \le \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |g|^2 e^{-\varphi}\, dV.$$

Proof. Apply the general estimates to the bundle E_φ deduced from E by multiplication of the metric by $e^{-\varphi}$; we have $\mathrm{i}\Theta(E_\varphi) = \mathrm{i}\Theta(E) + \mathrm{i}d'd''\varphi$. It is not necessary here to assume in addition that $g \in L^2_{n,q}(X, E_\varphi)$. In fact, g is in L^2_{loc} and we can exhaust X by the relatively compact weakly pseudoconvex domains

$$X_c = \big\{x \in X \ ;\ \psi(x) < c\big\}$$

where $\psi \in C^\infty(X,\mathbb{R})$ is a plurisubharmonic exhaustion function (note that $-\log(c-\psi)$ is also such a function on X_c). We get therefore solutions f_c on X_c with uniform L^2 bounds; any weak limit f gives the desired solution. $\square$

If estimates for (p,q)-forms instead of (n,q)-forms are needed, one can invoke the isomorphism $\Lambda^p T^\star_X \simeq \Lambda^{n-p} T_X \otimes \Lambda^n T^\star_X$ (obtained through contraction of n-forms by $(n-p)$-vectors) to get

$$\Lambda^{p,q} T^\star_X \otimes E \simeq \Lambda^{n,q} T^\star_X \otimes F, \qquad F = E \otimes \Lambda^{n-p} T_X.$$

Let us look more carefully to the case $p = 0$. The $(1,1)$-curvature form of $\Lambda^n T_X$ with respect to a hermitian metric ω on T_X is called the *Ricci curvature* of ω. We denote:

(6.6) Definition. $\mathrm{Ricci}(\omega) = \mathrm{i}\Theta(\Lambda^n T_X) = \mathrm{i}\ \mathrm{Tr}\ \Theta(T_X)$.

For any local coordinate system $(z_1,\ldots,z_n)$, the holomorphic n-form $dz_1 \wedge \ldots \wedge dz_n$ is a section of $\Lambda^n T^\star_X$, hence Formula V-13.3 implies

$$\text{(6.7)}\quad \mathrm{Ricci}(\omega) = \mathrm{i}d'd'' \log |dz_1 \wedge \ldots \wedge dz_n|^2_\omega = -\mathrm{i}d'd'' \log \det(\omega_{jk}).$$

The estimates of Th. 6.5 can therefore be applied to any $(0,q)$-form g, but $\lambda_1 \le \ldots \le \lambda_n$ must be replaced by the eigenvalues of the $(1,1)$-form

$$\mathrm{i}\Theta(E) + \mathrm{Ricci}(\omega) + \mathrm{i}d'd''\varphi \qquad (\text{supposed } \ge 0). \tag{6.8}$$

We consider now domains $\Omega \subset \mathbb{C}^n$ equipped with the euclidean metric of $\mathbb{C}^n$, and the trivial bundle $E = \Omega \times \mathbb{C}$. The following result is especially convenient because it requires only weak plurisubharmonicity and avoids to compute the curvature eigenvalues.

(6.9) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex open subset and φ an upper semi-continuous plurisubharmonic function on Ω. For every $\varepsilon \in \,]0,1]$ and every $g \in L^2_{p,q}(\Omega, \mathrm{loc})$ such that $d''g = 0$ and*

$$\int_\Omega \big(1+|z|^2\big)|g|^2\, e^{-\varphi} dV < +\infty,$$

we can find a L^2_{loc} form f of type $(p,q-1)$ such that $d''f = g$ and

$$\int_\Omega \big(1+|z|^2\big)^{-\varepsilon}\,|f|^2\, e^{-\varphi}\, dV \le \frac{4}{q\varepsilon^2}\int_\Omega \big(1+|z|^2\big)|g|^2\, e^{-\varphi}\, dV < +\infty.$$

Moreover f can be chosen smooth if g and φ are smooth.

Proof. Since $\Lambda^p T\Omega$ is a trivial bundle with trivial metric, the proof is immediately reduced to the case $p = 0$ (or equivalently $p = n$). Let us first suppose that φ is smooth. We replace φ by $\Phi = \varphi + \tau$ where

$$\tau(z) = \log\big(1 + (1+|z|^2)^\varepsilon\big).$$

(6.10) Lemma. *The smallest eigenvalue $\lambda_1(z)$ of $\mathrm{i}d'd''\tau(z)$ satisfies*

$$\lambda_1(z) \ge \frac{\varepsilon^2}{2(1+|z|^2)\big(1+(1+|z|^2)^\varepsilon\big)}.$$

In fact a brute force computation of the complex hessian $H\tau_z(\xi)$ and the Cauchy-Schwarz inequality yield

$$\begin{aligned}
&H\tau_z(\xi) = \\
&= \frac{\varepsilon(1+|z|^2)^{\varepsilon-1}|\xi|^2}{1+(1+|z|^2)^\varepsilon} + \frac{\varepsilon(\varepsilon-1)(1+|z|^2)^{\varepsilon-2}|\langle\xi,z\rangle|^2}{1+(1+|z|^2)^\varepsilon} - \frac{\varepsilon^2(1+|z|^2)^{2\varepsilon-2}|\langle\xi,z\rangle|^2}{\big(1+(1+|z|^2)^\varepsilon\big)^2} \\
&\ge \varepsilon\left(\frac{(1+|z|^2)^{\varepsilon-1}}{1+(1+|z|^2)^\varepsilon} - \frac{(1-\varepsilon)(1+|z|^2)^{\varepsilon-2}|z|^2}{1+(1+|z|^2)^\varepsilon} - \frac{\varepsilon(1+|z|^2)^{2\varepsilon-2}|z|^2}{\big(1+(1+|z|^2)^\varepsilon\big)^2}\right)|\xi|^2 \\
&= \varepsilon\,\frac{1+\varepsilon|z|^2+(1+|z|^2)^\varepsilon}{(1+|z|^2)^{2-\varepsilon}\big(1+(1+|z|^2)^\varepsilon\big)^2}\,|\xi|^2 \ge \frac{\varepsilon^2|\xi|^2}{(1+|z|^2)^{1-\varepsilon}\big(1+(1+|z|^2)^\varepsilon\big)^2} \\
&\ge \frac{\varepsilon^2}{2(1+|z|^2)\big(1+(1+|z|^2)^\varepsilon\big)}\,|\xi|^2. \qquad \square
\end{aligned}$$

The Lemma implies $e^{-\tau}/\lambda_1 \le 2(1+|z|^2)/\varepsilon^2$, thus Cor. 6.5 provides an f such that

$$\int_\Omega \big(1+(1+|z|^2)^\varepsilon\big)^{-1}\,|f|^2\,e^{-\varphi}\,dV \le \frac{2}{q\varepsilon^2}\int_\Omega \big(1+|z|^2\big)|g|^2\,e^{-\varphi}\,dV < +\infty,$$

and the required estimate follows. If φ is not smooth, apply the result to a sequence of regularized weights $\rho_\varepsilon \star \varphi \ge \varphi$ on an increasing sequence of domains $\Omega_c \subset\subset \Omega$, and extract a weakly convergent subsequence of solutions. □

7. Extension of Holomorphic Functions from Subvarieties

The existence theorems for solutions of the d'' operator easily lead to an extension theorem for sections of a holomorphic line bundle defined in a neighborhood of an analytic subset. The following result (Demailly 1982) is an improvement and a generalization of Jennane's extension theorem (Jennane 1976).

(7.1) Theorem. *Let (X,ω) be a weakly pseudoconvex Kähler manifold, L a hermitian line bundle and E a hermitian vector bundle over X. Let Y be an analytic subset of X such that $Y=\sigma^{-1}(0)$ for some section σ of E, and p the maximal codimension of the irreducible components of Y. Let f be a holomorphic section of $K_X\otimes L$ defined in the open set $U\supset Y$ of points $x\in X$ such that $|\sigma(x)|<1$. If $\int_U |f|^2 dV < +\infty$ and if the curvature form of L satisfies*

$$\mathrm{i}\Theta(L) \ge \Big(\frac{p}{|\sigma|^2}+\frac{\varepsilon}{1+|\sigma|^2}\Big)\{\mathrm{i}\Theta(E)\sigma,\sigma\}$$

for some $\varepsilon>0$, there is a section $F\in H^0(X,K_X\otimes L)$ such that $F_{\restriction Y}=f_{\restriction Y}$ and

$$\int_X \frac{|F|^2}{(1+|\sigma|^2)^{p+\varepsilon}}\,dV \le \Big(1+\frac{(p+1)^2}{\varepsilon}\Big)\int_U |f|^2\,dV.$$

The proof will involve a weight with logarithmic singularities along Y. We must therefore apply the existence theorem over $X\smallsetminus Y$. This requires to know whether $X\smallsetminus Y$ has a complete Kähler metric.

(7.2) Lemma. *Let (X,ω) be a Kähler manifold, and $Y=\sigma^{-1}(0)$ an analytic subset defined by a section of a hermitian vector bundle $E\to X$. If X is weakly pseudoconvex and exhausted by $X_c=\{x\in X\ ;\ \psi(x)<c\}$, then*

$X_c \smallsetminus Y$ has a complete Kähler metric for all $c \in \mathbb{R}$. The same conclusion holds for $X \smallsetminus Y$ if (X, ω) is complete and if for some constant $C \geq 0$ we have $\Theta_E \leq_{\mathrm{Grif}} C\,\omega \otimes \langle\ ,\ \rangle_E$ on X.

Proof. Set $\tau = \log|\sigma|^2$. Then $d'\tau = \{D'\sigma, \sigma\}/|\sigma|^2$ and $D''D'\sigma = D^2\sigma = \Theta(E)\sigma$, thus

$$\mathrm{i}d'd''\tau = \mathrm{i}\frac{\{D'\sigma, D'\sigma\}}{|\sigma|^2} - \mathrm{i}\frac{\{D'\sigma, \sigma\} \wedge \{\sigma, D'\sigma\}}{|\sigma|^4} - \frac{\{\mathrm{i}\Theta(E)\sigma, \sigma\}}{|\sigma|^2}.$$

For every $\xi \in T_X$, we find therefore

$$\begin{aligned} H\tau(\xi) &= \frac{|\sigma|^2\,|D'\sigma \cdot \xi|^2 - |\langle D'\sigma \cdot \xi, \sigma\rangle|^2}{|\sigma|^4} - \frac{\Theta_E(\xi \otimes \sigma, \xi \otimes \sigma)}{|\sigma|^2} \\ &\geq -\frac{\Theta_E(\xi \otimes \sigma, \xi \otimes \sigma)}{|\sigma|^2} \end{aligned}$$

by the Cauchy-Schwarz inequality. If C is a bound for the coefficients of Θ_E on the compact subset $\overline{X}_c$, we get $\mathrm{i}d'd''\tau \geq -C\omega$ on X_c. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a convex increasing function. We set

$$\widehat{\omega} = \omega + \mathrm{i}d'd''(\chi \circ \tau).$$

Formula 5.3 shows that $\widehat{\omega}$ is positive definite if $\chi' \leq 1/2C$ and that $\widehat{\omega}$ is complete near $Y = \tau^{-1}(-\infty)$ as soon as

$$\int_{-\infty}^{0} \sqrt{\chi''(t)}\,dt = +\infty.$$

One can choose for example χ such that $\chi(t) = \frac{1}{5C}(t - \log|t|)$ for $t \leq -1$. In order to obtain a complete Kähler metric on $X_c \smallsetminus Y$, we need also that the metric be complete near ∂X_c. Such a metric is given by

$$\begin{aligned} \widetilde{\omega} &= \widehat{\omega} + \mathrm{i}d'd'' \log(c - \psi)^{-1} = \widehat{\omega} + \frac{\mathrm{i}d'd''\psi}{c - \psi} + \frac{\mathrm{i}d'\psi \wedge d''\psi}{(c - \psi)^2} \\ &\geq \mathrm{i}d' \log(c - \psi)^{-1} \wedge d'' \log(c - \psi)^{-1}\ ; \end{aligned}$$

$\widetilde{\omega}$ is complete on $X_c \smallsetminus \Omega$ because $\log(c - \psi)^{-1}$ tends to $+\infty$ on ∂X_c. $\square$

Proof of Theorem 7.1. When we replace σ by $(1 + \eta)\sigma$ for some small $\eta > 0$ and let η tend to 0, we see that we can assume f defined in a neighborhood of $\overline{U}$. Let h be the continuous section of L such that $h = (1 - |\sigma|^{p+1})f$ on $U = \{|\sigma| < 1\}$ and $h = 0$ on $X \smallsetminus U$. We have $h_{\restriction Y} = f_{\restriction Y}$ and

$$d''h = -\frac{p+1}{2}\,|\sigma|^{p-1}\,\{\sigma, D'\sigma\}\,f \quad \text{on}\ \ U, \qquad d''h = 0 \quad \text{on}\ \ X \smallsetminus U.$$

We consider $g = d''h$ as a $(n, 1)$-form with values in the hermitian line bundle $L_\varphi = L$, endowed with the weight $e^{-\varphi}$ given by

$$\varphi = p\log|\sigma|^2 + \varepsilon\log(1+|\sigma|^2).$$

Notice that φ is singular along Y. The Cauchy-Schwarz inequality implies $\mathrm{i}\{D'\sigma,\sigma\}\wedge\{\sigma,D'\sigma\}\le \mathrm{i}\{D'\sigma,D'\sigma\}$ as in Lemma 7.2, and we find

$$\begin{aligned}\mathrm{i}d'd''\log(1+|\sigma|^2) = {}& \frac{(1+|\sigma|^2)\mathrm{i}\{D'\sigma,D'\sigma\} - \mathrm{i}\{D'\sigma,\sigma\}\wedge\{\sigma,D'\sigma\}}{(1+|\sigma|^2)^2}\\ & - \frac{\{\mathrm{i}\Theta(E)\sigma,\sigma\}}{1+|\sigma|^2} \ge \frac{\mathrm{i}\{D'\sigma,D'\sigma\}}{(1+|\sigma|^2)^2} - \frac{\{\mathrm{i}\Theta(E)\sigma,\sigma\}}{1+|\sigma|^2}.\end{aligned}$$

The inequality $\mathrm{i}d'd''\log|\sigma|^2 \ge -\{\mathrm{i}\Theta(E)\sigma,\sigma\}/|\sigma|^2$ obtained in Lemma 7.2 and the above one imply

$$\begin{aligned}\mathrm{i}\Theta(L_\varphi) &= \mathrm{i}\Theta(L) + p\,\mathrm{i}d'd''\log|\sigma|^2 + \varepsilon\,\mathrm{i}d'd''\log(1+|\sigma|^2)\\ &\ge \mathrm{i}\Theta(L) - \Big(\frac{p}{|\sigma|^2} + \frac{\varepsilon}{1+|\sigma|^2}\Big)\{\mathrm{i}\Theta(E)\sigma,\sigma\} + \varepsilon\frac{\mathrm{i}\{D'\sigma,D'\sigma\}}{(1+|\sigma|^2)^2}\\ &\ge \varepsilon\frac{\mathrm{i}\,\{D'\sigma,\sigma\}\wedge\{\sigma,D'\sigma\}}{|\sigma|^2\,(1+|\sigma|^2)^2},\end{aligned}$$

thanks to the hypothesis on the curvature of L and the Cauchy-Schwarz inequality. Set $\xi = (p+1)/2\,|\sigma|^{p-1}\{D'\sigma,\sigma\} = \sum\xi_j\,dz_j$ in an ω-orthonormal basis $\partial/\partial z_j$, and let $\widehat{\xi} = \sum\xi_j\partial/\partial\overline{z}_j$ be the dual $(0,1)$-vector field. For every $(n,1)$-form v with values in L_φ, we find

$$\begin{aligned}\big|\langle d''h,v\rangle\big| &= \big|\langle\overline{\xi}\wedge f,v\rangle\big| = \big|\langle f,\widehat{\xi}\lrcorner\, v\rangle\big| \le |f|\,|\widehat{\xi}\lrcorner\, v|,\\ \widehat{\xi}\lrcorner\, v &= \sum -\mathrm{i}\xi_j\,dz_j\wedge\Lambda v = -\mathrm{i}\xi\wedge\Lambda v,\\ |\langle d''h,v\rangle|^2 &\le |f|^2\,|\widehat{\xi}\lrcorner\, v|^2 = |f|^2\langle -\mathrm{i}\xi\wedge\Lambda v,\widehat{\xi}\lrcorner\, v\rangle\\ &= |f|^2\langle -\mathrm{i}\overline{\xi}\wedge\xi\wedge\Lambda v,v\rangle = |f|^2\langle[\mathrm{i}\xi\wedge\overline{\xi},\Lambda]v,v\rangle\\ &\le \frac{(p+1)^2}{4\varepsilon}\,|\sigma|^{2p}\,(1+|\sigma|^2)^2\,|f|^2\,\langle[\mathrm{i}\Theta(L_\varphi),\Lambda]v,v\rangle.\end{aligned}$$

Thus, in the notations of Th. 6.1, the form $g = d''h$ satisfies

$$\langle A_1^{-1}g,g\rangle \le \frac{(p+1)^2}{4\varepsilon}\,|\sigma|^{2p}(1+|\sigma|^2)^2\,|f|^2 \le \frac{(p+1)^2}{\varepsilon}\,|f|^2\,e^{\varphi},$$

where the last equality results from the fact that $(1+|\sigma|^2)^2\le 4$ on the support of g. Lemma 7.2 shows that the existence theorem 6.1 can be applied on each set $X_c\smallsetminus Y$. Letting c tend to infinity, we infer the existence of a $(n,0)$-form u with values in L such that $d''u = g$ on $X\smallsetminus Y$ and

$$\int_{X\smallsetminus Y}|u|^2\,e^{-\varphi}\,dV \le \int_{X\smallsetminus Y}\langle A_1^{-1}g,g\rangle e^{-\varphi},\qquad \text{thus}$$

$$\int_{X\smallsetminus Y}\frac{|u|^2}{|\sigma|^{2p}(1+|\sigma|^2)^{\varepsilon}}\,dV \le \frac{(p+1)^2}{\varepsilon}\int_U|f|^2\,dV.$$

This estimate implies in particular that u is locally L^2 near Y. As g is continuous over X, Lemma 7.3 below shows that the equality $d''u = g = d''h$ extends to X, thus $F = h - u$ is holomorphic everywhere. Hence $u = h - F$ is continuous on X. As $|\sigma(x)| \le C\, d(x, Y)$ in a neighborhood of every point of Y, we see that $|\sigma|^{-2p}$ is non integrable at every point $x_0 \in Y_{\mathrm{reg}}$, because $\operatorname{codim} Y \le p$. It follows that $u = 0$ on Y, so

$$F_{\restriction Y} = h_{\restriction Y} = f_{\restriction Y}.$$

The final L^2-estimate of Th. 7.1 follows from the inequality

$$|F|^2 = |h - u|^2 \le (1 + |\sigma|^{-2p})\, |u|^2 + (1 + |\sigma|^{2p})\, |f|^2$$

which implies

$$\frac{|F|^2}{(1 + |\sigma|^2)^p} \le \frac{|u|^2}{|\sigma|^{2p}} + |f|^2. \qquad \square$$

(7.3) Lemma. *Let Ω be an open subset of $\mathbb{C}^n$ and Y an analytic subset of Ω. Assume that v is a $(p, q-1)$-form with L^2_{loc} coefficients and w a (p,q)-form with L^1_{loc} coefficients such that $d''v = w$ on $\Omega \smallsetminus Y$ (in the sense of distribution theory). Then $d''v = w$ on Ω.*

Proof. An induction on the dimension of Y shows that it is sufficient to prove the result in a neighborhood of a regular point $a \in Y$. By using a local analytic isomorphism, the proof is reduced to the case where Y is contained in the hyperplane $z_1 = 0$, with $a = 0$. Let $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function such that $\lambda(t) = 0$ for $t \le \frac{1}{2}$ and $\lambda(t) = 1$ for $t \ge 1$. We must show that

$$(7.4) \qquad \int_\Omega w \wedge \alpha = (-1)^{p+q} \int_\Omega v \wedge d''\alpha$$

for all $\alpha \in \mathcal{D}_{n-p,n-q}(\Omega)$. Set $\lambda_\varepsilon(z) = \lambda(|z_1|/\varepsilon)$ and replace α in the integral by $\lambda_\varepsilon \alpha$. Then $\lambda_\varepsilon \alpha \in \mathcal{D}_{n-p,n-q}(\Omega \smallsetminus Y)$ and the hypotheses imply

$$\int_\Omega w \wedge \lambda_\varepsilon \alpha = (-1)^{p+q} \int_\Omega v \wedge d''(\lambda_\varepsilon \alpha) = (-1)^{p+q} \int_\Omega v \wedge (d''\lambda_\varepsilon \wedge \alpha + \lambda_\varepsilon d''\alpha).$$

As w and v have L^1_{loc} coefficients on Ω, the integrals of $w \wedge \lambda_\varepsilon \alpha$ and $v \wedge \lambda_\varepsilon d''\alpha$ converge respectively to the integrals of $w \wedge \alpha$ and $v \wedge d''\alpha$ as ε tends to 0. The remaining term can be estimated by means of the Cauchy-Schwarz inequality:

$$\Big| \int_\Omega v \wedge d''\lambda_\varepsilon \wedge \alpha \Big|^2 \le \int_{|z_1| \le \varepsilon} |v \wedge \alpha|^2\, dV. \int_{\operatorname{Supp} \alpha} |d''\lambda_\varepsilon|^2\, dV\,;$$

as $v \in L^2_{\mathrm{loc}}(\Omega)$, the integral $\int_{|z_1| \le \varepsilon} |v \wedge \alpha|^2\, dV$ converges to 0 with ε, whereas

$$\int_{\mathrm{Supp}\,\alpha} |d''\lambda_\varepsilon|^2\,dV \le \frac{C}{\varepsilon^2}\,\mathrm{Vol}\big(\mathrm{Supp}\ \alpha \cap \{|z_1| \le \varepsilon\}\big) \le C'.$$

Equality (7.4) follows when ε tends to 0. □

(7.5) Corollary. *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex domain and let φ, ψ be plurisubharmonic functions on Ω, where ψ is supposed to be finite and continuous. Let $\sigma = (\sigma_1, \ldots, \sigma_r)$ be a family of holomorphic functions on Ω, let $Y = \sigma^{-1}(0)$, $p =$ maximal codimension of Y and set*

a) $U = \{z \in \Omega\,;\ |\sigma(z)|^2 < e^{-\psi(z)}\}$, *resp.*

b) $U' = \{z \in \Omega\,;\ |\sigma(z)|^2 < e^{\psi(z)}\}$.

For every $\varepsilon > 0$ and every holomorphic function f on U, there exists a holomorphic function F on Ω such that $F_{\restriction Y} = f_{\restriction Y}$ and

a) $$\int_\Omega \frac{|F|^2\,e^{-\varphi+p\psi}}{(1+|\sigma|^2 e^\psi)^{p+\varepsilon}}\,dV \le \Big(1+\frac{(p+1)^2}{\varepsilon}\Big)\int_U |f|^2\,e^{-\varphi+p\psi}\,dV, \qquad \textit{resp.}$$

b) $$\int_\Omega \frac{|F|^2\,e^{-\varphi}}{(e^\psi+|\sigma|^2)^{p+\varepsilon}}\,dV \le \Big(1+\frac{(p+1)^2}{\varepsilon}\Big)\int_U |f|^2\,e^{-\varphi-(p+\varepsilon)\psi}\,dV.$$

Proof. After taking convolutions with smooth kernels on pseudoconvex subdomains $\Omega_c \subset\subset \Omega$, we may assume φ, ψ smooth. In either case a) or b), apply Th. 7.1 to

a) $E = \Omega \times \mathbb{C}^r$ with the weight e^ψ, $L = \Omega \times \mathbb{C}$ with the weight $e^{-\varphi+p\psi}$, and $U = \{|\sigma|^2 e^\psi < 1\}$. Then

$$\mathrm{i}\Theta(E) = -\mathrm{i}d'd''\psi \otimes \mathrm{Id}_E \le 0, \qquad \mathrm{i}\Theta(L) = \mathrm{i}d'd''\varphi - p\,\mathrm{i}d'd''\psi \ge p\,\mathrm{i}\Theta(E).$$

b) $E = \Omega \times \mathbb{C}^r$ with the weight $e^{-\psi}$, $L = \Omega \times \mathbb{C}$ with the weight $e^{-\varphi-(p+\varepsilon)\psi}$, and $U = \{|\sigma|^2 e^{-\psi} < 1\}$. Then

$$\mathrm{i}\Theta(E) = \mathrm{i}d'd''\psi \otimes \mathrm{Id}_E \ge 0, \qquad \mathrm{i}\Theta(L) = \mathrm{i}d'd''\varphi + (p+\varepsilon)\,\mathrm{i}d'd''\psi \ge (p+\varepsilon)\,\mathrm{i}\Theta(E).$$

The condition on $\Theta(L)$ is satisfied in both cases and K_Ω is trivial. □

(7.6) Hörmander-Bombieri-Skoda theorem. *Let $\Omega \subset \mathbb{C}^n$ be a weakly pseudoconvex domain and φ a plurisubharmonic function on Ω. For every $\varepsilon > 0$ and every point $z_0 \in \Omega$ such that $e^{-\varphi}$ is integrable in a neighborhood of z_0, there exists a holomorphic function F on Ω such that $F(z_0) = 1$ and*

$$\int_\Omega \frac{|F(z)|^2\,e^{-\varphi(z)}}{(1+|z|^2)^{n+\varepsilon}}\,dV < +\infty.$$

(Bombieri 1970) originally stated the theorem with the exponent $3n$ instead of $n+\varepsilon$; the improved exponent $n+\varepsilon$ is due to (Skoda 1975). The example $\Omega = \mathbb{C}^n$, $\varphi(z) = 0$ shows that one cannot replace ε by 0.

Proof. Apply Cor. 7.5 b) to $f \equiv 1$, $\sigma(z) = z - z_0$, $p = n$ and $\psi \equiv \log r^2$ where $U = B(z_0, r)$ is a ball such that $\int_U e^{-\varphi}\, dV < +\infty$. □

(7.7) Corollary. *Let φ be a plurisubharmonic function on a complex manifold X. Let A be the set of points $z \in X$ such that $e^{-\varphi}$ is not locally integrable in a neighborhood of z. Then A is an analytic subset of X.*

Proof. Let $\Omega \subset X$ be an open coordinate patch isomorphic to a ball of $\mathbb{C}^n$, with coordinates $(z_1, \ldots, z_n)$. Define $E \subset H^0(\Omega, \mathcal{O})$ to be the Hilbert space of holomorphic functions f on Ω such that

$$\int_\Omega |f(z)|^2 e^{-\varphi(z)}\, dV(z) < +\infty.$$

Then $A \cap \Omega = \bigcap_{f \in E} f^{-1}(0)$. In fact, every f in E must obviously vanish on A; conversely, if $z_0 \notin A$, Th. 7.6 shows that there exists $f \in E$ such that $f(z_0) \neq 0$. By Th. II-5.5, we conclude that A is analytic. □

8. Applications to Hypersurface Singularities

We first give some basic definitions and results concerning multiplicities of divisors on a complex manifold.

(8.1) Proposition. *Let X be a complex manifold and $\Delta = \sum \lambda_j [Z_j]$ a divisor on X with real coefficients $\lambda_j \geq 0$. Let $x \in X$ and $f_j = 0$, $1 \leq j \leq N$, irreducible equations of Z_j on a neighborhood U of x.*

a) *The multiplicity of Δ at x is defined by*

$$\mu(\Delta, x) = \sum \lambda_j \operatorname{ord}_x f_j.$$

b) *Δ is said to have normal crossings at a point $x \in \operatorname{Supp} \Delta$ if all hypersurfaces Z_j containing x are smooth at x and intersect transversally, i.e. if the linear forms df_j defining the corresponding tangent spaces $T_x Z_j$ are linearly independent at x. The set $\operatorname{nnc}(\Delta)$ of non normal crossing points is an analytic subset of X.*

c) *The non-integrability locus $\operatorname{nil}(\Delta)$ is defined as the set of points $x \in X$ such that $\prod |f_j|^{-2\lambda_j}$ is non integrable near x. Then $\operatorname{nil}(\Delta)$ is an analytic subset of X and there are inclusions*

$$\{x \in X\,;\, \mu(\Delta, x) \geq n\} \subset \operatorname{nil}(\Delta) \subset \{x \in X\,;\, \mu(\Delta, x) \geq 1\}.$$

Moreover $\operatorname{nil}(\Delta) \subset \operatorname{nnc}(\Delta)$ if all coefficients of Δ satisfy $\lambda_j < 1$.

Proof. b) The set $\operatorname{nnc}(\Delta) \cap U$ is the union of the analytic sets

$$f_{j_1} = \ldots = f_{j_p} = 0, \qquad df_{j_1} \wedge \ldots \wedge df_{j_p} = 0,$$

for each subset $\{j_1, \ldots, j_p\}$ of the index set $\{1, \ldots, N\}$. Thus $\mathrm{nnc}(\Delta)$ is analytic.

c) The analyticity of $\mathrm{nil}(\Delta)$ follows from Cor. 7.7 applied to the plurisubharmonic function $\varphi = \sum 2\lambda_j \log |f_j|$. Assume first that $\lambda_j < 1$ and that Δ has normal crossings at x. Let $f_{j_1}(x) = \ldots = f_{j_s}(x) = 0$ and $f_j(x) \neq 0$ for $j \neq j_l$. Then, we can choose local coordinates $(w_1, \ldots, w_n)$ on U such that $w_1 = f_{j_1}(z)$, ..., $w_s = f_{j_s}(z)$, and we have

$$\int_U \frac{d\lambda(z)}{\prod |f_j(z)|^{2\lambda_j}} \leq \int_U \frac{C\, d\lambda(w)}{|w_1|^{2\lambda_1} \ldots |w_s|^{2\lambda_s}} < +\infty.$$

It follows that $\mathrm{nil}(\Delta) \subset \mathrm{nnc}(\Delta)$. Let us prove now the statement relating $\mathrm{nil}(\Delta)$ with multiplicity sets. Near any point x, we have $|f_j(z)| \leq C_j |z - x|^{m_j}$ with $m_j = \mathrm{ord}_x f_j$, thus

$$\prod |f_j|^{-2\lambda_j} \geq C\, |z - x|^{-2\mu(\Delta, x)}.$$

It follows that $x \in \mathrm{nil}(\Delta)$ as soon as $\mu(\Delta, x) \geq n$. On the other hand, we are going to prove that $\mu(\Delta, x) < 1$ implies $x \notin \mathrm{nil}(\Delta)$, i.e. $\prod |f_j|^{-2\lambda_j}$ integrable near x. We may assume λ_j rational; otherwise replace each λ_j by a slightly larger rational number in such a way that $\mu(\Delta, x) < 1$ is still true. Set $f = \prod f_j^{k\lambda_j}$ where k is a common denominator. The result is then a consequence of the following lemma. □

(8.2) Lemma. *If $f \in \mathcal{O}_{X,x}$ is not identically 0, there exists a neighborhood U of x such that $\int_U |f|^{-2\lambda}\, dV$ converges for all $\lambda < 1/m$, $m = \mathrm{ord}_x f$.*

Proof. One can assume that f is a Weierstrass polynomial

$$f(z) = z_n^m + a_1(z') z_n^{m-1} + \cdots + a_m(z'), \qquad a_j(z') \in \mathcal{O}_{n-1}, \qquad a_j(0) = 0,$$

with respect to some coordinates $(z_1, \ldots, z_n)$ centered at x. Let $v_j(z')$, $1 \leq j \leq m$, denote the roots z_n of $f(z) = 0$. On a small neighborhood U of x we have $|v_j(z')| \leq 1$. The inequality between arithmetic and geometric mean implies

$$\begin{aligned}
\int_{\{|z_n| \leq 1\}} |f(z)|^{-2\lambda}\, dx_n dy_n &= \int_{\{|z_n| \leq 1\}} \prod_{1 \leq j \leq m} |z_n - v_j(z')|^{-2\lambda}\, dx_n dy_n \\
&\leq \frac{1}{m} \int_{\{|z_n| \leq 1\}} \sum_{1 \leq j \leq m} |z_n - v_j(z')|^{-2m\lambda}\, dx_n dy_n \\
&\leq \int_{\{|z_n| \leq 2\}} \frac{dx_n dy_n}{|z_n|^{2m\lambda}},
\end{aligned}$$

so the Lemma follows from the Fubini theorem. □

Another interesting application concerns the study of multiplicities of singular points for algebraic hypersurfaces in $\mathbb{P}^n$. Following (Waldschmidt 1975), we introduce the following definition.

(8.3) Definition. *Let S be a finite subset of $\mathbb{P}^n$. For any integer $t \geq 1$, we define $\omega_t(S)$ as the minimum of the degrees of non zero homogeneous polynomials $P \in \mathbb{C}[z_0, \ldots, z_n]$ which vanish at order t at every point of S, i.e. $D^\alpha P(w) = 0$ for every $w \in S$ and every multi-index $\alpha = (\alpha_0, \ldots, \alpha_n)$ of length $|\alpha| < t$.*

It is clear that $t \longmapsto \omega_t(S)$ is a non-decreasing and subadditive function, i.e. for all integers $t_1, t_2 \geq 1$ we have $\omega_{t_1+t_2}(S) \leq \omega_{t_1}(S) + \omega_{t_2}(S)$. One defines

$$(8.4) \quad \Omega(S) = \inf_{t \geq 1} \frac{\omega_t(S)}{t}.$$

For all integers $t, t' \geq 1$, the monotonicity and subadditivity of $\omega_t(S)$ show that

$$\omega_t(S) \leq ([t/t'] + 1)\, \omega_{t'}(S), \qquad \text{hence} \;\; \Omega(S) \leq \frac{\omega_t(S)}{t} \leq \Big(\frac{1}{t'} + \frac{1}{t}\Big)\, \omega_{t'}(S).$$

We find therefore

$$(8.5) \quad \Omega(S) = \lim_{t \to +\infty} \frac{\omega_t(S)}{t}.$$

Our goal is to find a lower bound of $\Omega(S)$ in terms of $\omega_t(S)$. For $n = 1$, it is obvious that $\Omega(S) = \omega_t(S)/t = \operatorname{card} S$ for all t. From now on, we assume that $n \geq 2$.

(8.6) Theorem. *Let $t_1, t_2 \geq 1$ be integers, let P be a homogeneous polynomial of degree $\omega_{t_2}(S)$ vanishing at order $\geq t_2$ at every point of S. If $P = P_1^{k_1} \ldots P_N^{k_N}$ is the decomposition of P in irreducible factors and $Z_j = P_j^{-1}(0)$, we set*

$$\alpha = \frac{t_1 + n - 1}{t_2}, \qquad \Delta = \sum (k_j \alpha - [k_j \alpha])\, [Z_j], \qquad a = \dim\big(\mathrm{nil}(\Delta)\big).$$

Then we have the inequality

$$\frac{\omega_{t_1}(S) + n - a - 1}{t_1 + n - 1} \leq \frac{\omega_{t_2}(S)}{t_2}.$$

Let us first make a few comments before giving the proof. If we let t_2 tend to infinity and observe that $\mathrm{nil}(\Delta) \subset \mathrm{nnc}(\Delta)$ by Prop. 8.1 c), we get $a \leq 2$ and

$$(8.7) \quad \frac{\omega_{t_1}(S) + 1}{t_1 + n - 1} \leq \Omega(S) \leq \frac{\omega_{t_2}(S)}{t_2}.$$

Such a result was first obtained by (Waldschmidt 1975, 1979) with the lower bound $\omega_{t_1}(S)/(t_1+n-1)$, as a consequence of the Hörmander-Bombieri-Skoda theorem. The above improved inequalities were then found by (Esnault-Viehweg 1983), who used rather deep tools of algebraic geometry. Our proof will consist in a refinement of the Bombieri-Waldschmidt method due to (Azhari 1990). It has been conjectured by (Chudnovsky 1979) that $\Omega(S) \geq (\omega_1(S)+n-1)/n$. Chudnovsky's conjecture is true for $n=2$ (as shown by (8.7)); this case was first verified independently by (Chudnovsky 1979) and (Demailly 1982). The conjecture can also be verified in case S is a complete polytope, and the lower bound of the conjecture is then optimal (see Demailly 1982a and ??.?.?). More generally, it is natural to ask whether the inequality

$$(8.8)\qquad \frac{\omega_{t_1}(S)+n-1}{t_1+n-1} \leq \Omega(S) \leq \frac{\omega_{t_2}(S)}{t_2}$$

always holds; this is the case if there are infinitely many t_2 for which P can be chosen in such a way that $\mathrm{nil}(\Delta)$ has dimension $a=0$.

(8.9) Bertini's lemma. *If $E \subset \mathbb{P}^n$ is an analytic subset of dimension a, there exists a dense subset in the grassmannian of k-codimensional linear subspaces Y of $\mathbb{P}^n$ such that* $\dim(E\cap Y) \leq a-k$ *(when $k>a$ this means that $E\cap Y=\emptyset$).*

Proof. By induction on n, it suffices to show that $\dim(E\cap H)\leq a-1$ for a generic hyperplane $H\subset\mathbb{P}^n$. Let E_j be the (finite) family of irreducible components of E, and $w_j\in E_j$ an arbitrary point. Then $E\cap H=\bigcup E_j\cap H$ and we have $\dim E_j\cap H<\dim E_j\leq a$ as soon as H avoids all points w_j. $\square$

Proof of Theorem 8.6. By Bertini's lemma, there exists a linear subspace $Y\subset\mathbb{P}^n$ of codimension $a+1$ such that $\mathrm{nil}(\Delta)\cap Y=\emptyset$. We consider P as a section of the line bundle $\mathcal{O}(D)$ over $\mathbb{P}^n$, where $D=\deg P$ (cf. Th. V-15.5). There are sections $\sigma_1,\ldots,\sigma_{a+1}$ of $\mathcal{O}(1)$ such that $Y=\sigma^{-1}(0)$. We shall apply Th. 7.1 to $E=\mathcal{O}(1)$ with its standard hermitian metric, and to $L=\mathcal{O}(k)$ equipped with the additional weight $\varphi=\alpha\log|P|^2$. We may assume that the open set $U=\{|\sigma|<1\}$ is such that $\mathrm{nil}(\Delta)\cap\overline{U}=\emptyset$, otherwise it suffices to multiply σ by a large constant. This implies that the polynomial $Q=\prod P_j^{[k_j\alpha]}$ satisfies

$$\int_U |Q|^2\,e^{-\varphi}\,dV=\int_U\prod|P_j|^{-2(k_j\alpha-[k_j\alpha])}\,dV<+\infty.$$

Set $\omega=ic\big(\mathcal{O}(1)\big)$. We have $\mathrm{i}d'd''\log|P|^2\geq -ic\big(\mathcal{O}(D)\big)=-D\omega$ by the Lelong-Poincaré equation, thus $\mathrm{i}\Theta(L_\varphi)\geq(k-\alpha D)\omega$. The desired curvature inequality $\mathrm{i}\Theta(L_\varphi)\geq(a+1+\varepsilon)\mathrm{i}\Theta(E)$ is satisfied if $k-\alpha D\geq(a+1+\varepsilon)$. We thus take

$$k = [\alpha D] + a + 2.$$

The section $f \in H^0(U, K_{\mathbb{P}^n} \otimes L) = H^0\big(U, \mathcal{O}(k-n-1)\big)$ is taken to be a multiple of Q by some polynomial. This is possible provided that

$$k - n - 1 \geq \deg Q \quad \Longleftrightarrow \quad \alpha D + a + 2 - n - 1 \geq \sum [k_j\alpha] \deg P_j,$$

or equivalently, as $D = \sum k_j \deg P_j$,

$$(8.10) \qquad \sum (k_j\alpha - [k_j\alpha]) \deg P_j \geq n - a - 1.$$

Then we get $f \in H^0(U, K_{\mathbb{P}^n} \otimes L)$ such that $\int_U |f|^2 e^{-\varphi}\, dV < +\infty$. Theorem 7.1 implies the existence of $F \in H^0(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes L)$, i.e. of a polynomial F of degree $k - n - 1$, such that

$$\int_{\mathbb{P}^n} |F|^2 e^{-\varphi}\, dV = \int_{\mathbb{P}^n} \frac{|F|^2}{|P|^{2\alpha}}\, dV < +\infty\,;$$

observe that $|\sigma|$ is bounded, for we are on a compact manifold. Near any $w \in S$, we have $|P(z)| \leq C|z-w|^{t_2}$, thus $|P(z)|^{2\alpha} \leq C|z-w|^{2(t_1+n-1)}$. This implies that the above integral can converge only if F vanishes at order $\geq t_1$ at each point $w \in S$. Therefore

$$\omega_{t_1}(S) \leq \deg F = k - n - 1 = [\alpha D] + a + 1 - n \leq \alpha\omega_{t_2}(S) + a + 1 - n,$$

which is the desired inequality.

However, the above proof only works under the additional assumption (8.10). Assume on the contrary that

$$\beta = \sum (k_j\alpha - [k_j\alpha]) \deg P_j < n - a - 1.$$

Then the polynomial Q has degree

$$\sum [k_j\alpha] \deg P_j = \alpha \deg P - \beta = \alpha D - \beta,$$

and Q vanishes at every point $w \in S$ with order

$$\begin{aligned} \mathrm{ord}_w Q &\geq \sum [k_j\alpha]\, \mathrm{ord}_w P_j = \alpha \sum k_j\, \mathrm{ord}_w P_j - \sum (k_j\alpha - [k_j\alpha])\, \mathrm{ord}_w P_j \\ &\geq \alpha\ \mathrm{ord}_w P - \beta \geq \alpha t_2 - \beta = t_1 - (\beta - n + 1). \end{aligned}$$

This implies $\mathrm{ord}_w Q \geq t_1 - [\beta - n + 1]$. As $[\beta - n + 1] < n - a - 1 - n + 1 = -a \leq 0$, we can take a derivative of order $-[\beta - n + 1]$ of Q to get a polynomial F with

$$\deg F = \alpha D - \beta + [\beta - n + 1] \leq \alpha D - n + 1,$$

which vanishes at order t_1 on S. In this case, we obtain therefore

$$\omega_{t_1}(S) \leq \alpha D - n + 1 = \frac{t_1 + n - 1}{t_2}\, \omega_{t_2}(S) - n + 1$$

and the proof of Th. 8.6 is complete. □

9. Skoda's L^2 Estimates for Surjective Bundle Morphisms

Let (X,ω) be a Kähler manifold, $\dim X = n$, and $g : E \longrightarrow Q$ a holomorphic morphism of hermitian vector bundles over X. Assume in the first instance that g is *surjective*. We are interested in conditions insuring for example that the induced morphism $g : H^k(X, K_X \otimes E) \longrightarrow H^k(X, K_X \otimes Q)$ is also surjective. For that purpose, it is natural to consider the subbundle $S = \operatorname{Ker} g \subset E$ and the exact sequence

$$(9.1)\quad 0 \longrightarrow S \longrightarrow E \xrightarrow{\ g\ } Q \longrightarrow 0.$$

Assume for the moment that S and Q are endowed with the metrics induced by that of E. Let L be a line bundle over X. We consider the tensor product of sequence (9.1) by L :

$$(9.2)\quad 0 \longrightarrow S\otimes L \longrightarrow E\otimes L \xrightarrow{\ g\ } Q\otimes L \longrightarrow 0.$$

(9.3) Theorem. *Let k be an integer such that $0 \le k \le n$. Set $r = \operatorname{rk} E$, $q = \mathrm{rkQ}$, $s = \operatorname{rk} S = r - q$ and*

$$m = \min\{n-k, s\} = \min\{n-k, r-q\}.$$

Assume that (X,ω) possesses also a complete Kähler metric $\widehat\omega$, that $E \ge_m 0$, and that $L \longrightarrow X$ is a hermitian line bundle such that

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\mathrm{i}\Theta(\det Q) \ge 0$$

for some $\varepsilon > 0$. Then for every D''-closed form f of type (n,k) with values in $Q \otimes L$ such that $\|f\| < +\infty$, there exists a D''-closed form h of type (n,k) with values in $E\otimes L$ such that $f = g\cdot h$ and

$$\|h\|^2 \le (1 + m/\varepsilon)\,\|f\|^2.$$

The idea of the proof is essentially due to (Skoda 1978), who actually proved the special case $k = 0$. The general case appeared in (Demailly 1982c).

Proof. Let $j : S \to E$ be the inclusion morphism, $g^\star : Q \to E$ and $j^\star : E \to S$ the adjoints of g, j, and

$$D_E = \begin{pmatrix} D_S & -\beta^\star \\ \beta & D_Q \end{pmatrix},\quad \beta \in C^\infty_{1,0}\big(X, \hom(S,Q)\big),\quad \beta^\star \in C^\infty_{0,1}\big(X, \hom(Q,S)\big),$$

the matrix of D_E with respect to the orthogonal splitting $E \simeq S \oplus Q$ (cf. §V-14). Then $g^\star f$ is a lifting of f in $E\otimes L$. We shall try to find h under the form

$$h = g^\star f + ju, \qquad u \in L^2_{n,k}(X, S \otimes L).$$

As the images of S and Q in E are orthogonal, we have $|h|^2 = |f|^2 + |u|^2$ at every point of X. On the other hand $D''_{Q\otimes L} f = 0$ by hypothesis and $D''g^\star = -j \circ \beta^\star$ by V-14.3 d), hence

$$D''_{E\otimes L} h = -j(\beta^\star \wedge f) + j\, D''_{S\otimes L} = j(D''_{S\otimes L} - \beta^\star \wedge f).$$

We are thus led to solve the equation

$$(9.4) \quad D''_{S\otimes L} u = \beta^\star \wedge f,$$

and for that, we apply Th. 4.5 to the $(n, k+1)$-form $\beta^\star \wedge f$. One observes now that the curvature of $S \otimes L$ can be expressed in terms of β. This remark will be used to prove:

(9.5) Lemma. $\langle A_k^{-1}(\beta^\star \wedge f), (\beta^\star \wedge f)\rangle \le (m/\varepsilon)\, |f|^2$.

If the Lemma is taken for granted, Th. 4.5 yields a solution u of (9.4) in $L^2_{n,q}(X, S \otimes L)$ such that $\|u\|^2 \le (m/\varepsilon)\, \|f\|^2$. As $\|h\|^2 = \|f\|^2 + \|u\|^2$, the proof of Th. 9.3 is complete. □

Proof of Lemma 9.5. Exactly as in the proof of Th. VII-10.3, formulas (V-14.6) and (V-14.7) yield

$$\mathrm{i}\Theta(S) \ge_m \mathrm{i}\beta^\star \wedge \beta, \qquad \mathrm{i}\Theta(\det Q) \ge \mathrm{Tr}_Q(\mathrm{i}\beta \wedge \beta^\star) = \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta).$$

Since $C^\infty_{1,1}(X, \mathrm{Herm}\, S) \ni \Theta := -\mathrm{i}\beta^\star \wedge \beta \ge_{\mathrm{Grif}} 0$, Prop. VII-10.1 implies

$$m\ \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S + \mathrm{i}\beta^\star \wedge \beta \ge_m 0.$$

From the hypothesis on the curvature of L we get

$$\begin{aligned} \mathrm{i}\Theta(S \otimes L) &\ge_m \mathrm{i}\Theta(S) \otimes \mathrm{Id}_L + (m+\varepsilon)\, \mathrm{i}\Theta(\det Q) \otimes \mathrm{Id}_{S\otimes L} \\ &\ge_m \big(\mathrm{i}\beta^\star \wedge \beta + (m+\varepsilon)\ \mathrm{Tr}_S(-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S\big) \otimes \mathrm{Id}_L \\ &\ge_m (\varepsilon/m)\, (-\mathrm{i}\beta^\star \wedge \beta) \otimes \mathrm{Id}_S \otimes \mathrm{Id}_L\,. \end{aligned}$$

For any $v \in \Lambda^{n,k+1} T^\star_X \otimes S \otimes L$, Lemma VII-7.2 implies

$$(9.6) \quad \langle A_{k,S\otimes L} v, v\rangle \ge (\varepsilon/m)\, \langle -\mathrm{i}\beta^\star \wedge \beta \wedge \Lambda v, v\rangle,$$

because $\mathrm{rk}(S \otimes L) = s$ and $m = \min\{n-k, s\}$. Let $(dz_1, \ldots, dz_n)$ be an orthonormal basis of $T^\star_X$ at a given point $x_0 \in X$ and set

$$\beta = \sum_{1\le j\le n} dz_j \otimes \beta_j, \qquad \beta_j \in \hom(S, Q).$$

The adjoint of the operator $\beta^\star \wedge \bullet = \sum d\overline{z}_j \wedge \beta_j^\star \bullet$ is the contraction $\beta \,\lrcorner\, \bullet$ defined by

$$\beta \lrcorner v = \sum \frac{\partial}{\partial \overline{z}_j} \lrcorner (\beta_j v) = \sum -\mathrm{i} dz_j \wedge \Lambda(\beta_j v) = -\mathrm{i}\beta \wedge \Lambda v.$$

We get consequently $\langle -\mathrm{i}\beta^\star \wedge \beta \wedge \Lambda v, v\rangle = |\beta \lrcorner v|^2$ and (9.6) implies

$$|\langle \beta^\star \wedge f, v\rangle|^2 = |\langle f, \beta \lrcorner v\rangle|^2 \leq |f|^2\, |\beta \lrcorner v|^2 \leq (m/\varepsilon)\langle A_{k,S\otimes L} v, v\rangle\, |f|^2. \quad \square$$

If X has a plurisubharmonic exhaustion function ψ, we can select a convex increasing function $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ and multiply the metric of L by the weight $\exp(-\chi\circ\psi)$ in order to make the L^2 norm of f converge. Theorem 9.3 implies therefore:

(9.7) Corollary. *Let (X,ω) be a weakly pseudoconvex Kähler manifold, let $g : E \to Q$ be a surjective bundle morphism with $r = \mathrm{rk}\, E$, $q = \mathrm{rk}\, Q$, let $m = \min\{n-k, r-q\}$ and let $L \to X$ be a hermitian line bundle. Suppose that $E \geq_m 0$ and*

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\,\mathrm{i}\Theta(\det Q) \geq 0$$

for some $\varepsilon > 0$. Then g induces a surjective map

$$H^k(X, K_X \otimes E \otimes L) \longrightarrow H^k(X, K_X \otimes Q \otimes L).$$

The most remarkable feature of this result is that it does not require any strict positivity assumption on the curvature (for instance E can be a flat bundle). A careful examination of the proof shows that it amounts to verify that the image of the coboundary morphism

$$-\beta^\star \wedge \bullet \;:\; H^k(X, K_X \otimes Q \otimes L) \longrightarrow H^{k+1}(X, K_X \otimes S \otimes L)$$

vanishes; however the cohomology group $H^{k+1}(X, K_X \otimes S \otimes L)$ itself does not vanish in general as it would do under a strict positivity assumption (cf. Th. VII-9.4).

We want now to get also estimates when Q is endowed with a metric given a priori, that can be distinct from the quotient metric of E by g. Then the map $g^\star(gg^\star)^{-1} \;:\; Q \longrightarrow E$ is the lifting of Q orthogonal to $S = \mathrm{Ker}\, g$. The quotient metric $|\bullet|'$ on Q is therefore defined in terms of the original metric $|\bullet|$ by

$$|v|'^2 = |g^\star(gg^\star)^{-1}v|^2 = \langle (gg^\star)^{-1}v, v\rangle = \det(gg^\star)^{-1}\, \langle \widetilde{gg^\star} v, v\rangle$$

where $\widetilde{gg^\star} \in \mathrm{End}(Q)$ denotes the endomorphism of Q whose matrix is the transposed of the comatrix of $gg^\star$. For every $w \in \det Q$, we find

$$|w|'^2 = \det(gg^\star)^{-1}\, |w|^2.$$

If Q' denotes the bundle Q with the quotient metric, we get

$$\mathrm{i}\Theta(\det Q') = \mathrm{i}\Theta(\det Q) + \mathrm{i}d'd'' \log\det(gg^\star).$$

In order that the hypotheses of Th. 9.3 be satisfied, we are led to define a new metric $|\bullet|'$ on L by $|u|'^2 = |u|^2 \left(\det(gg^\star)\right)^{-m-\varepsilon}$. Then

$$\mathrm{i}\Theta(L') = \mathrm{i}\Theta(L) + (m+\varepsilon)\,\mathrm{i}d'd'' \log\det(gg^\star) \geq (m+\varepsilon)\,\mathrm{i}\Theta(\det Q').$$

Theorem 9.3 applied to (E, Q', L') can now be reformulated:

(9.8) Theorem. *Let X be a complete Kähler manifold equipped with a Kähler metric ω on X, let $E \to Q$ be a surjective morphism of hermitian vector bundles and let $L \to X$ be a hermitian line bundle. Set $r = \mathrm{rk}\,E$, $q = \mathrm{rk}\,Q$ and $m = \min\{n-k, r-q\}$ and suppose $E \geq_m 0$,*

$$\mathrm{i}\Theta(L) - (m+\varepsilon)\mathrm{i}\Theta(\det Q) \geq 0$$

for some $\varepsilon > 0$. Then for every D''-closed form f of type (n,k) with values in $Q \otimes L$ such that

$$I = \int_X \langle \widetilde{gg^\star} f, f\rangle\,(\det gg^\star)^{-m-1-\varepsilon}\,dV < +\infty,$$

there exists a D''-closed form h of type (n,k) with values in $E \otimes L$ such that $f = g \cdot h$ and

$$\int_X |h|^2\,(\det gg^\star)^{-m-\varepsilon}\,dV \leq (1 + m/\varepsilon)\,I. \qquad \square$$

Our next goal is to extend Th. 9.8 in the case when $g : E \longrightarrow Q$ is only *generically* surjective; this means that the analytic set

$$Y = \{x \in X \;;\; g_x : E_x \longrightarrow Q_x \text{ is not surjective }\}$$

defined by the equation $\Lambda^q g = 0$ is nowhere dense in X. Here $\Lambda^q g$ is a section of the bundle $\mathrm{hom}(\Lambda^q E, \det Q)$.

(9.9) Theorem. *The existence statement and the estimates of Th. 9.8 remain true for a generically surjective morphism $g : E \to Q$ provided that X is weakly pseudoconvex.*

Proof. Apply Th. 9.8 to each relatively compact domain $X_c \smallsetminus Y$ (these domains are complete Kähler by Lemma 7.2). From a sequence of solutions on $X_c \smallsetminus Y$ we can extract a subsequence converging weakly on $X \smallsetminus Y$ as c tends to $+\infty$. One gets a form h satisfying the estimates, such that $D''h = 0$ on $X \smallsetminus Y$ and $f = g \cdot h$. In order to see that $D''h = 0$ on X, it suffices to apply Lemma 7.3 and to observe that h has L^2_{loc} coefficients on X by our estimates. $\square$

A very special but interesting case is obtained for the trivial bundles $E = \Omega \times \mathbb{C}^r$, $Q = \Omega \times \mathbb{C}$ over a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. Then the morphism g is given by a r-tuple $(g_1, \ldots, g_r)$ of holomorphic functions on Ω. Let us take $k = 0$ and $L = \Omega \times \mathbb{C}$ with the metric given by a weight $e^{-\varphi}$. If we observe that $\widetilde{gg^\star} = \mathrm{Id}$ when $\operatorname{rk} Q = 1$, Th. 9.8 applied on $X = \Omega \smallsetminus g^{-1}(0)$ and Lemmas 7.2, 7.3 give:

(9.10) Theorem (Skoda 1978). *Let Ω be a complete Kähler open subset of $\mathbb{C}^n$ and φ a plurisubharmonic function on Ω. Set $m = \min\{n, r-1\}$. Then for every holomorphic function f on Ω such that*

$$I = \int_{\Omega \smallsetminus Z} |f|^2\, |g|^{-2(m+1+\varepsilon)} e^{-\varphi}\, dV < +\infty,$$

where $Z = g^{-1}(0)$, there exist holomorphic functions $(h_1, \ldots, h_r)$ on Ω such that $f = \sum g_j h_j$ and

$$\int_{\Omega \smallsetminus Y} |h|^2\, |g|^{-2(m+\varepsilon)} e^{-\varphi}\, dV \le (1 + m/\varepsilon) I. \qquad \square$$

This last theorem can be used in order to obtain a quick solution of the Levi problem mentioned in §I-4. It can be used also to prove a result of (Diederich-Pflug 1981), relating the pseudoconvexity property and the existence of complete Kähler metrics for domains of $\mathbb{C}^n$.

(9.11) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be an open subset. Then:*

a) *Ω is a domain of holomorphy if and only if Ω is pseudoconvex ;*

b) *If $(\overline{\Omega})^\circ = \Omega$ and if Ω has a complete Kähler metric $\widehat{\omega}$, then Ω is pseudoconvex.*

Note that statement b) can be false if the assumption $(\overline{\Omega})^\circ = \Omega$ is omitted: in fact $\mathbb{C}^n \smallsetminus \{0\}$ is complete Kähler by Lemma 7.2, but it is not pseudoconvex if $n \ge 2$.

Proof. b) By Th. I-4.12, it is enough to verify that Ω is a domain of holomorphy, i.e. that for every connected open subset U such that $U \cap \partial\Omega \neq \emptyset$ and every connected component W of $U \cap \Omega$ there exists a holomorphic function h on Ω such that $h_{\restriction W}$ cannot be continued to U. Since $(\overline{\Omega})^\circ = \Omega$, the set $U \smallsetminus \overline{\Omega}$ is not empty. We select $a \in U \smallsetminus \overline{\Omega}$. Then the integral

$$\int_{\Omega} |z - a|^{-2(n+\varepsilon)}\, dV(z)$$

converges. By Th. 9.10 applied to $f(z) = 1$, $g_j(z) = z_j - a_j$ and $\varphi = 0$, there exist holomorphic functions h_j on Ω such that $\sum (z_j - a_j)\, h_j(z) = 1$. This shows that at least one of the functions h_j cannot be analytically continued at $a \in U$.

a) Assume that Ω is pseudoconvex. Given any open connected set U such that $U \cap \partial\Omega \neq \emptyset$, choose $a \in U \cap \partial\Omega$. By Th. I-4.14 c) the function

$$\varphi(z) = (n+\varepsilon)\big(\log(1+|z|^2) - 2\log d(z, \complement\Omega)\big)$$

is plurisubharmonic on Ω. Then the integral

$$\int_\Omega |z-a|^{-2(n+\varepsilon)}\, e^{-\varphi(z)}\, dV(z) \leq \int_\Omega (1+|z|^2)^{-n-\varepsilon}\, dV(z)$$

converges, and we conclude as for b). □

10. Application of Skoda's L^2 Estimates to Local Algebra

We apply here Th. 9.10 to the study of ideals in the ring $\mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ of germs of holomorphic functions on $(\mathbb{C}^n, 0)$. Let $\mathcal{I} = (g_1, \ldots, g_r) \neq (0)$ be an ideal of $\mathcal{O}_n$.

(10.1) Definition. *Let $k \in \mathbb{R}_+$. We associate to $\mathcal{I}$ the following ideals:*

a) *the ideal $\overline{\mathcal{I}}^{(k)}$ of germs $u \in \mathcal{O}_n$ such that $|u| \leq C|g|^k$ for some constant $C \geq 0$, where $|g|^2 = |g_1|^2 + \cdots + |g_r|^2$.*

b) *the ideal $\widehat{\mathcal{I}}^{(k)}$ of germs $u \in \mathcal{O}_n$ such that*

$$\int_\Omega |u|^2\, |g|^{-2(k+\varepsilon)}\, dV < +\infty$$

on a small ball Ω centered at 0, if $\varepsilon > 0$ is small enough.

(10.2) Proposition. *For all $k, l \in \mathbb{R}_+$ we have*

a) $\overline{\mathcal{I}}^{(k)} \subset \widehat{\mathcal{I}}^{(k)}$;

b) $\mathcal{I}^k \subset \overline{\mathcal{I}}^{(k)}$ *if $k \in \mathbb{N}$;*

c) $\overline{\mathcal{I}}^{(k)}.\overline{\mathcal{I}}^{(l)} \subset \overline{\mathcal{I}}^{(k+l)}$;

d) $\overline{\mathcal{I}}^{(k)}.\widehat{\mathcal{I}}^{(l)} \subset \widehat{\mathcal{I}}^{(k+l)}$.

All properties are immediate from the definitions except a) which is a consequence of Lemma 8.2. Before stating the main result, we need a simple lemma.

(10.3) Lemma. *If $\mathcal{J} = (g_1, \ldots, g_r)$ and $r > n$, we can find elements $\widetilde{g}_1, \ldots, \widetilde{g}_n \in \mathcal{J}$ such that $C^{-1}|g| \le |\widetilde{g}| \le C|g|$ on a neighborhood of 0. Each $\widetilde{g}_j$ can be taken to be a linear combination*

$$\widetilde{g}_j = a_j.\, g = \sum_{1 \le k \le r} a_{jk} g_k, \qquad a_j \in \mathbb{C}^r \smallsetminus \{0\}$$

where the coefficients $([a_1], \ldots, [a_n])$ are chosen in the complement of a proper analytic subset of $(\mathbb{P}^{r-1})^n$.

It follows from the Lemma that the ideal $\mathcal{J} = (\widetilde{g}_1, \ldots, \widetilde{g}_n) \subset \mathcal{J}$ satisfies $\overline{\mathcal{J}}^{(k)} = \overline{\mathcal{J}}^{(k)}$ and $\widehat{\mathcal{J}}^{(k)} = \widehat{\mathcal{J}}^{(k)}$ for all k.

Proof. Assume that $g \in \mathcal{O}(\Omega)^r$. Consider the analytic subsets in $\Omega \times (\mathbb{P}^{r-1})^n$ defined by

$$A = \big\{(z, [w_1], \ldots, [w_n])\,;\, w_j.\, g(z) = 0\big\},$$
$$A^\star = \bigcup \text{irreducible components of } A \text{ not contained in } g^{-1}(0) \times (\mathbb{P}^{r-1})^n.$$

For $z \notin g^{-1}(0)$ the fiber $A_z = \{([w_1], \ldots, [w_n])\,;\, w_j.\, g(z) = 0\} = A^\star_z$ is a product of n hyperplanes in $\mathbb{P}^{r-1}$, hence $A \cap (\Omega \smallsetminus g^{-1}(0)) \times (\mathbb{P}^{r-1})^n$ is a fiber bundle with base $\Omega \smallsetminus g^{-1}(0)$ and fiber $(\mathbb{P}^{r-2})^n$. As $A^\star$ is the closure of this set in $\Omega \times (\mathbb{P}^{r-1})^n$, we have

$$\dim A^\star = n + n(r-2) = n(r-1) = \dim(\mathbb{P}^{r-1})^n.$$

It follows that the zero fiber

$$A^\star_0 = A^\star \cap \big(\{0\} \times (\mathbb{P}^{r-1})^n\big)$$

is a proper subset of $\{0\} \times (\mathbb{P}^{r-1})^n$. Choose $(a_1, \ldots, a_n) \in (\mathbb{C}^r \smallsetminus \{0\})^n$ such that $(0, [a_1], \ldots, [a_n])$ is not in $A^\star_0$. By an easy compactness argument the set $A^\star \cap \big(\overline{B}(0,\varepsilon) \times (\mathbb{P}^{r-1})^n\big)$ is disjoint from the neighborhood $B(0,\varepsilon) \times \prod [B(a_j, \varepsilon)]$ of $(0, [a_1], \ldots, [a_n])$ for ε small enough. For $z \in B(0,\varepsilon)$ we have $|a_j.\, g(z)| \ge \varepsilon |g(z)|$ for some j, otherwise the inequality $|a_j.\, g(z)| < \varepsilon |g(z)|$ would imply the existence of $h_j \in \mathbb{C}^r$ with $|h_j| < \varepsilon$ and $a_j.\, g(z) = h_j.\, g(z)$. Since $g(z) \ne 0$, we would have

$$(z, [a_1 - h_1], \ldots, [a_n - h_n]) \in A^\star \cap \big(B(0,\varepsilon) \times (\mathbb{P}^{r-1})^n\big),$$

a contradiction. We obtain therefore

$$\varepsilon |g(z)| \le \max |a_j.\, g(z)| \le (\max |a_j|)\, |g(z)| \quad \text{on } B(0,\varepsilon). \qquad \square$$

(10.4) Theorem (Briançon-Skoda 1974)**.** *Set $p = \min\{n-1, r-1\}$. Then*

a) $\widehat{\mathcal{J}}^{(k+1)} = \mathcal{J}\,\widehat{\mathcal{J}}^{(k)} = \overline{\mathcal{J}}\,\widehat{\mathcal{J}}^{(k)}$ *for $k \ge p$.*

b) $\overline{\mathcal{J}}^{(k+p)} \subset \widehat{\mathcal{J}}^{(k+p)} \subset \mathcal{J}^k$ *for all* $k \in \mathbb{N}$.

Proof. a) The inclusions $\mathcal{J}\widehat{\mathcal{J}}^{(k)} \subset \overline{\mathcal{J}}\widehat{\mathcal{J}}^{(k)} \subset \widehat{\mathcal{J}}^{(k+1)}$ are obvious thanks to Prop. 10.2, so we only have to prove that $\widehat{\mathcal{J}}^{(k+1)} \subset \mathcal{J}\widehat{\mathcal{J}}^{(k)}$. Assume first that $r \le n$. Let $f \in \widehat{\mathcal{J}}^{(k+1)}$ be such that

$$\int_\Omega |f|^2\, |g|^{-2(k+1+\varepsilon)}\, dV < +\infty.$$

For $k \ge p-1$, we can apply Th. 9.10 with $m = r-1$ and with the weight $\varphi = (k-m)\log|g|^2$. Hence f can be written $f = \sum g_j h_j$ with

$$\int_\Omega |h|^2\, |g|^{-2(k+\varepsilon)}\, dV < +\infty,$$

thus $h_j \in \widehat{\mathcal{J}}^{(k)}$ and $f \in \mathcal{J}\widehat{\mathcal{J}}^{(k)}$. When $r > n$, Lemma 10.3 shows that there is an ideal $\mathcal{G} \subset \mathcal{J}$ with n generators such that $\widehat{\mathcal{G}}^{(k)} = \widehat{\mathcal{J}}^{(k)}$. We find

$$\widehat{\mathcal{J}}^{(k+1)} = \widehat{\mathcal{G}}^{(k+1)} \subset \mathcal{G}\,\widehat{\mathcal{G}}^{(k)} \subset \mathcal{J}\,\widehat{\mathcal{J}}^{(k)} \qquad \text{for} \quad k \ge n-1.$$

b) Property a) implies inductively $\widehat{\mathcal{J}}^{(k+p)} = \mathcal{J}^k\,\widehat{\mathcal{J}}^{(p)}$ for all $k \in \mathbb{N}$. This gives in particular $\widehat{\mathcal{J}}^{(k+p)} \subset \mathcal{J}^k$. □

(10.5) Corollary.

a) *The ideal* $\overline{\mathcal{J}}$ *is the integral closure of* $\mathcal{J}$, *i.e. by definition the set of germs* $u \in \mathcal{O}_n$ *which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \qquad a_s \in \mathcal{J}^s, \qquad 1 \le s \le d.$$

b) *Similarly,* $\overline{\mathcal{J}}^{(k)}$ *is the set of germs* $u \in \mathcal{O}_n$ *which satisfy an equation*

$$u^d + a_1 u^{d-1} + \cdots + a_d = 0, \qquad a_s \in \mathcal{J}^{]ks[}, \qquad 1 \le s \le d,$$

where $]t[$ *denotes the smallest integer* $\ge t$.

As the ideal $\overline{\mathcal{J}}^{(k)}$ is finitely generated, property b) shows that there always exists a rational number $l \ge k$ such that $\overline{\mathcal{J}}^{(l)} = \overline{\mathcal{J}}^{(k)}$.

Proof. a) If $u \in \mathcal{O}_n$ satisfies a polynomial equation with coefficients $a_s \in \mathcal{J}^s$, then clearly $|a_s| \le C_s\, |g|^s$ and Lemma II-4.10 implies $|u| \le C\,|g|$.

Conversely, assume that $u \in \overline{\mathcal{J}}$. The ring $\mathcal{O}_n$ is Noetherian, so the ideal $\widehat{\mathcal{J}}^{(p)}$ has a finite number of generators $v_1, \ldots, v_N$. For every j we have $uv_j \in \overline{\mathcal{J}}\,\widehat{\mathcal{J}}^{(p)} = \mathcal{J}\,\widehat{\mathcal{J}}^{(p)}$, hence there exist elements $b_{jk} \in \mathcal{J}$ such that

$$uv_j = \sum_{1 \le k \le N} b_{jk} v_k.$$

The matrix $(u\delta_{jk} - b_{jk})$ has the non zero vector (v_j) in its kernel, thus u satisfies the equation $\det(u\delta_{jk} - b_{jk}) = 0$, which is of the required type.

b) Observe that $v_1, \ldots, v_N$ satisfy simultaneously some integrability condition $\int_\Omega |v_j|^{-2(p+\varepsilon)} < +\infty$, thus $\widehat{\mathcal{J}}^{(p)} = \widehat{\mathcal{J}}^{(p+\eta)}$ for $\eta \in [0, \varepsilon[$. Let $u \in \overline{\mathcal{J}}^{(k)}$. For every integer $m \in \mathbb{N}$ we have

$$u^m v_j \in \overline{\mathcal{J}}^{(km)}\, \widehat{\mathcal{J}}^{(p+\eta)} \subset \widehat{\mathcal{J}}^{(km+\eta+p)}.$$

If $k \notin \mathbb{Q}$, we can find m such that $d(km + \varepsilon/2, \mathbb{Z}) < \varepsilon/2$, thus $km + \eta \in \mathbb{N}$ for some $\eta \in]0, \varepsilon[$. If $k \in \mathbb{Q}$, we take m such that $km \in \mathbb{N}$ and $\eta = 0$. Then

$$u^m v_j \in \widehat{\mathcal{J}}^{(N+p)} = \mathcal{J}^N \widehat{\mathcal{J}}^{(p)} \quad \text{with} \quad N = km + \eta \in \mathbb{N},$$

and the reasoning made in a) gives $\det(u^m \delta_{jk} - b_{jk}) = 0$ for some $b_{jk} \in \mathcal{J}^N$. This is an equation of the type described in b), where the coefficients a_s vanish when s is not a multiple of m and $a_{ms} \in \mathcal{J}^{Ns} \subset \mathcal{J}^{]kms[}$. □

Let us mention that Briançon and Skoda's result 10.4 b) is optimal for $k = 1$. Take for example $\mathcal{J} = (g_1, \ldots, g_r)$ with $g_j(z) = z_j^r$, $1 \le j \le r$, and $f(z) = z_1 \ldots z_r$. Then $|f| \le C|g|$ and 10.4 b) yields $f^r \in \mathcal{J}$; however, it is easy to verify that $f^{r-1} \notin \mathcal{J}$. The theorem also gives an answer to the following conjecture made by J. Mather.

(10.6) Corollary. *Let $f \in \mathcal{O}_n$ and $\mathcal{J}_f = (z_1 \partial f/\partial z_1, \ldots, z_n \partial f/\partial z_n)$. Then $f \in \overline{\mathcal{J}}_f$, and for every integer $k \ge 0$, $f^{k+n-1} \in \mathcal{J}_f^k$.*

The Corollary is also optimal for $k = 1$: for example, one can verify that the function $f(z) = (z_1 \ldots z_n)^3 + z_1^{3n-1} + \ldots + z_n^{3n-1}$ is such that $f^{n-1} \notin \mathcal{J}_f$.

Proof. Set $g_j(z) = z_j\, \partial f/\partial z_j$, $1 \le j \le n$. By 10.4 b), it suffices to show that $|f| \le C|g|$. For every germ of analytic curve $\mathbb{C} \ni t \longmapsto \gamma(t)$, $\gamma \not\equiv 0$, the vanishing order of $f \circ \gamma(t)$ at $t = 0$ is the same as that of

$$t\, \frac{d(f \circ \gamma)}{dt} = \sum_{1 \le j \le n} t\, \gamma_j'(t)\, \frac{\partial f}{\partial z_j}\big(\gamma(t)\big).$$

We thus obtain

$$|f \circ \gamma(t)| \le C_1\, |t|\, \left|\frac{d(f \circ \gamma)}{dt}\right| \le C_2 \sum_{1 \le j \le n} |t\, \gamma_j'(t)|\, \left|\frac{\partial f}{\partial z_j}\big(\gamma(t)\big)\right| \le C_3\, |g \circ \gamma(t)|$$

and conclude by the following elementary lemma. □

(10.7) Lemma. *Let $f, g_1, \ldots, g_r \in \mathcal{O}_n$ be germs of holomorphic functions vanishing at 0. Then we have $|f| \le C|g|$ for some constant C if and only if* *for every germ of analytic curve* γ *through* 0 *there exists a constant* C_γ *such that* $|f\circ\gamma| \le C_\gamma |g\circ\gamma|$.

Proof. If the inequality $|f| \le C|g|$ does not hold on any neighborhood of 0, the germ of analytic set $(A, 0) \subset (\mathbb{C}^{n+r}, 0)$ defined by

$$g_j(z) = f(z)z_{n+j}, \qquad 1 \le j \le r,$$

contains a sequence of points $\big(z_\nu, g_j(z_\nu)/f(z_\nu)\big)$ converging to 0 as ν tends to $+\infty$, with $f(z_\nu) \ne 0$. Hence $(A, 0)$ contains an irreducible component on which $f \not\equiv 0$ and there is a germ of curve $\widetilde{\gamma} = (\gamma, \gamma_{n+j}) : (\mathbb{C}, 0) \to (\mathbb{C}^{n+r}, 0)$ contained in $(A, 0)$ such that $f\circ\gamma \not\equiv 0$. We get $g_j \circ \gamma = (f\circ\gamma)\gamma_{n+j}$, hence $|g\circ\gamma(t)| \le C|t|\,|f\circ\gamma(t)|$ and the inequality $|f\circ\gamma| \le C_\gamma|g\circ\gamma|$ does not hold. □

11. Integrability of Almost Complex Structures

Let M be a C^∞ manifold of real dimension $m = 2n$. An *almost complex structure* on M is by definition an endomorphism $J \in \mathrm{End}(TM)$ of class C^∞ such that $J^2 = -\mathrm{Id}$. Then TM becomes a complex vector bundle for which the scalar multiplication by i is given by J. The pair (M, J) is said to be an *almost complex manifold*. For such a manifold, the complexified tangent space $T_{\mathbb{C}}M = \mathbb{C} \otimes_{\mathbb{R}} TM$ splits into conjugate complex subspaces

$$(11.1) \quad T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M, \qquad \dim_{\mathbb{C}} T^{1,0}M = \dim_{\mathbb{C}} T^{0,1}M = n,$$

where $T^{1,0}M$, $T^{0,1}M \subset T_{\mathbb{C}}M$ are the eigenspaces of $\mathrm{Id}\otimes J$ corresponding to the eigenvalues i and $-i$. The complexified exterior algebra $\mathbb{C} \otimes_{\mathbb{R}} \Lambda^\bullet T^\star M = \Lambda^\bullet T^\star_{\mathbb{C}} M$ has a corresponding splitting

$$(11.2) \quad \Lambda^k T^\star_{\mathbb{C}} M = \bigoplus_{p+q=k} \Lambda^{p,q} T^\star_{\mathbb{C}} M$$

where we denote by definition

$$(11.3) \quad \Lambda^{p,q} T^\star_{\mathbb{C}} M = \Lambda^p (T^{1,0}M)^\star \otimes_{\mathbb{C}} \Lambda^q (T^{0,1}M)^\star.$$

As for complex manifolds, we let $C^s_{p,q}(M, E)$ be the space of differential forms of class C^s and bidegree (p, q) on M with values in a complex vector bundle E. There is a natural antisymmetric bilinear map

$$\theta \;:\; C^\infty(M, T^{1,0}M) \times C^\infty(M, T^{1,0}M) \longrightarrow C^\infty(M, T^{0,1}M)$$

which associates to a pair (ξ, η) of $(1,0)$-vector fields the $(0,1)$-component of the Lie bracket $[\xi, \eta]$. Since

$$[\xi, f\eta] = f[\xi, \eta] + (\xi.f)\,\eta, \qquad \forall f \in C^\infty(M, \mathbb{C})$$

we see that $\theta(\xi, f\eta) = f\,\theta(\xi,\eta)$. It follows that θ is in fact a $(2,0)$-form on M with values in $T^{0,1}M$.

If M is a *complex analytic manifold* and J its natural almost complex structure, we have in fact $\theta = 0$, because $[\partial/\partial z_j, \partial/\partial z_k] = 0$, $1 \le j,k \le n$, for any holomorphic local coordinate system $(z_1,\ldots,z_n)$.

(11.4) Definition. *The form $\theta \in C^\infty_{2,0}(M, T^{0,1}M)$ is called the torsion form of J. The almost complex structure J is said to be integrable if $\theta = 0$.*

(11.5) Example. If M is of real dimension $m = 2$, every almost complex structure is integrable, because $n = 1$ and alternate $(2,0)$-forms must be zero. Assume that M is a smooth oriented surface. To any Riemannian metric g we can associate the endomorphism $J \in \mathrm{End}(TM)$ equal to the rotation of $+\pi/2$. A change of orientation changes J into the conjugate structure $-J$. Conversely, if J is given, TM is a complex line bundle, so M is oriented, and a Riemannian metric g is associated to J if and only if g is J-hermitian. As a consequence, there is a one-to-one correspondence between conformal classes of Riemannian metrics on M and almost complex structures corresponding to a given orientation. □

If (M,J) is an almost complex manifold and $u \in C^\infty_{p,q}(M,\mathbb{C})$, we let $d'u$, $d''u$ be the components of type $(p+1,q)$ and $(p,q+1)$ in the exterior derivative du. Let $(\xi_1,\ldots,\xi_n)$ be a frame of $T^{1,0}M_{\restriction\Omega}$. The torsion form θ can be written

$$\theta = \sum_{1\le j\le n} \alpha_j \otimes \overline{\xi}_j, \qquad \alpha_j \in C^\infty_{2,0}(\Omega,\mathbb{C}).$$

Then θ yields conjugate operators θ', θ'' on $\Lambda^\bullet T^\star_{\mathbb{C}}M$ such that

$$(11.6)\qquad \theta' u = \sum_{1\le j\le n} \alpha_j \wedge (\overline{\xi}_j \;\lrcorner\; u), \qquad \theta'' u = \sum_{1\le j\le n} \overline{\alpha}_j \wedge (\xi_j \;\lrcorner\; u).$$

If u is of bidegree (p,q), then $\theta' u$ and $\theta'' u$ are of bidegree $(p+2,q-1)$ and $(p-1,q+2)$. It is clear that θ', θ'' are derivations, i.e.

$$\theta'(u\wedge v) = (\theta' u)\wedge v + (-1)^{\deg u} u \wedge (\theta' v)$$

for all smooth forms u, v, and similarly for θ''.

(11.7) Proposition. *We have* $d = d' + d'' - \theta' - \theta''$.

Proof. Since all operators occuring in the formula are derivations, it is sufficient to check the formula for forms of degree 0 or 1. If u is of degree 0, the result is obvious because $\theta' u = \theta'' u = 0$ and du can only have components of types $(1,0)$ or $(0,1)$. If u is a 1-form and ξ, η are complex vector fields, we have

$$du(\xi,\eta) = \xi.u(\eta) - \eta.du(\xi) - u([\xi,\eta]).$$

When u is of type $(0,1)$ and ξ, η of type $(1,0)$, we find

$$(du)^{2,0}(\xi,\eta) = -u\big(\theta(\xi,\eta)\big)$$

thus $(du)^{2,0} = -\theta' u$, and of course $(du)^{1,1} = d'u$, $(du)^{0,2} = d''u$, $\theta''u = 0$ by definition. The case of a $(1,0)$-form u follows by conjugation. □

Proposition 11.7 shows that J is integrable if and only if $d = d' + d''$. In this case, we infer immediately

$$d'^2 = 0, \qquad d'd'' + d''d' = 0, \qquad d''^2 = 0.$$

For an integrable almost complex structure, we thus have the same formalism as for a complex analytic structure, and indeed we shall prove:

(11.8) Newlander-Nirenberg theorem (1957). *Every integrable almost complex structure J on M is defined by a unique analytic structure.*

The proof we shall give follows rather closely that of (Hörmander 1966), which was itself based on previous ideas of (Kohn 1963, 1964). A function $f \in C^1(\Omega,\mathbb{C})$, $\Omega \subset M$, is said to be J-holomorphic if $d''f = 0$. Let $f_1,\ldots,f_p \in C^1(\Omega,\mathbb{C})$ and let h be a function of class C^1 on an open subset of $\mathbb{C}^p$ containing the range of $f = (f_1,\ldots,f_p)$. An easy computation gives

$$\text{(11.9)} \quad d''(h\circ f) = \sum_{1\le j\le p} \Big(\frac{\partial h}{\partial z_j}\circ f\Big) d''f_j + \Big(\frac{\partial h}{\partial \overline{z}_j}\circ f\Big)\overline{d'f_j},$$

in particular $h \circ f$ is J-holomorphic as soon as $f_1,\ldots,f_p$ are J-holomorphic and h holomorphic in the usual sense.

Constructing a complex analytic structure on M amounts to show the existence of J-holomorphic complex coordinates $(z_1,\ldots,z_n)$ on a neighborhood Ω of every point $a \in M$. Formula (11.9) then shows that all coordinate changes $h : (z_k) \mapsto (w_k)$ are holomorphic in the usual sense, so that M is furnished with a complex analytic atlas. The uniqueness of the analytic structure associated to J is clear, since the holomorphic functions are characterized by the condition $d''f = 0$. In order to show the existence, we need a lemma.

(11.10) Lemma. *For every point $a \in M$ and every integer $s \ge 1$, there exist C^∞ complex coordinates $(z_1,\ldots,z_n)$ centered at a such that*

$$d''z_j = O(|z|^s), \qquad 1 \le j \le n.$$

Proof. By induction on s. Let $(\xi_1^\star,\ldots,\xi_n^\star)$ be a basis of $\Lambda^{1,0}T^\star_{\mathbb{C}}M$. One can find complex functions z_j such that $dz_j(a) = \xi_j^\star$, i.e.

$$d'z_j(a) = \xi_j^\star, \qquad d''z_j(a) = 0.$$

Then $(z_1, \ldots, z_n)$ satisfy the conclusions of the Lemma for $s = 1$. If $(z_1, \ldots, z_n)$ are already constructed for the integer s, we have a Taylor expansion

$$d''z_j = \sum_{1\le k\le n} P_{jk}(z, \overline{z})\, \overline{d'z_k} + \mathrm{O}(|z|^{s+1})$$

where $P_{jk}(z, w)$ is a homogeneous polynomial in $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$ of total degree s. As J is integrable, we have

$$\begin{aligned}
0 = d''^2 z_j &= \sum_{1\le k,l\le n} \frac{\partial P_{jk}}{\partial z_l}\, d''z_l \wedge \overline{d'z_k} + \frac{\partial P_{jk}}{\partial \overline{z}_l}\, \overline{d'z_l} \wedge \overline{d'z_k} + \mathrm{O}(|z|^s) \\
&= \sum_{1\le k<l\le n} \Big[\frac{\partial P_{jk}}{\partial \overline{z}_l} - \frac{\partial P_{jl}}{\partial \overline{z}_k}\Big]\, \overline{d'z_l} \wedge \overline{d'z_k} + \mathrm{O}(|z|^s)
\end{aligned}$$

because $\partial P_{jk}/\partial z_l$ is of degree $s-1$ and $d''z_l = \mathrm{O}(|z|^s)$. Since the polynomial between brackets is of degree $s-1$, we must have

$$\frac{\partial P_{jk}}{\partial \overline{z}_l} - \frac{\partial P_{jl}}{\partial \overline{z}_k} = 0, \qquad \forall j, k, l.$$

We define polynomials Q_j of degree $s+1$

$$Q_j(z, \overline{z}) = \int_0^1 \sum_{1\le l\le n} \overline{z}_l\, P_{jl}(z, t\overline{z})\, dt.$$

Trivial computations show that

$$\begin{aligned}
\frac{\partial Q_j}{\partial \overline{z}_k} &= \int_0^1 \Big(P_{jk} + \sum_{1\le l\le n} \overline{z}_l \frac{\partial P_{jl}}{\partial \overline{z}_k}\Big)(z, t\overline{z})\, dt \\
&= \int_0^1 \frac{d}{dt}\Big[t\, P_{jk}(z, t\overline{z})\Big]\, dt = P_{jk}(z, \overline{z}), \\
d''\big(z_j - Q_j(z, \overline{z})\big) &= d''z_j - \sum_{1\le k\le n} \frac{\partial Q_j}{\partial \overline{z}_k}\, \overline{d'z_k} - \sum_{1\le k\le n} \frac{\partial Q_j}{\partial z_k}\, d''z_k \\
&= -\sum_{1\le k\le n} \frac{\partial Q_j}{\partial z_k}\, d''z_k + \mathrm{O}(|z|^{s+1}) = \mathrm{O}(|z|^{s+1})
\end{aligned}$$

because $\partial Q_j/\partial z_k$ is of degree s and $d''z_l = \mathrm{O}(|z|)$. The new coordinates

$$\widetilde{z}_j = z_j - Q_j(z, \overline{z}), \qquad 1 \le j \le n$$

fulfill the Lemma at step $s+1$. $\square$

All usual notions defined on complex analytic manifolds can be extended to integrable almost complex manifolds. For example, a smooth function φ is said to be strictly plurisubharmonic if $\mathrm{i}d'd''\varphi$ is a positive definite $(1,1)$-form. Then $\omega = \mathrm{i}d'd''\varphi$ is a Kähler metric on (M, J).

In this context, all L^2 estimates proved in the previous paragraphs still apply to an integrable almost complex manifold; remember that the proof of the Bochner-Kodaira-Nakano identity used only Taylor developments of order ≤ 2, and the coordinates given by Lemma 11.10 work perfectly well for that purpose. In particular, Th. 6.5 is still valid.

(11.11) Lemma. *Let $(z_1, \ldots, z_n)$ be coordinates centered at a point $a \in M$ with $d''z_j = \mathrm{O}(|z|^s)$, $s \ge 3$. Then the functions*

$$\psi(z) = |z|^2, \qquad \varphi_\varepsilon(z) = |z|^2 + \log(|z|^2 + \varepsilon^2), \qquad \varepsilon \in \,]0,1]$$

are strictly plurisubharmonic on a small ball $|z| < r_0$.

Proof. We have

$$\mathrm{i}d'd''\psi = \mathrm{i}\sum_{1\le j\le n} d'z_j \wedge \overline{d'z_j} + d'\overline{z}_j \wedge d''z_j + z_j\, d'd''\overline{z}_j + \overline{z}_j\, d'd''z_j.$$

The last three terms are $\mathrm{O}(|z|^s)$ and the first one is positive definite at $z = 0$, so the result is clear for ψ. Moreover

$$\mathrm{i}d'd''\varphi_\varepsilon = \mathrm{i}d'd''\psi + i\frac{(|z|^2+\varepsilon^2)\sum d'z_j \wedge \overline{d'z_j} - \sum \overline{z}_j d'z_j \wedge \overline{\sum \overline{z}_j d'z_j}}{(|z|^2+\varepsilon^2)^2} + \frac{\mathrm{O}(|z|^s)}{|z|^2+\varepsilon^2} + \frac{\mathrm{O}(|z|^{s+2})}{(|z|^2+\varepsilon^2)^2}.$$

We observe that the first two terms are positive definite, whereas the remainder is $\mathrm{O}(|z|)$ uniformly in ε.

Proof of theorem 11.8. With the notations of the previous lemmas, consider the pseudoconvex open set

$$\Omega = \{|z| < r\} = \{\psi(z) - r^2 < 0\}, \qquad r < r_0,$$

endowed with the Kähler metric $\omega = \mathrm{i}d'd''\psi$. Let $h \in \mathcal{D}(\Omega)$ be a cut-off function with $0 \le h \le 1$ and $h = 1$ on a neighborhood of $z = 0$. We apply Th. 6.5 to the $(0,1)$-forms

$$g_j = d''\big(z_j h(z)\big) \in C^\infty_{0,1}(\Omega, \mathbb{C})$$

for the weight

$$\varphi(z) = A|z|^2 + (n+1)\log|z|^2 = \lim_{\varepsilon\to 0} A|z|^2 + (n+1)\log(|z|^2+\varepsilon^2).$$

Lemma 11.11 shows that φ is plurisubharmonic for $A \geq n+1$, and for A large enough we obtain

$$\mathrm{i}d'd''\varphi + \mathrm{Ricci}(\omega) \geq \omega \quad \text{on } \Omega.$$

By Remark (6.8) we get a function f_j such that $d''f_j = g_j$ and

$$\int_\Omega |f_j|^2 e^{-\varphi} dV \leq \int_\Omega |g_j|^2 e^{-\varphi} dV.$$

As $g_j = d''z_j = \mathrm{O}(|z|^s)$ and $e^{-\varphi} = \mathrm{O}(|z|^{-2n-2})$ near $z=0$, the integral of g_j converges provided that $s \geq 2$. Then $\int |f_j(z)|^2 |z|^{-2n-2} dV$ converges also at $z=0$. Since the solution f_j is smooth, we must have $f_j(0) = df_j(0) = 0$. We set

$$\widetilde{z}_j = z_j h(z) - f_j, \qquad 1 \leq j \leq n.$$

Then $\widetilde{z}_j$ is J-holomorphic and $d\widetilde{z}_j(0) = dz_j(0)$, so $(z_1, \ldots, z_n)$ is a J-holomorphic coordinate system at $z=0$. □

In particular, any Riemannian metric on an oriented 2-dimensional real manifold defines a unique analytic structure. This fact will be used in order to obtain a simple proof of the well-known:

(11.12) Uniformization theorem. *Every simply connected Riemann surface X is biholomorphic either to $\mathbb{P}^1$, $\mathbb{C}$ or the unit disk Δ.*

Proof. We will merely use the fact that $H^1(X, \mathbb{R}) = 0$. If X is compact, then X is a complex curve of genus 0, so $X \simeq \mathbb{P}^1$ by Th. VI-14.16. On the other hand, the elementary Riemann mapping theorem says that an open set $\Omega \subset \mathbb{C}$ with $H^1(\Omega, \mathbb{R}) = 0$ is either equal to $\mathbb{C}$ or biholomorphic to the unit disk. Thus, all we have to show is that a non compact Riemann surface X with $H^1(X, \mathbb{R}) = 0$ can be embedded in the complex plane $\mathbb{C}$.

Let Ω_ν be an exhausting sequence of relatively compact connected open sets with smooth boundary in X. We may assume that $X \smallsetminus \Omega_\nu$ has no relatively compact connected components, otherwise we "fill the holes" of Ω_ν by taking the union with all such components. We let Y_ν be the double of the manifold with boundary $(\overline{\Omega}_\nu, \partial\Omega_\nu)$, i.e. the union of two copies of $\overline{\Omega}_\nu$ with opposite orientations and the boundaries identified. Then Y_ν is a compact oriented surface without boundary.

(11.13) Lemma. *We have $H^1(Y_\nu, \mathbb{R}) = 0$.*

Proof. Let us first compute $H^1_c(\Omega_\nu, \mathbb{R})$. Let u be a closed 1-form with compact support in Ω_ν. By Poincaré duality $H^1_c(X, \mathbb{R}) = 0$, so $u = df$ for some function $f \in \mathcal{D}(X)$. As $df = 0$ on a neighborhood of $X \smallsetminus \Omega_\nu$ and as all connected components of this set are non compact, f must be equal to the constant zero near $X \smallsetminus \Omega_\nu$. Hence $u = df$ is the zero class in $H^1_c(\Omega_\nu, \mathbb{R})$ and we get $H^1_c(\Omega_\nu, \mathbb{R}) = H^1(\Omega_\nu, \mathbb{R}) = 0$. The exact sequence of the pair $(\overline{\Omega}_\nu, \partial\Omega_\nu)$ yields

$$\mathbb{R} = H^0(\overline{\Omega}_\nu, \mathbb{R}) \longrightarrow H^0(\partial\Omega_\nu, \mathbb{R}) \longrightarrow H^1(\overline{\Omega}_\nu, \partial\Omega_\nu\,;\mathbb{R}) \simeq H^1_c(\Omega_\nu, \mathbb{R}) = 0,$$

thus $H^0(\partial\Omega_\nu, \mathbb{R}) = \mathbb{R}$. Finally, the Mayer-Vietoris sequence applied to small neighborhoods of the two copies of $\overline{\Omega}_\nu$ in Y_ν gives an exact sequence

$$H^0(\overline{\Omega}_\nu, \mathbb{R})^{\oplus 2} \longrightarrow H^0(\partial\Omega_\nu, \mathbb{R}) \longrightarrow H^1(Y_\nu, \mathbb{R}) \longrightarrow H^1(\overline{\Omega}_\nu, \mathbb{R})^{\oplus 2} = 0$$

where the first map is onto. Hence $H^1(Y_\nu, \mathbb{R}) = 0$. □

Proof End of the proof of the uniformization theorem. Extend the almost complex structure of $\overline{\Omega}_\nu$ in an arbitrary way to Y_ν, e.g. by an extension of a Riemannian metric. Then Y_ν becomes a compact Riemann surface of genus 0, thus $Y_\nu \simeq \mathbb{P}^1$ and we obtain in particular a holomorphic embedding $\Phi_\nu : \Omega_\nu \longrightarrow \mathbb{C}$. Fix a point $a \in \Omega_0$ and a non zero linear form $\xi^\star \in T_aX$. We can take the composition of Φ_ν with an affine linear map $\mathbb{C} \to \mathbb{C}$ so that $\Phi_\nu(a) = 0$ and $d\Phi_\nu(a) = \xi^\star$. By the well-known properties of injective holomorphic maps, (Φ_ν) is then uniformly bounded on every small disk centered at a, thus also on every compact subset of X by a connectedness argument. Hence (Φ_ν) has a subsequence converging towards an injective holomorphic map $\Phi : X \longrightarrow \mathbb{C}$. □

Chapter IX
Finiteness Theorems for q-Convex Spaces and Stein Spaces

1. Topological Preliminaries

1.A. Krull Topology of $\mathcal{O}_n$-Modules

We shall use in an essential way different kind of topological results. The first of these concern the topology of modules over a local ring and depend on the Artin-Rees and Krull lemmas. Let R be a noetherian local ring; "local" means that R has a unique maximal ideal $\mathfrak{m}$, or equivalently, that R has an ideal $\mathfrak{m}$ such that every element $\alpha \in R \smallsetminus \mathfrak{m}$ is invertible.

(1.1) Nakayama lemma. *Let E be a finitely generated R-module such that $\mathfrak{m}E = E$. Then $E = \{0\}$.*

Proof. By induction on the number of generators of E: if E is generated by $x_1, \ldots, x_p$, the hypothesis $E = \mathfrak{m}E$ shows that $x_p = \alpha_1 x_1 + \cdots + \alpha_p x_p$ with $\alpha_j \in \mathfrak{m}$; as $1 - \alpha_p \in R \smallsetminus \mathfrak{m}$ is invertible, we see that x_p can be expressed in terms of $x_1, \ldots, x_{p-1}$ if $p > 1$ and that $x_1 = 0$ if $p = 1$. □

(1.2) Artin-Rees lemma. *Let F be a finitely generated R-module and let E be a submodule. There exists an integer s such that*

$$E \cap \mathfrak{m}^k F = \mathfrak{m}^{k-s}(E \cap \mathfrak{m}^s F) \qquad \text{for} \quad k \geq s.$$

Proof. Let R_t be the polynomial ring $R[\mathfrak{m}t] = R + \mathfrak{m}t + \cdots + \mathfrak{m}^k t^k + \cdots$ where t is an indeterminate. If $g_1, \ldots, g_p$ is a set of generators of the ideal $\mathfrak{m}$ over R, we see that the ring R_t is generated by $g_1 t, \ldots, g_p t$ over R, hence R_t is also noetherian. Now, we consider the R_t-modules

$$E_t = \bigoplus E\, t^k, \qquad F_t = \bigoplus\, (\mathfrak{m}^k F)\, t^k.$$

Then F_t is generated over R_t by the generators of F over R, hence the submodule $E_t \cap F_t$ is finitely generated. Let s be the highest exponent of t in a set of generators $P_1(t), \ldots, P_N(t)$ of $E_t \cap F_t$. If we identify the components of t^k in the extreme terms of the equality

$$\bigoplus\ (E \cap \mathfrak{m}^k F)\, t^k = E_t \cap F_t = \sum_j \Big(\bigoplus_k \mathfrak{m}^k\, t^k \Big) P_j(t),$$

we get

$$E \cap \mathfrak{m}^k F \subset \sum_{l \leq s} \mathfrak{m}^{k-l}(E \cap \mathfrak{m}^l F) \subset \mathfrak{m}^{k-s}(E \cap \mathfrak{m}^s F).$$

The opposite inclusion is clear. □

(1.3) Krull lemma. *Let F be a finitely generated R-module and let E be a submodule. Then*

a) $\bigcap_{k \geq 0} \mathfrak{m}^k F = \{0\}$.

b) $\bigcap_{k \geq 0} (E + \mathfrak{m}^k F) = E$.

Proof. a) Put $G = \bigcap_{k \geq 0} \mathfrak{m}^k F \subset F$. By the Artin-Rees lemma, there exists $s \in \mathbb{N}$ such that $G \cap \mathfrak{m}^k F = \mathfrak{m}^{k-s}(G \cap \mathfrak{m}^s F)$. Taking $k = s+1$, we find $G \subset \mathfrak{m} G$, hence $\mathfrak{m} G = G$ and $G = \{0\}$ by the Nakayama lemma.

b) By applying a) to the quotient module F/E we get $\bigcap \mathfrak{m}^k (F/E) = \{0\}$. Property b) follows. □

Now assume that $R = \mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ and $\mathfrak{m} = (z_1, \ldots, z_n)$. Then $\mathcal{O}_n/\mathfrak{m}^k$ is a finite dimensional vector space generated by the monomials z^α, $|\alpha| < k$. It follows that $E/\mathfrak{m}^k E$ is a finite dimensional vector space for any finitely generated $\mathcal{O}_n$-module E. As $\bigcap \mathfrak{m}^k E = \{0\}$ by 1.3 a), there is an injection

$$(1.4) \quad E \hookrightarrow \prod_{k \in \mathbb{N}} E/\mathfrak{m}^k E.$$

We endow E with the Hausdorff topology induced by the product, i.e. with the weakest topology that makes all projections $E \longrightarrow E/\mathfrak{m}^k E$ continuous for the complex vector space topology on $E/\mathfrak{m}^k E$. This topology is called the *Krull topology* (or rather, the *analytic Krull topology*; the "algebraic" Krull topology would be obtained by taking the discrete topology on $E/\mathfrak{m}^k E$). For $E = \mathcal{O}_n$, this is the topology of simple convergence on coefficients, defined by the collection of semi-norms $\sum c_\alpha z^\alpha \longmapsto |c_\alpha|$. Observe that this topology is not complete: the completion of $\mathcal{O}_n$ can be identified with the ring of formal power series $\mathbb{C}[[z_1, \ldots, z_n]]$. In general, the completion is the inverse limit $\widehat{E} = \varprojlim E/\mathfrak{m}^k E$. Every $\mathcal{O}_n$-homomorphism $E \longrightarrow F$ is continuous, because the induced finite dimensional linear maps $E/\mathfrak{m}^k E \longrightarrow F/\mathfrak{m}^k F$ are continuous.

(1.5) Theorem. *Let $E \subset F$ be finitely generated $\mathcal{O}_n$-modules. Then:*

a) *The map $F \longrightarrow G = F/E$ is open, i.e. the Krull topology of G is the quotient of the Krull topology of F;*

b) *E is closed in F and the topology induced by F on E coincides with the Krull topology of E.*

Proof. a) is an immediate consequence of the fact that the surjective finite dimensional linear maps $F/\mathfrak{m}^k F \longrightarrow G/\mathfrak{m}^k G$ are open.

b) Let $\overline{E}$ be the closure of E in F. The image of $\overline{E}$ in $F/\mathfrak{m}^k F$ is mapped into the closure of the image of E. As every subspace of a finite dimensional space is closed, the images of E and $\overline{E}$ must coincide, i.e. $\overline{E} + \mathfrak{m}^k F = E + \mathfrak{m}^k F$. Therefore

$$E \subset \overline{E} \subset \bigcap (E + \mathfrak{m}^k F) = E$$

thanks to 1.3 b). The topology induced by F on E is the weakest that makes all projections $E \longrightarrow E/E \cap \mathfrak{m}^k F$ continuous (via the injections $E/E \cap \mathfrak{m}^k F \hookrightarrow F/\mathfrak{m}^k F$). However, the Artin-Rees lemma gives

$$\mathfrak{m}^k E \subset E \cap \mathfrak{m}^k F = \mathfrak{m}^{k-s}(E \cap \mathfrak{m}^s F) \subset \mathfrak{m}^{k-s} E \quad \text{for} \quad k \geq s,$$

so the topology induced by F coincides with that induced by $\prod E/\mathfrak{m}^k E$. □

1.B. Compact Pertubations of Linear Operators

We now recall some basic results in the perturbation theory of linear operators. These results will be needed in order to obtain a finiteness criterion for cohomology groups.

(1.6) Definition. *Let E, F be Hausdorff locally convex topological vector spaces and $g : E \longrightarrow F$ a continuous linear operator.*

a) *g is said to be compact if there exists a neighborhood U of 0 in E such that the image $\overline{g(U)}$ is compact in F.*

b) *g is said to be a monomorphism if g is a topological isomorphism of E onto a closed subspace of F, and a quasi-monomorphism if $\ker g$ is finite dimensional and $\widetilde{g} : E/\ker g \longrightarrow F$ a monomorphism.*

c) *g is said to be an epimorphism if g is surjective and open, and a quasi-epimorphism if g is an epimorphism of E onto a closed finite codimensional subspace $F' \subset F$.*

d) *g is said to be a quasi-isomorphism if g is simultaneously a quasi-monomorphism and a quasi-epimorphism.*

(1.7) Lemma. *Assume that E, F are Fréchet spaces. Then*

a) *g is a (quasi-) monomorphism if and only if $g(E)$ is closed in F and g is injective (resp. and* $\ker g$ *is finite dimensional).*

b) *g is a (quasi-) epimorphism if and only if g is surjective (resp. $g(E)$ is finite codimensional).*

Proof. a) If $g(E)$ is closed, the map $\widetilde{g} : E/\ker g \longrightarrow g(E)$ is a continuous bijective linear map between Fréchet spaces, so $\widetilde{g}$ is a topological isomorphism by Banach's theorem.

b) If g is surjective, Banach's theorem implies that g is open, thus g is an epimorphism. If $g(E)$ is finite codimensional, let S be a supplementary subspace of $g(E)$ in F, $\dim S < +\infty$. Then the map

$$G \ : \ (E/\ker g) \oplus S \longrightarrow F, \qquad \widetilde{x} \oplus y \longmapsto \widetilde{g}(\widetilde{x}) + y$$

is a bijective linear map between Fréchet spaces, so it is a topological isomorphism. In particular $g(E) = G\big((E/\ker g) \oplus \{0\}\big)$ is closed as an image of a closed subspace. Hence $g(E)$ is also a Fréchet space and $g : E \longrightarrow g(E)$ is an epimorphism. □

(1.8) Theorem. *Let $h : E \longrightarrow F$ be a compact linear operator.*

a) *If $g : E \longrightarrow F$ is a quasi-monomorphism, then $g + h$ is a quasi-monomorphism.*

b) *If E, F are Fréchet spaces and if $g : E \longrightarrow F$ is a quasi-epimorphism, then $g + h$ is a quasi-epimorphism.*

Proof. Set $f = g + h$ and let U be an open convex symmetric neighborhood of 0 in E such that $K = \overline{h(U)}$ is compact.

a) It is sufficient to show that there is a finite dimensional subspace $E' \subset E$ such that $f_{\restriction E'}$ is a monomorphism. If we take E' equal to a supplementary subspace of $\ker g$, we see that we may assume g injective. Then g is a monomorphism, so we may assume in fact that E is a subspace of F and that g is the inclusion. Let V be an open convex symmetric neighborhood of 0 in F such that $U = V \cap E$. There exists a closed finite codimensional subspace $F' \subset F$ such that $K \cap F' \subset 2^{-1}V$ because $\bigcap_{F'} K \cap F' = \{0\}$. If we replace E by $E' = h^{-1}(F')$ and U by $U' = U \cap E'$, we get

$$K' := \overline{h(U')} \subset K \cap F' \subset 2^{-1}V.$$

Hence, we may assume without loss of generality that $K \subset 2^{-1}V$. Then we show that $f = g + h$ is actually a monomorphism. If Ω is an arbitrary open neighborhood of 0 in E, we have to check that there exists a neighborhood W of 0 in F such that $f(x) \in W \Longrightarrow x \in \Omega$. There is an integer N such that $2^{-N}K \cap E \subset \Omega$. We choose W convex and so small that

$$(W + 2^{-N}K) \cap E \subset \Omega \quad \text{and} \quad 2^N W + K \subset 2^{-1}V.$$

Let $x \in E$ be such that $f(x) \in W$. Then $x \in 2^n U$ for n large enough and we infer

$$x = f(x) - h(x) \in W + 2^n K \subset 2^{n-1}V \quad \text{provided that} \quad n \geq -N.$$

Thus $x \in 2^{n-1}V \cap E = 2^{n-1}U$. By induction we finally get $x \in 2^{-N}U$, so

$$x \in (W + 2^{-N}K) \cap E \subset \Omega.$$

b) By Lemma 1.7 b), we only have to show that there is a finite dimensional subspace $S \subset F$ such that the induced map

$$\widetilde{f} : E \longrightarrow F \longrightarrow F/S$$

is surjective. If we take S equal to a supplementary subspace of $g(E)$ and replace g, h by the induced maps $\widetilde{g}, \widetilde{h} : E \longrightarrow F/S$, we may assume that g itself is surjective. Then g is open, so $V = g(U)$ is a convex open neighborhood of 0 in F. As K is compact, there exists a finite set of elements $b_1, \ldots, b_N \in K$ such that $K \subset \bigcup(b_j + 2^{-1}V)$. If we take now $S = \mathrm{Vect}(b_1, \ldots, b_N)$, we obtain $\widetilde{K} \subset 2^{-1}\widetilde{V}$ where $\widetilde{K}$ is the closure of $\widetilde{h}(U)$ and $V = \widetilde{g}(U)$, so we may assume in addition that $K \subset 2^{-1}V$. Then we show that $f = g + h$ is actually surjective. Let $y_0 \in V$. There exists $x_0 \in U$ such that $g(x_0) = y_0$, thus

$$y_1 = y_0 - f(x_0) = -h(x_0) \in K \subset 2^{-1}V.$$

By induction, we construct $x_n \in 2^{-n}U$ such that $g(x_n) = y_n$ and

$$y_{n+1} = y_n - f(x_n) = -h(x_n) \in 2^{-n}K \subset 2^{-n-1}V.$$

Hence $y_{n+1} = y_0 - f(x_0 + \cdots + x_n)$ tends to 0 in F, but we still have to make sure that the series $\sum x_n$ converges in E. Let U_p be a fundamental system of convex neighborhoods of 0 in E such that $U_{p+1} \subset 2^{-1}U_p$. For each p, K is contained in the union of the open sets $g(2^n U_p \cap 2^{-1}U)$ when $n \in \mathbb{N}$, equal to $g(2^{-1}U) = 2^{-1}V$. There exists an increasing sequence $N(p)$ such that $K \subset g(2^{N(p)}U_p \cap 2^{-1}U)$, thus

$$2^{1-n}K \subset g(2^{N(p)+1-n}U_p \cap 2^{-n}U).$$

As $y_n \in 2^{1-n}K$, we see that we can choose $x_n \in 2^{N(p)+1-n}\, U_p \cap 2^{-n}\, U$ for $N(p) < n \leq N(p+1)$; then

$$x_{N(p)+1} + \cdots + x_{N(p+1)} \in (1 + 2^{-1} + \cdots\ )\, U_p \subset 2\, U_p.$$

As E is complete, the series $x = \sum x_n$ converges towards an element x such that $f(x) = y_0$, and f is surjective. □

The following important finiteness theorem due to L. Schwartz can be easily deduced from this.

(1.9) Theorem. *Let $(E^\bullet, d)$ and $(F^\bullet, \delta)$ be complexes of Fréchet spaces with continuous differentials, and $\rho^\bullet : E^\bullet \longrightarrow F^\bullet$ a continuous complex morphism. If ρ^q is compact and $H^q(\rho^\bullet) : H^q(E^\bullet) \longrightarrow H^q(F^\bullet)$ surjective, then $H^q(F^\bullet)$ is a Hausdorff finite dimensional space.*

Proof. Consider the operators

$$g, h \ : \ Z^q(E^\bullet) \oplus F^{q-1} \longrightarrow Z^q(F^\bullet),$$
$$g(x \oplus y) = \rho^q(x) + \delta^{q-1}(y), \qquad h(x \oplus y) = -\rho^q(x).$$

As $Z^q(E^\bullet) \subset E^q$, $Z^q(F^\bullet) \subset F^q$ are closed, all our spaces are Fréchet spaces. Moreover the hypotheses imply that h is compact and g is surjective since $H^q(\rho^\bullet)$ is surjective. Hence g is an epimorphism and $f = g + h = 0 \oplus \delta^{q-1}$ is a quasi-epimorphism by 1.8 b). Therefore $B^q(F^\bullet)$ is closed and finite codimensional in $Z^q(F^\bullet)$, thus $H^q(F^\bullet)$ is Hausdorff and finite dimensional. □

(1.10) Remark. If $\rho^\bullet : E^\bullet \longrightarrow F^\bullet$ is a continuous morphism of Fréchet complexes and if $H^q(\rho^\bullet)$ is surjective, then $H^q(\rho^\bullet)$ is in fact open, because the above map g is open. If $H^q(\rho^\bullet)$ is bijective, it follows that $H^q(\rho^\bullet)$ is necessarily a topological isomorphism (however $H^q(E^\bullet)$ and $H^q(F^\bullet)$ need not be Hausdorff). □

1.C. Abstract Mittag-Leffler Theorem

We will also need the following abstract Mittag-Leffler theorem, which is a very efficient tool in order to deal with cohomology groups of inverse limits.

(1.11) Proposition. *Let $(E^\bullet_\nu, \delta)_{\nu\in\mathbb{N}}$ be a sequence of Fréchet complexes together with morphisms $E^\bullet_{\nu+1} \longrightarrow E^\bullet_\nu$. We assume that the image of $E^\bullet_{\nu+1}$ in $E^\bullet_\nu$ is dense and we let $E^\bullet = \varprojlim E^\bullet_\nu$ be the inverse limit complex.*

a) *If all maps $H^q(E^\bullet_{\nu+1}) \longrightarrow H^q(E^\bullet_\nu)$, $\nu \in \mathbb{N}$, are surjective, then the limit $H^q(E^\bullet) \longrightarrow H^q(E^\bullet_0)$ is surjective.*
b) *If all maps $H^q(E^\bullet_{\nu+1}) \longrightarrow H^q(E^\bullet_\nu)$, $\nu \in \mathbb{N}$, have a dense range, then $H^q(E^\bullet) \longrightarrow H^q(E^\bullet_0)$ has a dense range.*
c) *If all maps $H^{q-1}(E^\bullet_{\nu+1}) \longrightarrow H^{q-1}(E^\bullet_\nu)$ have a dense range and all maps $H^q(E^\bullet_{\nu+1}) \longrightarrow H^q(E^\bullet_\nu)$ are injective, $\nu \in \mathbb{N}$, then $H^q(E^\bullet) \longrightarrow H^q(E^\bullet_0)$ is injective.*
d) *Let $\varphi^\bullet : F^\bullet \longrightarrow E^\bullet$ be a morphism of Fréchet complexes that has a dense range. If every map $H^q(F^\bullet) \longrightarrow H^q(E^\bullet_\nu)$ has a dense range, then $H^q(F^\bullet) \longrightarrow H^q(E^\bullet)$ has a dense range.*

Proof. If x is an element of $E^\bullet$ or of $E^\bullet_\mu$, $\mu \geq \nu$, we denote by x^ν its canonical image in $E^\bullet_\nu$. Let d_ν be a translation invariant distance that defines the topology of $E^\bullet_\nu$. After replacement of $d_\nu(x, y)$ by

$$d'_\nu(x,y) = \max_{\mu \le \nu} \{d_\mu(x^\mu, y^\mu)\}, \qquad x,y \in E^\bullet_\nu,$$

we may assume that all maps $E^\bullet_{\nu+1} \longrightarrow E^\bullet_\nu$ are Lipschitz continuous with coefficient 1.

a) Let $x_0 \in Z^q(E^\bullet_0)$ represent a given cohomology class $\overline{x}_0 \in H^q(E^\bullet_0)$. We construct by induction a convergent sequence $x_\nu \in Z^q(E^\bullet_\nu)$ such that $\overline{x}_\nu$ is mapped onto $\overline{x}_0$. If x_ν is already chosen, we can find by assumption $x_{\nu+1} \in Z^q(E^\bullet_{\nu+1})$ such that $\overline{x}^\nu_{\nu+1} = \overline{x}_\nu$, i.e. $x^\nu_{\nu+1} = x_\nu + \delta y_\nu$ for some $y_\nu \in E^{q-1}_\nu$. If we replace $x_{\nu+1}$ by $x_{\nu+1} - \delta y_{\nu+1}$ where $y_{\nu+1} \in E^{q-1}_{\nu+1}$ yields an approximation $y^\nu_{\nu+1}$ of y_ν, we may assume that $\max\{d_\nu(y_\nu, 0), d_\nu(\delta y_\nu, 0)\} \le 2^{-\nu}$. Then (x_ν) converges to a limit $\xi \in Z^q(E^\bullet)$ and we have $\xi^0 = x_0 + \delta \sum y^0_\nu$.

b) The density assumption for cohomology groups implies that the map

$$Z^q(E^\bullet_{\nu+1}) \times E^{q-1}_\nu \longrightarrow Z^q(E^\bullet_\nu), \qquad (x_{\nu+1}, y_\nu) \longmapsto x^\nu_{\nu+1} + \delta y_\nu$$

has a dense range. If we approximate y_ν by elements coming from $E^{q-1}_{\nu+1}$, we see that the map $Z^q(E^\bullet_{\nu+1}) \longrightarrow Z^q(E^\bullet_\nu)$ has also a dense range. If $x_0 \in Z^q(E^\bullet_0)$, we can find inductively a sequence $x_\nu \in Z^q(E^\bullet_\nu)$ such that $d_\nu(x^\nu_{\nu+1}, x_\nu) \le \varepsilon 2^{-\nu-1}$ for all ν, thus (x_ν) converges to an element $\xi \in Z^q(E^\bullet)$ such that $d_0(\xi^0, x_0) \le \varepsilon$ and $Z^q(E^\bullet) \longrightarrow Z^q(E^\bullet_0)$ has a dense range.

c) Let $x \in Z^q(E^\bullet)$ be such that $\overline{x}^0 \in H^q(E^\bullet_0)$ is zero. By assumption, the image of $\overline{x}$ in $H^q(E^\bullet_\nu)$ must be also zero, so we can write $x^\nu = dy_\nu$, $y_\nu \in E^{q-1}_\nu$. We have $z_\nu = y^\nu_{\nu+1} - y_\nu \in Z^{q-1}(E^\bullet_\nu)$. Let $z_{\nu+1} \in Z^{q-1}(E^\bullet_{\nu+1})$ be such that $z^\nu_{\nu+1}$ approximates z_ν. If we replace $y_{\nu+1}$ by $y_{\nu+1} - z_{\nu+1}$, we still have $x^{\nu+1} = dy_{\nu+1}$ and we may assume in addition that $d_\nu(y^\nu_{\nu+1}, y_\nu) \le 2^{-\nu}$. Then (y_ν) converges towards an element $y \in E^{q-1}$ such that $x = dy$, thus $\overline{x} = 0$ and $H^q(E^\bullet) \longrightarrow H^q(E^\bullet_0)$ is injective.

d) For every class $\overline{y} \in H^q(E^\bullet)$, the hypothesis implies the existence of a sequence $x_\nu \in Z^q(F^\bullet)$ such that $\varphi^q(\overline{x}_\nu)^\nu$ converges to $\overline{y}^\nu$, that is, $d_\nu(y^\nu, \varphi^q(x_\nu)^\nu + \delta z_\nu)$ tends to 0 for some sequence $z_\nu \in E^{q-1}_\nu$. Approximate z_ν by $\varphi^{q-1}(w_\nu)^\nu$ for some $w_\nu \in F^{q-1}$ and replace x_ν by $x'_\nu = x_\nu + \delta w_\nu$. Then $\varphi^q(x'_\nu)$ converges to y in $Z^q(E^\bullet)$. □

2. q-Convex Spaces

2.A. q-Convex Functions

The concept of q-convexity, first introduced in (Rothstein 1955) and further developed by (Andreotti-Grauert 1962), generalizes the concepts of pseudoconvexity already considered in chapters 1 and 8. Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$. A function $v \in C^2(M, \mathbb{R})$ is said to be strongly (resp. weakly) q-convex at a point $x \in M$ if $id'd''v(x)$ has at least $(n - q + 1)$

strictly positive (resp. nonnegative) eigenvalues, or equivalently if there exists a $(n-q+1)$-dimensional subspace $F \subset T_xM$ on which the complex Hessian H_xv is positive definite (resp. semi-positive). Weak 1-convexity is thus equivalent to plurisubharmonicity. Some authors use different conventions for the number of positive eigenvalues in q-convexity. The reason why we introduce the number $n-q+1$ instead of q is mainly due to the following result:

(2.1) Proposition. *If $v \in C^2(M,\mathbb{R})$ is strongly (weakly) q-convex and if Y is a submanifold of M, then $v_{\restriction Y}$ is strongly (weakly) q-convex.*

Proof. Let $d = \dim Y$. For every $x \in Y$, there exists $F \subset T_xM$ with $\dim F = n-q+1$ such that Hv is (semi-) positive on F. Then $G = F \cap T_xY$ has dimension $\geq (n-q+1)-(n-d) = d-q+1$, and $H(v_{\restriction Y})$ is (semi-) positive on $G \subset T_xY$. Hence $v_{\restriction Y}$ is strongly (weakly) q-convex at x. □

(2.2) Proposition. *Let $v_j \in C^2(M,\mathbb{R})$ be a weakly (strongly) q_j-convex function, $1 \leq j \leq s$, and $\chi \in C^2(\mathbb{R}^s,\mathbb{R})$ a convex function that is increasing (strictly increasing) in all variables. Then $v = \chi(v_1,\ldots,v_s)$ is weakly (strongly) q-convex with $q-1 = \sum(q_j-1)$. In particular $v_1+\cdots+v_s$ is weakly (strongly) q-convex.*

Proof. A simple computation gives

$$Hv = \sum_j \frac{\partial\chi}{\partial t_j}(v_1,\ldots,v_s)\,Hv_j + \sum_{j,k}\frac{\partial^2\chi}{\partial t_j\partial t_k}(v_1,\ldots,v_s)\,d'v_j\otimes\overline{d'v_k}, \tag{2.3}$$

and the second sum defines a semi-positive hermitian form. In every tangent space T_xM there exists a subspace F_j of codimension q_j-1 on which Hv_j is semi-positive (positive definite). Then $F = \bigcap F_j$ has codimension $\leq q-1$ and Hv is semi-positive (positive definite) on F. □

The above result cannot be improved, as shown by the trivial example

$$v_1(z) = -2|z_1|^2+|z_2|^2+|z_3|^2, \qquad v_2(z) = |z_1|^2-2|z_2|^2+|z_3|^2 \quad \text{on} \quad \mathbb{C}^3,$$

in which case $q_1 = q_2 = 2$ but v_1+v_2 is only 3-convex. However, formula (2.3) implies the following result.

(2.4) Proposition. *Let $v_j \in C^2(M,\mathbb{R})$, $1 \leq j \leq s$, be such that every convex linear combination $\sum \alpha_j v_j$, $\alpha_j \geq 0$, $\sum\alpha_j = 1$, is weakly (strongly) q-convex. If $\chi \in C^2(\mathbb{R}^s,\mathbb{R})$ is a convex function that is increasing (strictly increasing) in all variables, then $\chi(v_1,\ldots,v_s)$ is weakly (strongly) q-convex.* □

The invariance property of Prop. 2.1 immediately suggests the definition of q-convexity on complex spaces or analytic schemes:

(2.5) Definition. *Let $(X, \mathcal{O}_X)$ be an analytic scheme. A function v on X is said to be strongly (resp. weakly) q-convex of class C^k on X if X can be covered by patches $G : U \xrightarrow{\simeq} A$, $A \subset \Omega \subset \mathbb{C}^N$ such that for each patch there exists a function $\widetilde{v}$ on Ω with $\widetilde{v}_{\restriction A} \circ G = v_{\restriction U}$, which is strongly (resp. weakly) q-convex of class C^k.*

The notion of q-convexity on a patch U does not depend on the way U is embedded in $\mathbb{C}^N$, as shown by the following lemma.

(2.6) Lemma. *Let $G : U \longrightarrow A \subset \Omega \subset \mathbb{C}^N$ and $G' : U' \longrightarrow A' \subset \Omega' \subset \mathbb{C}^{N'}$ be two patches of X. Let $\widetilde{v}$ be a strongly (weakly) q-convex function on Ω and $v = \widetilde{v}_{\restriction A} \circ G$. For every $x \in U \cap U'$ there exists a strongly (weakly) q-convex function $\widetilde{v}'$ on a neighborhood $W' \subset \Omega'$ of $G'(x)$ such that $\widetilde{v}'_{\restriction A' \cap W'} \circ G'$ coincides with v on $G'^{-1}(W')$.*

Proof. The isomorphisms

$$G' \circ G^{-1} : A \supset G(U \cap U') \longrightarrow G'(U \cap U') \subset A'$$
$$G \circ G'^{-1} : A' \supset G'(U \cap U') \longrightarrow G(U \cap U') \subset A$$

are restrictions of holomorphic maps $H : W \longrightarrow \Omega'$, $H' : W' \longrightarrow \Omega$ defined on neighborhoods $W \ni G(x)$, $W' \ni G'(x)$; we can shrink W' so that $H'(W') \subset W$. If we compose the automorphism $(z, z') \longmapsto (z, z' - H(z))$ of $W \times \mathbb{C}^{N'}$ with the function $v(z) + |z'|^2$ we see that the function $\varphi(z, z') = \widetilde{v}(z) + |z' - H(z)|^2$ is strongly (weakly) q-convex on $W \times \Omega'$. Now, W' can be embedded in $W \times \Omega'$ via the map $z' \longmapsto \big(H'(z'), z'\big)$, so that the composite function

$$\widetilde{v}'(z') = \varphi\big(H'(z'), z'\big) = \widetilde{v}\big(H'(z')\big) + |z' - H \circ H'(z')|^2$$

is strongly (weakly) q-convex on W' by Prop. 2.1. Since $H \circ G = G'$ and $H' \circ G' = G$ on $G'^{-1}(W')$, we have $\widetilde{v}' \circ G' = \widetilde{v} \circ G = v$ on $G'^{-1}(W')$ and the lemma follows. $\square$

A consequence of this lemma is that Prop. 2.2 is still valid for an analytic scheme X (all the extensions $\widetilde{v}_j$ near a given point $x \in X$ can be obtained with respect to the same local embedding).

(2.7) Definition. *An analytic scheme $(X, \mathcal{O}_X)$ is said to be strongly (resp. weakly) q-convex if X has a C^∞ exhaustion function ψ which is strongly (resp. weakly) q-convex outside an exceptional compact set $K \subset X$. We say that X is strongly q-complete if ψ can be chosen so that $K = \emptyset$. By convention, a compact scheme X is said to be strongly 0-complete, with exceptional compact set $K = X$.*

We consider the sublevel sets

(2.8) $X_c = \{x \in X \ ; \ \psi(x) < c\}, \qquad c \in \mathbb{R}.$

If $K \subset X_c$, we may select a convex increasing function χ such that $\chi = 0$ on $]-\infty, c]$ and $\chi' > 0$ on $]c, +\infty[$. Then $\chi \circ \psi = 0$ on X_c, so that $\chi \circ \psi$ is weakly q-convex everywhere in virtue of (2.3). In the weakly q-convex case, we may therefore always assume $K = \emptyset$. The following properties are almost immediate consequences of the definition:

(2.9) Theorem.

a) *A scheme X is strongly (weakly) q-convex if and only if the reduced space X_{red} is strongly (weakly) q-convex.*

b) *If X is strongly (weakly) q-convex, every closed analytic subset Y of X_{red} is strongly (weakly) q-convex.*

c) *If X is strongly (weakly) q-convex, every sublevel set X_c containing the exceptional compact set K is strongly (weakly) q-convex.*

d) *If U_j is a weakly q_j-convex open subset of X, $1 \le j \le s$, the intersection $U = U_1 \cap \ldots \cap U_s$ is weakly q-convex with $q - 1 = \sum(q_j - 1)$; U is strongly q-convex (resp. q-complete) as soon as one of the sets U_j is strongly q_j-convex (resp. q_j-complete).*

Proof. a) is clear, since Def. 2.5 does not involve the structure sheaf $\mathcal{O}_X$. In cases b) and c), let ψ be an exhaustion of the required type on X. Then $\psi_{\restriction Y}$ and $1/(c - \psi)$ are exhaustions on Y and X_c respectively (this is so only if Y is closed). Moreover, these functions are strongly (weakly) q-convex on $Y \smallsetminus (K \cap Y)$ and $X_c \smallsetminus K$, thanks to Prop. 2.1 and 2.2. For property d), note that a sum $\psi = \psi_1 + \cdots + \psi_s$ of exhaustion functions on the sets U_j is an exhaustion on U, choose the ψ_j's weakly q_j-convex everywhere, and apply Prop. 2.2. □

(2.10) Corollary. *Any finite intersection $U = U_1 \cap \ldots \cap U_s$ of weakly 1-convex open subsets is weakly 1-convex. The set U is strongly 1-convex (resp. 1-complete) as soon as one of the sets U_j is strongly 1-convex (resp. 1-complete).*

2.B. Neighborhoods of q-complete subspaces

We prove now a rather useful result asserting the existence of q-complete neighborhoods for q-complete subvarieties. The case $q = 1$ goes back to (Siu 1976), who used a much more complicated method. The first step is an approximation-extension theorem for strongly q-convex functions.

(2.11) Proposition. *Let Y be an analytic set in a complex space X and ψ a strongly q-convex C^∞ function on Y. For every continuous function $\delta > 0$ on Y, there exists a strongly q-convex C^∞ function φ on a neighborhood V of Y such that $\psi \le \varphi_{\restriction Y} < \psi + \delta$.*

Proof. Let Z_k be a stratication of Y as given by Prop. II.5.6, i.e. Z_k is an increasing sequence of analytic subsets of Y such that $Y = \bigcup Z_k$ and $Z_k \smallsetminus Z_{k-1}$ is a smooth k-dimensional manifold (possibly empty for some k's). We shall prove by induction on k the following statement:

There exists a C^∞ function φ_k on X which is strongly q-convex along Y and on a closed neighborhood $\overline{V}_k$ of Z_k in X, such that $\psi \le \varphi_{k\restriction Y} < \psi + \delta$.

We first observe that any smooth extension φ_{-1} of ψ to X satisfies the requirements with $Z_{-1} = V_{-1} = \emptyset$. Assume that V_{k-1} and φ_{k-1} have been constructed. Then $Z_k \smallsetminus V_{k-1} \subset Z_k \smallsetminus Z_{k-1}$ is contained in $Z_{k,\mathrm{reg}}$. The closed set $Z_k \smallsetminus V_{k-1}$ has a locally finite covering (A_λ) in X by open coordinate patches $A_\lambda \subset \Omega_\lambda \subset \mathbb{C}^{N_\lambda}$ in which Z_k is given by equations $z'_\lambda = (z_{\lambda,k+1}, \ldots, z_{\lambda,N_\lambda}) = 0$. Let θ_λ be C^∞ functions with compact support in A_λ such that $0 \le \theta_\lambda \le 1$ and $\sum \theta_\lambda = 1$ on $Z_k \smallsetminus V_{k-1}$. We set

$$\varphi_k(x) = \varphi_{k-1}(x) + \sum \theta_\lambda(x)\, \varepsilon_\lambda^3 \log(1 + \varepsilon_\lambda^{-4}|z'_\lambda|^2) \quad \text{on} \quad X.$$

For $\varepsilon_\lambda > 0$ small enough, we will have $\psi \le \varphi_{k-1\restriction Y} \le \varphi_{k\restriction Y} < \psi + \delta$. Now, we check that φ_k is still strongly q-convex along Y and near any $x_0 \in \overline{V}_{k-1}$, and that φ_k becomes strongly q-convex near any $x_0 \in Z_k \smallsetminus V_{k-1}$. We may assume that $x_0 \in \operatorname{Supp}\theta_\mu$ for some μ, otherwise φ_k coincides with φ_{k-1} in a neighborhood of x_0. Select μ and a small neighborhood $W \subset\subset \Omega_\mu$ of x_0 such that

a) if $x_0 \in Z_k \smallsetminus V_{k-1}$, then $\theta_\mu(x_0) > 0$ and $A_\mu \cap W \subset\subset \{\theta_\mu > 0\}$;

b) if $x_0 \in A_\lambda$ for some λ (there is only a finite set I of such λ's), then $A_\mu \cap W \subset\subset A_\lambda$ and $z_{\lambda\restriction A_\mu \cap W}$ has a holomorphic extension $\widetilde{z}_\lambda$ to $\overline{W}$;

c) if $x_0 \in \overline{V}_{k-1}$, then $\varphi_{k-1\restriction A_\mu \cap W}$ has a strongly q-convex extension $\widetilde{\varphi}_{k-1}$ to $\overline{W}$;

d) if $x_0 \in Y \smallsetminus \overline{V}_{k-1}$, then $\varphi_{k-1\restriction Y\cap W}$ has a strongly q-convex extension $\widetilde{\varphi}_{k-1}$ to $\overline{W}$.

Otherwise take an arbitrary smooth extension $\widetilde{\varphi}_{k-1}$ of $\varphi_{k-1\restriction A_\mu \cap W}$ to $\overline{W}$ and let $\widetilde{\theta}_\lambda$ be an extension of $\theta_{\lambda\restriction A_\mu\cap W}$ to $\overline{W}$. Then

$$\widetilde{\varphi}_k = \widetilde{\varphi}_{k-1} + \sum \widetilde{\theta}_\lambda\, \varepsilon_\lambda^3 \log(1 + \varepsilon_\lambda^{-4}|\widetilde{z}'_\lambda|^2)$$

is an extension of $\varphi_{k\restriction A_\mu\cap W}$ to $\overline{W}$, resp. of $\varphi_{k\restriction Y\cap W}$ to $\overline{W}$ in case d). As the function $\log(1 + \varepsilon_\lambda^{-4}|\widetilde{z}'_\lambda|^2)$ is plurisubharmonic and as its first derivative $\langle \widetilde{z}'_\lambda, d\widetilde{z}'_\lambda\rangle\, (\varepsilon_\lambda^4 + |\widetilde{z}'_\lambda|^2)^{-1}$ is bounded by $O(\varepsilon_\lambda^{-2})$, we see that

$$id'd''\widetilde{\varphi}_k \ge id'd''\widetilde{\varphi}_{k-1} - O(\textstyle\sum \varepsilon_\lambda).$$

Therefore, for ε_λ small enough, $\widetilde{\varphi}_k$ remains q-convex on $\overline{W}$ in cases c) and d). Since all functions $\widetilde{z}'_\lambda$ vanish along $Z_k \cap W$, we have

$$id'd''\widetilde{\varphi}_k \ge id'd''\widetilde{\varphi}_{k-1} + \sum_{\lambda\in I} \theta_\lambda\, \varepsilon_\lambda^{-1}\, id'd''|\widetilde{z}'_\lambda|^2 \ge id'd''\widetilde{\varphi}_{k-1} + \theta_\mu\, \varepsilon_\mu^{-1}\, id'd''|z'_\mu|^2$$

at every point of $Z_k \cap W$. Moreover $id'd''\widetilde{\varphi}_{k-1}$ has at most $(q-1)$-negative eigenvalues on TZ_k since $Z_k \subset Y$, whereas $id'd''|z'_\mu|^2$ is positive definite in the normal directions to Z_k in Ω_μ. In case a), we thus find that $\widetilde{\varphi}_k$ is strongly q-convex on $\overline{W}$ for ε_μ small enough; we also observe that only finitely many conditions are required on each ε_λ if we choose a locally finite covering of $\bigcup \operatorname{Supp} \theta_\lambda$ by neighborhoods W as above. Therefore, for ε_λ small enough, φ_k is strongly q-convex on a neighborhood $\overline{V}'_k$ of $Z_k \smallsetminus V_{k-1}$. The function φ_k and the set $V_k = V'_k \cup V_{k-1}$ satisfy the requirements at order k. It is clear that we can choose the sequence φ_k stationary on every compact subset of X ; the limit φ and the open set $V = \bigcup V_k$ fulfill the proposition. □

The second step is the existence of almost plurisubharmonic functions having poles along a prescribed analytic set. By an almost plurisubharmonic function on a manifold, we mean a function that is locally equal to the sum of a plurisubharmonic function and of a smooth function, or equivalently, a function whose complex Hessian has bounded negative part. On a complex space, we require that our function can be locally extended as an almost plurisubharmonic function in the ambient space of an embedding.

(2.12) Lemma. *Let Y be an analytic subvariety in a complex space X. There is an almost plurisubharmonic function v on X such that $v = -\infty$ on Y with logarithmic poles and $v \in C^\infty(X \smallsetminus Y)$.*

Proof. Since $\mathcal{I}_Y \subset \mathcal{O}_X$ is a coherent subsheaf, there is a locally finite covering of X by patches A_λ isomorphic to analytic sets in balls $B(0, r_\lambda) \subset \mathbb{C}^{N_\lambda}$, such that $\mathcal{I}_Y$ admits a system of generators $g_\lambda = (g_{\lambda,j})$ on a neighborhood of each set $\overline{A}_\lambda$. We set

$$v_\lambda(z) = \log |g_\lambda(z)|^2 - \frac{1}{r_\lambda^2 - |z - z_\lambda|^2} \quad \text{on} \quad A_\lambda,$$
$$v(z) = M_{(1,\ldots,1)}\big(\ldots, v_\lambda(z), \ldots\big) \quad \text{for} \;\; \lambda \;\; \text{such that} \;\; A_\lambda \ni z,$$

where M_η is the regularized max function defined in I-3.37. As the generators $(g_{\lambda,j})$ can be expressed in terms of one another on a neighborhood of $\overline{A}_\lambda \cap \overline{A}_\mu$, we see that the quotient $|g_\lambda|/|g_\mu|$ remains bounded on this set. Therefore none of the values $v_\lambda(z)$ for $A_\lambda \ni z$ and z near ∂A_λ contributes to the value of v, since $1/(r_\lambda^2 - |z - z_\lambda|^2)$ tends to $+\infty$ on ∂A_λ. It follows that v is smooth on $X \smallsetminus Y$; as each v_λ is almost plurisubharmonic on A_λ, we also see that v is almost plurisubharmonic on X. □

(2.13) Theorem. *Let X be a complex space and Y a strongly q-complete analytic subset. Then Y has a fundamental family of strongly q-complete neighborhoods V in X.*

Proof. By Prop. 2.11 applied to a strongly q-convex exhaustion of Y and $\delta = 1$, there exists a strongly q-convex function φ on a neighborhood W_0 of Y such that $\varphi_{\restriction Y}$ is an exhaustion. Let W_1 be a neighborhood of Y such that $\overline{W}_1 \subset W_0$ and such that $\varphi_{\restriction \overline{W}_1}$ is an exhaustion. We are going to show that every neighborhood $W \subset W_1$ of Y contains a strongly q-complete neighborhood V. If v is the function given by Lemma 2.12, we set

$$\widetilde{v} = v + \chi \circ \varphi \quad \text{on} \quad \overline{W}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth convex increasing function. If χ grows fast enough, we get $\widetilde{v} > 0$ on ∂W and the $(q-1)$-codimensional subspace on which $id'd''\varphi$ is positive definite (in some ambient space) is also positive definite for $id'd''\widetilde{v}$ provided that χ' be large enough to compensate the bounded negative part of $id'd''v$. Then $\widetilde{v}$ is strongly q-convex. Let θ be a smooth convex increasing function on $]-\infty, 0[$ such that $\theta(t) = 0$ for $t < -3$ and $\theta(t) = -1/t$ on $]-1, 0[$. The open set $V = \{z \in W\,;\, \widetilde{v}(z) < 0\}$ is a neighborhood of Y and $\widetilde{\psi} = \varphi + \theta \circ \widetilde{v}$ is a strongly q-convex exhaustion of V. □

2.C. Runge Open Subsets

In order to extend the classical Runge theorem into an approximation result for sheaf cohomology groups, we need the concept of a q-Runge open subset.

(2.14) Definition. *An open subset U of a complex space X is said to be q-Runge (resp. q-Runge complete) in X if for every compact subset $L \subset U$ there exists a smooth exhaustion function ψ on X and a sublevel set X_b of ψ such that $L \subset X_b \subset\subset U$ and ψ is strongly q-convex on $X \smallsetminus \overline{X}_b$ (resp. on the whole space X).*

(2.15) Example. If X is strongly q-complete and if ψ is a strongly q-convex exhaustion function of X, then every sublevel set X_c of ψ is q-Runge complete in X: every compact set $L \subset X_c$ satisfies $L \subset X_b \subset\subset X_c$ for some $b < c$. More generally, if X is strongly q-convex and if ψ is strongly q-convex on $X \smallsetminus K$, every sublevel set X_c containing K is q-Runge in X.

Later on, we shall need the following technical result.

(2.16) Proposition. *Let Y be an analytic subset of a complex space X. If U is a q-Runge complete open subset of Y and L a compact subset, there exist a neighborhood V of Y in X and a strongly q-convex exhaustion $\widetilde{\psi}$ on V such that $U = Y \cap V$ and $L \subset Y \cap V_b \subset\subset U$ for some sublevel set V_b of $\widetilde{\psi}$.*

Proof. Let ψ be a strongly q-convex exhaustion on Y with $L \subset \{\psi < b\} \subset\subset U$ as in Def. 2.14. Then $L \subset \{\psi < b - \delta\}$ for some number $\delta > 0$ and Lemma 2.11 gives a strongly q-convex function φ on a neighborhood W_0 of Y so that $\psi \le \varphi_{\restriction Y} < \psi + \delta$. The neighborhood V and the function $\widetilde{\psi} = \varphi + \theta \circ \widetilde{v}$ constructed in the proof of Th. 2.13 are the desired ones: we have $\psi \le \widetilde{\psi}_{\restriction Y} = \varphi_{\restriction Y} < \psi+\delta$, thus

$$L \subset Y \cap V_{b-\delta} \subset \{\psi < b\} \subset\subset U. \qquad \square$$

3. q-Convexity Properties in Top Degrees

It is obvious by definition that a n-dimensional complex manifold M is strongly q-complete for $q \ge n+1$ (an arbitrary smooth function is then strongly q-convex !). If M is connected and non compact, (Greene and Wu 1975) have shown that M is strongly n-complete, i.e. there is a smooth exhaustion function ψ on M such that $id'd''\psi$ has at least one positive eigenvalue everywhere. We need the following lemmas.

(3.1) Lemma. *Let ψ be a strongly q-convex function on M and $\varepsilon > 0$ a given number. There exists a hermitian metric ω on M such that the eigenvalues $\gamma_1 \le \ldots \le \gamma_n$ of the Hessian form $id'd''\psi$ with respect to ω satisfy $\gamma_1 \ge -\varepsilon$ and $\gamma_q = \ldots = \gamma_n = 1$.*

Proof. Let ω_0 be a fixed hermitian metric, $A_0 \in C^\infty(\mathrm{End}\, TM)$ the hermitian endomorphism associated to the hermitian form $id'd''\psi$ with respect to ω_0, and $\gamma_1^0 \le \ldots \le \gamma_n^0$ the eigenvalues of A_0 (or $id'd''\psi$). We can choose a function $\eta \in C^\infty(M, \mathbb{R})$ such that $0 < \eta(x) \le \gamma_q^0(x)$ at each point $x \in M$. Select a positive function $\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\theta(t) \ge |t|/\varepsilon \ \text{ for } \ t \le 0, \quad \theta(t) \ge t \ \text{ for } \ t \ge 0, \quad \theta(t) = t \ \text{ for } \ t \ge 1.$$

We let ω be the hermitian metric defined by the hermitian endomorphism

$$A(x) = \eta(x)\,\theta[(\eta(x))^{-1}A_0(x)]$$

where $\theta[\eta^{-1}A_0] \in C^\infty(\mathrm{End}\, TM)$ is defined as in Lemma VII-6.2. By construction, the eigenvalues of $A(x)$ are $\alpha_j(x) = \eta(x)\theta\big(\gamma_j^0(x)/\eta(x)\big) > 0$ and we have

$$\begin{aligned} &\alpha_j(x) \ge |\gamma_j^0(x)|/\varepsilon && \text{for } \ \gamma_j^0(x) \le 0, \\ &\alpha_j(x) \ge \gamma_j^0(x) && \text{for } \ \gamma_j^0(x) \ge 0, \\ &\alpha_j(x) = \gamma_j^0(x) && \text{for } \ j \ge q \quad \big(\text{then } \gamma_j^0(x) \ge \eta(x)\big). \end{aligned}$$

The eigenvalues of $id'd''\psi$ with respect to ω are $\gamma_j(x) = \gamma_j^0(x)/\alpha_j(x)$ and they have the required properties. $\square$

On a hermitian manifold (M, ω), we consider the Laplace operator Δ_ω defined by

$$(3.2)\quad \Delta_\omega v = \mathrm{Trace}_\omega(id'd''v) = \sum_{1\le j,k\le n} \omega^{jk}(z)\frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}$$

where (ω^{jk}) is the conjugate of the inverse matrix of (ω_{jk}). Note that Δ_ω may differ from the usual Laplace-Beltrami operator if ω is not Kähler. We say that v is strongly ω-subharmonic if $\Delta_\omega v > 0$. This property implies clearly that v is strongly n-convex; however, as

$$\Delta_\omega \chi(v_1,\ldots,v_s) = \sum_j \frac{\partial\chi}{\partial t_j}(v_1,\ldots,v_s)\,\Delta_\omega v_j + \sum_{j,k}\frac{\partial^2\chi}{\partial t_j\partial t_k}(v_1,\ldots,v_s)\,\langle d'v_j, d'v_k\rangle_\omega,$$

subharmonicity has the advantage of being preserved by all convex increasing transformations. Conversely, if ψ is strongly n-convex and ω chosen as in Lemma 3.1 with ε small enough, we get $\Delta_\omega\psi \ge 1-(n-1)\varepsilon > 0$, thus ψ is strongly subharmonic for a suitable metric ω.

(3.3) Lemma. *Let $U, W \subset M$ be open sets such that for every connected component U_s of U there is a connected component $W_{t(s)}$ of W such that $W_{t(s)} \cap U_s \ne \emptyset$ and $W_{t(s)} \smallsetminus \overline{U}_s \ne \emptyset$. Then there exists a function $v \in C^\infty(M,\mathbb{R})$, $v \ge 0$, with support contained in $\overline{U}\cup\overline{W}$, such that v is strongly ω-subharmonic and > 0 on U.*

Proof. We first prove that the result is true when U, W are small cylinders with the same radius and axis. Let $a_0 \in M$ be a given point and $z_1,\ldots,z_n$ holomorphic coordinates centered at a_0. We set $\mathrm{Re}\,z_j = x_{2j-1}$, $\mathrm{Im}\,z_j = x_{2j}$, $x' = (x_2,\ldots,x_{2n})$ and $\omega = \sum \widetilde{\omega}_{jk}(x)dx_j\otimes dx_k$. Let U be the cylinder $|x_1| < r$, $|x'| < r$, and W the cylinder $r-\varepsilon < x_1 < r+\varepsilon$, $|x'| < r$. There are constants $c, C > 0$ such that

$$\sum \widetilde{\omega}^{jk}(x)\xi_j\xi_k \ge c|\xi|^2 \quad\text{and}\quad \sum|\widetilde{\omega}^{jk}(x)| \le C \quad\text{on } \overline{U}.$$

Let $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ be a nonnegative function equal to 0 on $]-\infty,-r]\cup[r+\varepsilon,+\infty[$ and strictly convex on $]-r,r]$. We take explicitly $\chi(x_1) = (x_1+r)\exp(-1/(x_1+r)^2)$ on $]-r,r]$ and

$$v(x) = \chi(x_1)\exp\big(1/(|x'|^2-r^2)\big)\quad\text{on } U\cup W,\quad v=0\quad\text{on } M\smallsetminus(U\cup W).$$

We have $v \in C^\infty(M,\mathbb{R})$, $v > 0$ on U, and a simple computation gives

$$\begin{aligned}\frac{\Delta_\omega v(x)}{v(x)} &= \widetilde{\omega}^{11}(x)\big(4(x_1+r)^{-5} - 2(x_1+r)^{-3}\big)\\ &+ \sum_{j>1}\widetilde{\omega}^{1j}(x)\big(1+2(x_1+r)^{-2}\big)(-2x_j)(r^2-|x'|^2)^{-2}\\ &+ \sum_{j,k>1}\widetilde{\omega}^{jk}(x)\Big(x_jx_k\big(4-8(r^2-|x'|^2)\big) - 2(r^2-|x'|^2)^2\delta_{jk}\Big)(r^2-|x'|^2)^{-4}.\end{aligned}$$

For r small, we get

$$\frac{\Delta_\omega v(x)}{v(x)} \geq 2c(x_1+r)^{-5} - C_1(x_1+r)^{-2}|x'|(r^2-|x'|^2)^{-2} + (2c|x'|^2 - C_2 r^4)(r^2-|x'|^2)^{-4}$$

with constants C_1, C_2 independent of r. The negative term is bounded by $C_3(x_1+r)^{-4} + c|x'|^2(r^2-|x'|^2)^{-4}$, hence

$$\Delta_\omega v/v(x) \geq c(x_1+r)^{-5} + (c|x'|^2 - C_2 r^4)(r^2-|x'|^2)^{-4}.$$

The last term is negative only when $|x'| < C_4 r^2$, in which case it is bounded by $C_5 r^{-4} < c(x_1+r)^{-5}$. Hence v is strongly ω-subharmonic on U.

Next, assume that U and W are connected. Then $U \cup W$ is connected. Fix a point $a \in W \smallsetminus \overline{U}$. If $z_0 \in U$ is given, we choose a path $\Gamma \subset U \cup W$ from z_0 to a which is piecewise linear with respect to holomorphic coordinate patches. Then we can find a finite sequence of cylinders (U_j, W_j) of the type described above, $1 \leq j \leq N$, whose axes are segments contained in Γ, such that

$$U_j \cup W_j \subset U \cup W, \quad \overline{W}_j \subset U_{j+1} \quad \text{and} \quad z_0 \in U_0, \quad a \in W_N \subset W \smallsetminus \overline{U}.$$

For each such pair, we have a function $v_j \in C^\infty(M)$ with support in $\overline{U}_j \cup \overline{W}_j$, $v_j \geq 0$, strongly ω-subharmonic and > 0 on U_j. By induction, we can find constants $C_j > 0$ such that $v_0 + C_1 v_1 + \cdots + C_j v_j$ is strongly ω-subharmonic on $U_0 \cup \ldots \cup U_j$ and ω-subharmonic on $M \smallsetminus \overline{W}_j$. Then

$$w_{z_0} = v_0 + C_1 v_1 + \ldots + C_N v_N \geq 0$$

is ω-subharmonic on U and strongly ω-subharmonic > 0 on a neighborhood Ω_0 of the given point z_0. Select a denumerable covering of U by such neighborhoods Ω_p and set $v(z) = \sum \varepsilon_p w_{z_p}(z)$ where ε_p is a sequence converging sufficiently fast to 0 so that $v \in C^\infty(M, \mathbb{R})$. Then v has the required properties.

In the general case, we find for each pair $(U_s, W_{t(s)})$ a function v_s with support in $\overline{U}_s \cup \overline{W}_{t(s)}$, strongly ω-subharmonic and > 0 on U_s. Any convergent series $v = \sum \varepsilon_s v_s$ yields a function with the desired properties. □

(3.4) Lemma. *Let X be a connected, locally connected and locally compact topological space. If U is a relatively compact open subset of X, we let $\widetilde{U}$ be the union of U with all compact connected components of $X \smallsetminus U$. Then $\widetilde{U}$ is open and relatively compact in X, and $X \smallsetminus \widetilde{U}$ has only finitely many connected components, all non compact.*

Proof. A rather easy exercise of general topology. Intuitively, $\widetilde{U}$ is obtained by "filling the holes" of U in X. □

(3.5) Theorem (Greene-Wu 1975). *Every n-dimensional connected non compact complex manifold M has a strongly subharmonic exhaustion function with respect to any hermitian metric ω. In particular, M is strongly n-complete.*

Proof. Let $\varphi \in C^\infty(M, \mathbb{R})$ be an arbitrary exhaustion function. There exists a sequence of connected smoothly bounded open sets $\Omega'_\nu \subset\subset M$ such that $\overline{\Omega'_\nu} \subset \Omega'_{\nu+1}$ and $M = \bigcup \Omega'_\nu$. Let $\Omega_\nu = \widetilde{\Omega'_\nu}$ be the relatively compact open set given by Lemma 3.4. Then $\overline{\Omega_\nu} \subset \Omega_{\nu+1}$, $M = \bigcup \Omega_\nu$ and $M \smallsetminus \Omega_\nu$ has no compact connected component. We set

$$U_1 = \Omega_2, \qquad U_\nu = \Omega_{\nu+1} \smallsetminus \overline{\Omega_{\nu-2}} \quad \text{for} \quad \nu \geq 2.$$

Then $\partial U_\nu = \partial\Omega_{\nu+1} \cup \partial\Omega_{\nu-2}$; any connected component $U_{\nu,s}$ of U_ν has its boundary $\partial U_{\nu,s} \not\subset \partial\Omega_{\nu-2}$, otherwise $\overline{U}_{\nu,s}$ would be open and closed in $M \smallsetminus \Omega_{\nu-2}$, hence $\overline{U}_{\nu,s}$ would be a compact component of $M \smallsetminus \Omega_{\nu-2}$. Therefore $\partial U_{\nu,s}$ intersects $\partial\Omega_{\nu+1} \subset U_{\nu+1}$. If $\partial U_{\nu+1,t(s)}$ is a connected component of $U_{\nu+1}$ containing a point of $\partial U_{\nu,s}$, then $U_{\nu+1,t(s)} \cap U_{\nu,s} \neq \emptyset$ and $U_{\nu+1,t(s)} \smallsetminus \overline{U}_{\nu,s} \neq \emptyset$. Lemma 7 implies that there is a nonnegative function $v_\nu \in C^\infty(M, \mathbb{R})$ with support in $U_\nu \cup U_{\nu+1}$, which is strongly ω-subharmonic on U_ν. An induction yields constants C_ν such that

$$\psi_\nu = \varphi + C_1 v_1 + \cdots + C_\nu v_\nu$$

is strongly ω-subharmonic on $\overline{\Omega_\nu} \subset U_0 \cup \ldots \cup U_\nu$, thus $\psi = \varphi + \sum C_\nu v_\nu$ is a strongly ω-subharmonic exhaustion function on M. □

By an induction on the dimension, the above result can be generalized to an arbitrary complex space (or analytic scheme), as was first shown by T. Ohsawa.

(3.6) Theorem (Ohsawa 1984). *Let X be a complex space of maximal dimension n.*

a) *X is always strongly $(n+1)$-complete.*

b) *If X has no compact irreducible component of dimension n, then X is strongly n-complete.*

c) *If X has only finitely many irreducible components of dimension n, then X is strongly n-convex.*

Proof. We prove a) and b) by induction on $n = \dim X$. For $n = 0$, property b) is void and a) is obvious (any function can then be considered as strongly 1-convex). Assume that a) has been proved in dimension $\leq n-1$. Let X' be the union of X_{sing} and of the irreducible components of X of dimension at most $n-1$, and $M = X \smallsetminus X'$ the n-dimensional part of X_{reg}. As $\dim X' \leq n-1$, the induction hypothesis shows that X' is strongly n-complete. By Th. 2.13, there exists a strongly n-convex exhaustion function φ' on a neighborhood V' of X'. Take a closed neighborhood $\overline{V} \subset V'$ and an arbitrary exhaustion φ on X that extends $\varphi'_{\restriction \overline{V}}$. Since every function on a n-dimensional manifold is strongly $(n+1)$-convex, we conclude that X is at worst $(n+1)$-complete, as stated in a).

In case b), the hypothesis means that the connected components M_j of $M = X \smallsetminus X'$ have non compact closure $\overline{M}_j$ in X. On the other hand, Lemma 3.1 shows that there exists a hermitian metric ω on M such that $\varphi_{\restriction M \cap V}$ is strongly ω-subharmonic. Consider the open sets $U_{j,\nu} \subset M_j$ provided by Lemma 3.7 below. By the arguments already used in Th. 3.5, we can find a strongly ω-subharmonic exhaustion $\psi = \varphi + \sum_{j,\nu} C_{j,\nu} v_{j,\nu}$ on X, with $v_{j,\nu}$ strongly ω-subharmonic on $U_{j,\nu}$, $\operatorname{Supp} v_{j,\nu} \subset U_{j,\nu} \cup U_{j,\nu+1}$ and $C_{j,\nu}$ large. Then ψ is strongly n-convex on X.

(3.7) Lemma. *For each j, there exists a sequence of open sets $U_{j,\nu} \subset\subset M_j$, $\nu \in \mathbb{N}$, such that*

a) *$M_j \smallsetminus V' \subset \bigcup_\nu U_{j,\nu}$ and $(U_{j,\nu})$ is locally finite in $\overline{M}_j$;*

b) *for every connected component $U_{j,\nu,s}$ of $U_{j,\nu}$ there is a connected component $U_{j,\nu+1,t(s)}$ of $U_{j,\nu+1}$ such that $U_{j,\nu+1,t(s)} \cap U_{j,\nu,s} \neq \emptyset$ and $U_{j,\nu+1,t(s)} \smallsetminus \overline{U}_{j,\nu,s} \neq \emptyset$.*

Proof. By Lemma 3.4 applied to the space $\overline{M}_j$, there exists a sequence of relatively compact connected open sets $\Omega_{j,\nu}$ in $\overline{M}_j$ such that $\overline{M}_j \smallsetminus \Omega_{j,\nu}$ has no compact connected component, $\overline{\Omega}_{j,\nu} \subset \Omega_{j,\nu+1}$ and $\overline{M}_j = \bigcup \Omega_{j,\nu}$. We define a compact set $K_{j,\nu} \subset M_j$ and an open set $W_{j,\nu} \subset \overline{M}_j$ containing $K_{j,\nu}$ by

$$K_{j,\nu} = (\overline{\Omega}_{j,\nu} \smallsetminus \Omega_{j,\nu-1}) \smallsetminus V', \qquad W_{j,\nu} = \Omega_{j,\nu+1} \smallsetminus \overline{\Omega}_{j,\nu-2}.$$

By induction on ν, we construct an open set $U_{j,\nu} \subset\subset W_{j,\nu} \smallsetminus X' \subset M_j$ and a finite set $F_{j,\nu} \subset \partial U_{j,\nu} \smallsetminus \overline{\Omega}_{j,\nu}$. We let $F_{j,-1} = \emptyset$. If these sets are already constructed for $\nu-1$, the compact set $K_{j,\nu} \cup F_{j,\nu-1}$ is contained in the open set $W_{j,\nu}$, thus contained in a finite union of connected components $W_{j,\nu,s}$. We can write $K_{j,\nu} \cup F_{j,\nu-1} = \bigcup L_{j,\nu,s}$ where $L_{j,\nu,s}$ is contained in $W_{j,\nu,s} \smallsetminus X' \subset M_j$. The open set $W_{j,\nu,s} \smallsetminus X'$ is connected and non contained in $\overline{\Omega}_{j,\nu} \cup L_{j,\nu,s}$, otherwise its closure $\overline{W}_{j,\nu,s}$ would have no boundary point $\in \partial \Omega_{j,\nu+1}$, thus would be open and compact in $\overline{M}_j \smallsetminus \Omega_{j,\nu-2}$, contradiction. We select a point $a_s \in (W_{j,\nu,s} \smallsetminus X') \smallsetminus (\overline{\Omega}_{j,\nu} \cup L_{j,\nu,s})$ and a smoothly bounded connected open set $U_{j,\nu,s} \subset\subset W_{j,\nu,s} \smallsetminus X'$ containing $L_{j,\nu,s}$ with $a_s \in \partial U_{j,\nu,s}$. Finally, we set $U_{j,\nu} = \bigcup_s U_{j,\nu,s}$ and let $F_{j,\nu}$ be the set of all points a_s. By construction, we have $U_{j,\nu} \supset K_{j,\nu} \cup F_{j,\nu-1}$, thus $\bigcup U_{j,\nu} \supset \bigcup K_{j,\nu} = M_j \smallsetminus V'$, and $\partial U_{j,\nu,s} \ni a_s$ with $a_s \in F_{j,\nu} \subset U_{j,\nu+1}$. Property b) follows. □

Proof of Theorem 3.6 c) (end). Let $Y \subset X$ be the union of X_{sing} with all irreducible components of X that are non compact or of dimension $< n$.

Then $\dim Y \le n-1$, so Y is n-convex and Th. 2.13 implies that there is an exhaustion function $\psi \in C^\infty(X,\mathbb{R})$ such that ψ is strongly n-convex on a neighborhood V of Y. Then the complement $K = X \smallsetminus V$ is compact and ψ is strongly n-convex on $X \smallsetminus K$. □

(3.8) Proposition. *Let M be a connected non compact n-dimensional complex manifold and U an open subset of M. Then U is n-Runge complete in M if and only if $M \smallsetminus U$ has no compact connected component.* □

Proof. First observe that a strongly n-convex function cannot have any local maximum, so it satisfies the maximum principle. If $M \smallsetminus U$ has a compact connected component T, then T has a compact neighborhood L in M such that $\partial L \subset U$. We have $\max_L \psi = \max_{\partial L} \psi$ for every strongly n-convex function, thus $\partial L \subset M_b$ implies $L \subset M_b$; thus we cannot find a sublevel set M_b such that $\partial L \subset M_b \subset\subset U$, and U is not n-Runge in M.

On the other hand, assume that $M \smallsetminus U$ has no compact connected component and let L be a compact subset of U. Let ω be any hermitian metric on M and φ a strongly ω-subharmonic exhaustion function on M. Set $b = 1+\sup_L \varphi$ and

$$P = \{x \in M \smallsetminus U\,;\, \varphi(x) \le b\}.$$

As $M \smallsetminus U$ has no compact connected component, all its components T_α contain a point y_α in

$$W = \{x \in X\,;\, \varphi(x) > b+1\}.$$

For every point $x \in P$ with $x \in T_\alpha$, there exists a connected open set $V_x \subset\subset M \smallsetminus L$ containing x such that $\partial V_x \ni y_\alpha$ ($M \smallsetminus L$ is a neighborhood of $M \smallsetminus U$ and we can consider a tubular neighborhood of a path from x to y_α in $M \smallsetminus L$). The compact set P can be covered by a finite number of open sets V_{x_j}. Then Lemma 3.3 yields functions v_j with support in $\overline{V}_{x_j} \cup \overline{W}$ which are strongly ω-subharmonic on V_{x_j}. Let χ be a convex increasing function such that $\chi(t) = 0$ on $]-\infty, b]$ and $\chi'(t) > 0$ on $]b, +\infty[$. Consider the function

$$\psi = \varphi + \sum C_j v_j + \chi \circ \varphi.$$

First, choose C_j large enough so that $\psi \ge b$ on P. Then choose χ increasing fast enough so that ψ is strongly ω-subharmonic on $\overline{W}$. Then ψ is a strongly n-convex exhaustion function on M, and as $\psi \ge \varphi$ on M and $\psi = \varphi$ on L, we see that

$$L \subset \{x \in M\,;\, \psi(x) < b\} \subset U.$$

This proves that U is n-Runge complete in M. □

4. Andreotti-Grauert Finiteness Theorems

4.A. Case of Vector Bundles over Manifolds

The crucial point in the proof of the Andreotti-Grauert theorems is the following special case, which is easily obtained by the methods of chapter 8.

(4.1) Proposition. *Let M be a strongly q-complete manifold with $q \geq 1$, and E a holomorphic vector bundle over M. Then:*

a) $H^k\big(M, \mathcal{O}(E)\big) = 0$ *for $k \geq q$.*

b) *Let U be a q-Runge complete open subset of M. Every d''-closed form $h \in C^\infty_{0,q-1}(U, E)$ can be approximated uniformly with all derivatives on every compact subset of U by a sequence of global d''-closed forms $\widetilde{h}_\nu \in C^\infty_{0,q-1}(M, E)$.*

Proof. We replace E by $\widetilde{E} = \Lambda^n TM \otimes E$; then we can work with forms of bidegree (n, k) instead of $(0, k)$. Let ψ be a strongly q-convex exhaustion function on M and ω the metric given by Lemma 3.1. Select a function $\rho \in C^\infty(M, \mathbb{R})$ which increases rapidly at infinity so that the hermitian metric $\widetilde{\omega} = e^\rho \omega$ is complete on M. Denote by E_χ the bundle E endowed with the hermitian metric obtained by multiplication of a fixed metric of E by the weight $\exp(-\rho \circ \psi)$ where $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ is a convex increasing function. We apply Th. VIII-4.5 for the bundle E_χ over the complete hermitian manifold $(M, \widetilde{\omega})$. Then

$$ic(E_\chi) = ic(E) + id'd''(\chi \circ \psi) \otimes \mathrm{Id}_E \geq_{\mathrm{Nak}} ic(E) + \chi' \circ \psi \; id'd''\psi \otimes \mathrm{Id}_E\,.$$

The eigenvalues of $id'd''\psi$ with respect to $\widetilde{\omega}$ are $e^{-\rho}\gamma_j$, so Lemma VII-7.2 and Prop. VI-8.3 yield

$$\begin{aligned}[ic(E_\chi), \Lambda] + T_{\widetilde{\omega}} &\geq [ic(E), \Lambda] + T_{\widetilde{\omega}} + \chi' \circ \psi \; [id'd''\psi, \Lambda] \otimes \mathrm{Id}_E \\ &\geq [ic(E), \Lambda] + T_{\widetilde{\omega}} + \chi' \circ \psi \; e^{-\rho}(\gamma_1 + \cdots + \gamma_k) \otimes \mathrm{Id}_E\end{aligned}$$

when this curvature tensor acts on (n, k)-forms. For $k \geq q$, we have

$$\gamma_1 + \cdots + \gamma_k \geq 1 - (q-1)\varepsilon > 0 \quad \text{ if } \varepsilon \leq 1/q.$$

We choose χ_0 increasing fast enough so that all the eigenvalues of the above curvature tensor are ≥ 1 when $\chi = \chi_0$. Then for every $g \in C^\infty_{n,k}(M, E)$ with $D''g = 0$ the equation $D''f = g$ can be solved with an estimate

$$\int_M |f|^2 e^{-\chi \circ \psi} dV \leq \int_M |g|^2 e^{-\chi \circ \psi} dV,$$

where $\chi = \chi_0 + \chi_1$ and where χ_1 is a convex increasing function chosen so that the integral of g converges. This gives a). In order to prove b), let

$h \in C^\infty_{n,q-1}(U,E)$ be such that $D''h = 0$ and let L be an arbitrary compact subset of U. Thanks to Def. 2.14, we can choose ψ such that there is a sublevel set M_b with $L \subset M_b \subset\subset U$. Select $b_0 < b$ so that $L \subset M_{b_0}$, and let $\theta \in C^\infty(\mathbb{R},\mathbb{R})$ be a convex increasing function such that $\theta = 0$ on $]-\infty, b_0[$ and $\theta \geq 1$ on $]b,+\infty[$. Let $\eta \in \mathcal{D}(U)$ be a cut-off function such that $\eta = 1$ on M_b. We solve the equation $D''f = g$ for $g = D''(\eta h)$ with the weight $\chi = \chi_0 + \nu\theta\circ\psi$ and let ν tend to infinity. As g has compact support in $U \smallsetminus M_b$ and $\chi\circ\psi \geq \chi_0\circ\psi + \nu$ on this set, we find a solution f_ν such that

$$\int_{M_{b_0}} |f_\nu|^2 e^{-\chi_0\circ\psi} dV \leq \int_M |f_\nu|^2 e^{-\chi\circ\psi} dV \leq \int_{U\smallsetminus M_b} |g|^2 e^{-\chi\circ\psi} dV \leq Ce^{-\nu},$$

thus f_ν converges to 0 in $L^2(M_{b_0})$ and $h_\nu = \eta h - f_\nu \in C^\infty_{n,q-1}(M,E)$ is a D''-closed form converging to h in $L^2(M_{b_0})$. However, if we choose the minimal solution such that $\delta''_\chi f_\nu = 0$ as in Rem. VIII-4.6, we get $\Delta''_\chi f_\nu = \delta''_\chi g$ on M and in particular $\Delta''_{\chi_0} f_\nu = 0$ on M_{b_0}. Gårding's inequality VI-3.3 applied to the elliptic operator Δ''_{χ_0} shows that f_ν converges to 0 with all derivatives on L, hence h_ν converges to h on L. Now, replace L by an exhaustion L_ν of U by compact sets; some diagonal subsequence h_ν converges to h in $C^\infty_{n,q-1}(U,E)$. □

4.B. A Local Vanishing Result for Sheaves

Let $(X, \mathcal{O}_X)$ be an analytic scheme and $\mathcal{S}$ a coherent sheaf of $\mathcal{O}_X$-modules. We wish to extend Prop. 4.1 to the cohomology groups $H^k(X,\mathcal{S})$. The first step is to show that the result holds on small open sets, and this is done by means of local resolutions of $\mathcal{S}$.

For a given point $x \in X$, we choose a patch $(A, \mathcal{O}_\Omega/\mathcal{J})$ of X containing x, where A is an analytic subset of $\Omega \subset \mathbb{C}^N$ and $\mathcal{J}$ a sheaf of ideals with zero set A. Let $i_A : A \longrightarrow \Omega$ be the inclusion. Then $(i_A)_\star\mathcal{S}$ is a coherent $\mathcal{O}_\Omega$-module supported on A. In particular there is a neighborhood $W_0 \subset \Omega$ of x and a surjective sheaf morphism

$$\mathcal{O}^{p_0} \longrightarrow (i_A)_\star\mathcal{S} \quad \text{on} \quad W_0, \qquad (u_1,\ldots,u_{p_0}) \longmapsto \sum_{1\leq j\leq p_0} u_j G_j$$

where $G_1,\ldots,G_{p_0} \in \mathcal{S}(A\cap W_0)$ are generators of $(i_A)_\star\mathcal{S}$ on W_0. If we repeat the procedure inductively for the kernel of the above surjective morphism, we get a *homological free resolution* of $(i_A)_\star\mathcal{S}$:

$$(4.3)\qquad \mathcal{O}^{p_l} \longrightarrow \cdots \longrightarrow \mathcal{O}^{p_1} \longrightarrow \mathcal{O}^{p_0} \longrightarrow (i_A)_\star\mathcal{S} \longrightarrow 0 \quad \text{on} \quad W_l$$

of arbitrary large length l, on neighborhoods $W_l \subset W_{l-1} \subset \ldots \subset W_0$. In particular, after replacing Ω by W_{2N} and A by $A\cap W_{2N}$, we may assume that $(i_A)_\star\mathcal{S}$ has a resolution of length $2N$ on Ω. In this case, we shall say that $A \subset \Omega$ is a $\mathcal{S}$-*distinguished patch* of X.

(4.4) Lemma. *Let $A \subset \Omega$ be a $\mathcal{S}$-distinguished patch of X and U a strongly q-convex open subset of A. Then*

$$H^k(U, \mathcal{S}) = 0 \quad \text{for } k \geq q.$$

Proof. Theorem 2.13 shows that there exists a strongly q-convex open set $V \subset \Omega$ such that $U = A \cap V$. Let us denote by $\mathcal{Z}^l$ the kernel of $\mathcal{O}^{p_l} \longrightarrow \mathcal{O}^{p_{l-1}}$ for $l \geq 1$ and $\mathcal{Z}^0 = \ker\big(\mathcal{O}^{p_0} \longrightarrow (i_A)_\star \mathcal{S}\big)$. There are exact sequences

$$0 \longrightarrow \mathcal{Z}^0 \longrightarrow \mathcal{O}^{p_0} \longrightarrow (i_A)_\star \mathcal{S} \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{Z}^l \longrightarrow \mathcal{O}^{p_l} \longrightarrow \mathcal{Z}^{l-1} \longrightarrow 0, \qquad 1 \leq l \leq 2N.$$

For $k \geq q$, Prop. 4.1 a) gives $H^k(V, \mathcal{O}^{p_l}) = 0$, therefore we get

$$H^k(U, \mathcal{S}) \simeq H^k\big(V, (i_A)_\star \mathcal{S}\big) \simeq H^{k+1}(V, \mathcal{Z}^0) \simeq \ldots \simeq H^{k+2N+1}(V, \mathcal{Z}^{2N}),$$

and the last group vanishes because $\operatorname{topdim} V \leq \dim_{\mathbb{R}} V = 2N$. $\square$

4.C. Topological Structure on Spaces of Sections and on Cohomology Groups

Let $V \subset \Omega$ be a strongly 1-complete open set relatively to a $\mathcal{S}$-distinguished patch $A \subset \Omega$ and let $U = A \cap V$. By the proof of Lemma 4.4, we have

$$H^1(V, \mathcal{Z}^0) \simeq H^{2N+1}(V, \mathcal{Z}^{2N}) = 0,$$

hence we get an exact sequence

$$(4.5) \quad 0 \longrightarrow \mathcal{Z}^0(V) \longrightarrow \mathcal{O}^{p_0}(V) \longrightarrow \mathcal{S}(U) \longrightarrow 0.$$

We are going to show that the Fréchet space structure on $\mathcal{O}^{p_0}(V)$ induces a natural Fréchet space structure on the groups of sections of $\mathcal{S}$ over any open subset. We first note that $\mathcal{Z}^0(V)$ is closed in $\mathcal{O}^{p_0}(V)$. Indeed, let $f_\nu \in \mathcal{Z}^0(V)$ be a sequence converging to a limit $f \in \mathcal{O}^{p_0}(V)$ uniformly on compact subsets of V. For every $x \in V$, the germs $(f_\nu)_x$ converge to f_x with respect to the topology defined by (1.4) on $\mathcal{O}^{p_0}$. As $\mathcal{Z}^0_x$ is closed in $\mathcal{O}^{p_0}_x$ in view of Th. 1.5 b), we get $f_x \in \mathcal{Z}^0_x$ for all $x \in V$, thus $f \in \mathcal{Z}^0(V)$.

(4.6) Proposition. *The quotient topology on $\mathcal{S}(U)$ is independent of the choices made above.*

Proof. For a smaller set $U' = A \cap V'$ where V' is a strongly 1-convex open subset of V, the restriction map $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{O}^{p_0}(V')$ is continuous, thus $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is continuous. If (V_α) is a countable covering of V by such sets and $U_\alpha = A \cap V_\alpha$, we get an injection of $\mathcal{S}(U)$ onto the closed subspace of the product $\prod \mathcal{S}(U_\alpha)$ consisting of families which are compatible in the intersections. Therefore, the Fréchet topology induced by the product coincides with the original topology of $\mathcal{S}(U)$. If we choose other generators $H_1, \ldots, H_{q_0}$ for $(i_A)_\star \mathcal{S}$, the germs $H_{j,x}$ can be expressed in terms of the $G_{j,x}$'s, thus we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^{p_0}(V) & \xrightarrow{G} & \mathcal{S}(U) & \longrightarrow & 0 \\ \downarrow & & \| & & \\ \mathcal{O}^{q_0}(V) & \xrightarrow{H} & \mathcal{S}(U) & \longrightarrow & 0 \end{array}$$

provided that U and V are small enough. If we express the generators G_j in terms of the H_j's, we find a similar diagram with opposite vertical arrows and we conclude easily that the topology obtained in both cases is the same. Finally, it remains to show that the topology of $\mathcal{S}(U)$ is independent of the embedding $A \subset \Omega$ near a given point $x \in X$. We compare the given embedding with the Zariski embedding $(A, x) \subset \Omega'$ of minimal dimension d. After shrinking A and changing coordinates, we may assume $\Omega = \Omega' \times \mathbb{C}^{N-d}$ and that the embedding $i_A : A \longrightarrow \Omega$ is the composite of $i'_A : A \longrightarrow \Omega'$ and of the inclusion $j : \Omega' \longrightarrow \Omega' \times \{0\} \subset \Omega$. For $V' \subset \Omega'$ sufficient small and $U' = A \cap V'$, we have a surjective map $G' : \mathcal{O}^{p_0}(V') \longrightarrow \mathcal{S}(U')$ obtained by choosing generators G'_j of $(i'_A)^\star \mathcal{S}$ on a neighborhood of x in Ω'. Then we consider the open set $V = V' \times \mathbb{C}^{N-d} \subset \Omega$ and the surjective map onto $\mathcal{S}(U')$ equal to the composite

$$\mathcal{O}^{p_0}(V) \xrightarrow{j^\star} \mathcal{O}^{p_0}(V') \xrightarrow{G'} \mathcal{S}(U).$$

This map corresponds to a choice of generators $G_j \in (i_A)^\star \mathcal{S}(V)$ equal to the functions G'_j, considered as functions independent of the last variables $z_{d+1}, \ldots, z_N$. Since $j^\star$ is open, it is obvious that the quotient topology on $\mathcal{S}(U')$ is the same for both embeddings. $\square$

Now, there is a natural topology on the cohomology groups $H^k(X, \mathcal{S})$. In fact, let (U_α) be a countable covering of X by strongly 1-complete open sets, such that each U_α is contained in a $\mathcal{S}$-distinguished patch. Since the intersections $U_{\alpha_0 \ldots \alpha_k}$ are again strongly 1-complete, the covering $\mathcal{U}$ is acyclic by Lemma 4.4 and Leray's theorem shows that $H^k(X, \mathcal{S})$ is isomorphic to $\check{H}^q(\mathcal{U}, \mathcal{S})$. We consider the product topology on the spaces of Čech cochains $C^k(\mathcal{U}, \mathcal{S}) = \prod \mathcal{S}(U_{\alpha_0 \ldots \alpha_k})$ and the quotient topology on $\check{H}^k(\mathcal{U}, \mathcal{S})$. It is clear that $\check{H}^0(\mathcal{U}, \mathcal{S})$ is a Fréchet space; however the higher cohomology groups $\check{H}^k(\mathcal{U}, \mathcal{S})$ need not be Hausdorff because the coboundary groups may be non closed in the cocycle groups. The resulting topology on $H^k(X, \mathcal{S})$ is independent of the choice of the covering: in fact we only have to check that the bijective continuous map $\check{H}^k(\mathcal{U}, \mathcal{S}) \longrightarrow \check{H}^k(\mathcal{U}', \mathcal{S})$ is a topological isomorphism if $\mathcal{U}'$ is a refinement of $\mathcal{U}$, and this follows from Rem. 1.10 applied to the morphism of Čech complexes $C^\bullet(\mathcal{U}, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{U}', \mathcal{S})$.

Finally, observe that when $\mathcal{S}$ is the locally free sheaf associated to a holomorphic vector bundle E on a smooth manifold X, the topology on $H^k\big(X, \mathcal{O}(E)\big)$ is the same as the topology associated to the Fréchet space structure on the Dolbeault complex $\big(C^\infty_{0,\bullet}(X,E), d''\big)$: by the analogue of formula (IV-6.11) we have a bijective continuous map

$$\begin{aligned} \check H^k\big(\mathcal{U}, \mathcal{O}(E)\big) &\longrightarrow H^k\big(C^\infty_{0,\bullet}(X,E)\big) \\ \{(c_{\alpha_0\ldots\alpha_k})\} &\longmapsto f(z) = \sum_{\alpha_0,\ldots,\alpha_q} c_{\alpha_0\ldots\alpha_q}(z)\,\theta_{\alpha_q}\,d''\theta_{\alpha_0}\wedge\ldots\wedge d''\theta_{\alpha_{q-1}} \end{aligned}$$

where (θ_α) is a partition of unity subordinate to $\mathcal{U}$. As in Rem. 1.10, the continuity of the inverse follows by the open mapping theorem applied to the surjective map

$$Z^k\big(C^\bullet(\mathcal{U}, \mathcal{O}(E))\big) \oplus C^\infty_{0,k-1}(X,E) \longrightarrow Z^k\big(C^\infty_{0,\bullet}(X,E)\big).$$

We shall need a few simple additional results.

(4.7) Proposition. *The following properties hold:*

a) *For every $x \in X$, the map $\mathcal{S}(X) \longrightarrow \mathcal{S}_x$ is continuous with respect to the topology of $\mathcal{S}_x$ defined by* (1.4).

b) *If $\mathcal{S}'$ is a coherent analytic subsheaf of $\mathcal{S}$, the space of global sections $\mathcal{S}'(X)$ is closed in $\mathcal{S}(X)$.*

c) *If $U' \subset U$ are open in X, the restriction maps $H^k(U,\mathcal{S}) \longrightarrow H^k(U',\mathcal{S})$ are continuous.*

d) *If U' is relatively compact in U, the restriction operator $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is compact.*

e) *Let $\mathcal{S} \longrightarrow \mathcal{S}'$ be a morphism of coherent sheaves over X. Then the induced maps $H^k(X,\mathcal{S}) \longrightarrow H^k(X,\mathcal{S}')$ are continuous.*

Proof. a) Let $V \subset \Omega$ be a strongly 1-convex open neighborhood of x relatively to a $\mathcal{S}$-distinguished patch $A \subset \Omega$. The map $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{O}^{p_0}_x$ is continuous, and the same is true for $\mathcal{O}^{p_0}_x \longrightarrow \mathcal{S}_x$ by §1. Therefore the composite $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{S}_x$ and its factorization $\mathcal{S}(U) \longrightarrow \mathcal{S}_x$ are continuous.

b) is a consequence of the above property a) and of the fact that each stalk $\mathcal{S}'_x$ is closed in $\mathcal{S}_x$ (cf. 1.5 b)).

c) The restriction map $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is continuous, and the case of higher cohomology groups follows immediately.

d) Assume first that $U = A \cap V$ and $U' = A \cap V'$, where $A \subset \Omega$ is a $\mathcal{S}$-distinguished patch and $V' \subset\subset V$ are strongly 1-convex open subsets of Ω. The operator $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{O}^{p_0}(V')$ is compact by Montel's theorem, thus $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is also compact. In the general case, select a finite family of strongly 1-convex sets $U'_\alpha \subset\subset U_\alpha \subset U$ such that (U'_α) covers $\overline{U}'$ and U_α is contained in some distinguished patch. There is a commutative diagram

$$\begin{array}{ccccc} \mathcal{S}(U) & & \longrightarrow & & \mathcal{S}(U') \\ \downarrow & & & & \downarrow \\ \prod \mathcal{S}(U_\alpha) & \longrightarrow & \prod \mathcal{S}(U'_\alpha) & \longrightarrow & \prod \mathcal{S}(U' \cap U'_\alpha) \end{array}$$

where the right vertical arrow is a monomorphism and where the first arrow in the bottom line is compact. Thus $\mathcal{S}(U) \longrightarrow \mathcal{S}(U')$ is compact.

e) It is enough to check that $\mathcal{S}(U) \longrightarrow \mathcal{S}'(U)$ is continuous, and for this we may assume that $U = A \cap V$ where V is a small neighborhood of a given point x. Let $G_1, \ldots, G_{p_0}$ be generators of $\mathcal{S}_x$, $G'_1, \ldots, G'_{p_0}$ their images in $\mathcal{S}'_x$. Complete these elements in order to obtain a system of generators $(G'_1, \ldots, G'_{q_0})$ of $\mathcal{S}'_x$. For V small enough, the map $\mathcal{S}(U) \longrightarrow \mathcal{S}'(U)$ is induced by the inclusion $\mathcal{O}^{p_0}(V) \longrightarrow \mathcal{O}^{p_0}(V) \times \{0\} \subset \mathcal{O}^{q_0}(V)$, hence continuous. □

4.D. Cartan-Serre Finiteness Theorem

The above results enable us to prove a finiteness theorem for cohomology groups over compact analytic schemes.

(4.8) Theorem (Cartan-Serre)**.** *Let $\mathcal{S}$ be a coherent analytic sheaf over an analytic scheme $(X, \mathcal{O}_X)$. If X is compact, all cohomology groups $H^k(X, \mathcal{S})$ are finite dimensional (and Hausdorff).*

Proof. There exist finitely many strongly 1-complete open sets $U'_\alpha \subset\subset U_\alpha$ such that each U_α is contained in some $\mathcal{S}$-distinguished patch and such that $\bigcup U'_\alpha = X$. By Prop. 4.7 d), the restriction map on Čech cochains

$$C^\bullet(\mathcal{U}, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{U}', \mathcal{S})$$

defines a compact morphism of complexes of Fréchet spaces. As the coverings $\mathcal{U} = (U_\alpha)$ and $\mathcal{U}' = (U'_\alpha)$ are acyclic by 4.4, the induced map

$$\check{H}^k(\mathcal{U}, \mathcal{S}) \longrightarrow \check{H}^k(\mathcal{U}', \mathcal{S})$$

is an isomorphism, both spaces being isomorphic to $H^k(X, \mathcal{S})$. We conclude by Schwartz' theorem 1.9. □

4.E. Local Approximation Theorem

We show that a local analogue of the approximation result 4.1 b) holds for a sheaf $\mathcal{S}$ over an analytic scheme $(X, \mathcal{O}_X)$.

(4.9) Lemma. *Let $A \subset \Omega$ be a $\mathcal{S}$-distinguished patch of X, and $U' \subset U \subset A$ open subsets such that U' is q-Runge complete in U. Then the restriction map*

$$H^{q-1}(U,\mathcal{S}) \longrightarrow H^{q-1}(U',\mathcal{S})$$

has a dense range.

Proof. Let L be an arbitrary compact subset of U'. Proposition 2.16 applied with $Y = U$ embedded in some neighborhood in Ω shows that there is a neighborhood V of U in Ω such that $A \cap V = U$ and a strongly q-convex function ψ on V such that $L \subset U_b \subset\subset U'$ for some $U_b = A \cap V_b$. The proof of Lemma 4.4 gives $H^q(V, \mathcal{Z}^0) = H^q(V_b, \mathcal{Z}^0) = 0$ and the cohomology exact sequences of $0 \to \mathcal{Z}^0 \to \mathcal{O}^{p_0} \to i_A^\star \mathcal{S} \to 0$ over V and V_b yield a commutative diagram of continuous maps

$$\begin{array}{ccccc}
H^{q-1}\big(V,\mathcal{O}^{p_0}\big) & \longrightarrow & H^{q-1}\big(V,i_A^\star\mathcal{S}\big) & = & H^{q-1}\big(U,\mathcal{S}\big)\\
\downarrow & & \downarrow & & \downarrow\\
H^{q-1}\big(V_b,\mathcal{O}^{p_0}\big) & \longrightarrow & H^{q-1}\big(V_b,i_A^\star\mathcal{S}\big) & = & H^{q-1}\big(U_b,\mathcal{S}\big)
\end{array}$$

where the horizontal arrows are surjective. Since V_b is q-Runge complete in V, the left vertical arrow has a dense range by Prop. 4.1 b). As U' is the union of an increasing sequence of sets U_{b_ν}, we only have to show that the range remains dense in the inverse limit $H^{q-1}(U', \mathcal{S})$. For that, we apply Property 1.11 d) on a suitable covering of U. Let $\mathcal{W}$ be a countable basis of the topology of U, consisting of strongly 1-convex open subsets contained in $\mathcal{S}$-distinguished patches. We let $\mathcal{W}'$ (resp. $\mathcal{W}_\nu$) be the subfamily of $W \in \mathcal{W}$ such that $W \subset\subset U'$ (resp. $W \subset\subset U_{b_\nu}$). Then $\mathcal{W}$, $\mathcal{W}'$, $\mathcal{W}_\nu$ are acyclic coverings of U, U', U_{b_ν} and each restriction map $C^\bullet(\mathcal{W},\mathcal{S}) \longrightarrow C^\bullet(\mathcal{W}_\nu,\mathcal{S})$ is surjective. Property 1.11 d) can thus be applied and the lemma follows. □

4.F. Statement and Proof of the Andreotti-Grauert Theorem

(4.10) Theorem (Andreotti-Grauert 1962). *Let $\mathcal{S}$ be a coherent analytic sheaf over a strongly q-convex analytic scheme $(X, \mathcal{O}_X)$. Then*

a) $H^k(X,\mathcal{S})$ *is Hausdorff and finite dimensional for $k \ge q$.*

Moreover, let U be a q-Runge open subset of X, $q \ge 1$. Then

b) *the restriction map $H^k(X,\mathcal{S}) \to H^k(U,\mathcal{S})$ is an isomorphism for $k \ge q$;*

c) *the restriction map $H^{q-1}(X,\mathcal{S}) \to H^{q-1}(U,\mathcal{S})$ has a dense range.*

The compact case $q = 0$ of 4.10 a) is precisely the Cartan-Serre finiteness theorem. For $q \ge 1$, the special case when X is strongly q-complete and $U = \emptyset$ yields the following very important consequence.

(4.11) Corollary. *If X is strongly q-complete, then*

$$H^k(X,\mathcal{S}) = 0 \quad \textit{for} \;\; k \ge q.$$

Assume that $q \geq 1$ and let ψ be a smooth exhaustion on X that is strongly q-convex on $X \smallsetminus K$. We first consider sublevel sets $X_d \supset X_c \supset K$, $d > c$, and verify assertions 4.10 b), c) for all restriction maps

$$H^k(X_d, \mathcal{S}) \longrightarrow H^k(X_c, \mathcal{S}), \qquad k \geq q-1.$$

The basic idea, already contained in (Andreotti-Grauert 1962), is to deform X_c into X_d through a sequence of strongly q-convex open sets (G_j) such that G_{j+1} is obtained from G_j by making a small bump.

(4.12) Lemma. *There exist a sequence of strongly q-convex open sets $G_0 \subset \ldots \subset G_s$ and a sequence of strongly q-complete open sets $U_0, \ldots, U_{s-1}$ in X such that*

a) $G_0 = X_c$, $\quad G_s = X_d$, $\quad G_{j+1} = G_j \cup U_j$ *for* $0 \leq j \leq s-1$;

b) $G_j = \{x \in X\,;\, \psi_j(x) < c_j\}$ *where ψ_j is an exhaustion function on X that is strongly q-convex on $X \smallsetminus K$;*

c) *U_j is contained in a $\mathcal{S}$-distinguished patch $A_j \subset \Omega_j$ of X;*

d) *$G_j \cap U_j$ is strongly q-complete and q-Runge complete in U_j.*

Proof. There exists a finite covering of the compact set $\overline{X}_d \smallsetminus X_c$ by $\mathcal{S}$-distinguished patches $A_j \subset \Omega_j$, $0 \leq j < s$, where $\Omega_j \subset \mathbb{C}^{N_j}$ is a euclidean ball and $K \cap A_j = \emptyset$. Let $\theta_j \in \mathcal{D}(X)$ be a family of functions such that $\operatorname{Supp}\theta_j \subset A_j$, $\theta_j \geq 0$, $\sum \theta_j \leq 1$ and $\sum \theta_j = 1$ on a neighborhood of $\overline{X}_d \smallsetminus X_c$. We can find $\varepsilon_0 > 0$ so small that

$$\psi_j = \psi - \varepsilon \sum_{0 \leq k < j} \theta_k$$

is still strongly q-convex on $X \smallsetminus K$ for $0 \leq j \leq s$ and $\varepsilon \leq \varepsilon_0$. We have $\psi_0 = \psi$ and $\psi_s = \psi - \varepsilon$ on $\overline{X}_d \smallsetminus X_c$, thus

$$G_j = \{x \in X\,;\, \psi_j(x) < c\}, \qquad 0 \leq j \leq s$$

is an increasing sequence of strongly q-convex open sets such that $G_0 = X_c$, $G_s = X_{c+\varepsilon}$. Moreover, as $\psi_{j+1} - \psi_j = -\varepsilon\theta_j$ has support in A_j, we have

$$G_{j+1} = G_j \cup U_j \quad \text{where } U_j = G_{j+1} \cap A_j.$$

It follows that conditions a), b), c) are satisfied with $c+\varepsilon$ instead of d. Finally, the functions

$$\varphi_j = 1/(c - \psi_{j+1}) + 1/(r_j^2 - |z - z_j|^2), \qquad \widetilde{\varphi}_j = 1/(c - \psi_j) + 1/(r_j^2 - |z - z_j|^2)$$

are strongly q-convex exhaustions on U_j and $G_j \cap U_j = G_j \cap A_j$. Let L be an arbitrary compact subset of $G_j \cap U_j$ and $a = \sup_L \psi_j < c$. Select $b \in]a, c[$ and set

$$\psi_{j,\eta} = \psi_j + \eta\varphi_j \quad \text{on} \ \ U_j, \qquad \eta > 0.$$

Then $\psi_{j,\eta}$ is an exhaustion of U_j. As φ_j is bounded below, we have

$$L \subset \{\psi_{j,\eta} < b\} \subset\subset \{\psi_j < c\} \cap U_j = G_j \cap U_j$$

for η small enough. Moreover

$$(1-\alpha)\psi_j + \alpha\psi_{j+1} = \psi - \varepsilon \sum_{0\le k<j} \theta_k - \alpha\varepsilon\,\theta_j$$

is strongly q-convex for all $\alpha \in [0,1]$ and $\varepsilon \le \varepsilon_0$ small enough, so Prop. 2.4 implies that $\psi_{j,\eta}$ is strongly q-convex. By definition, $G_j \cap U_j$ is thus q-Runge complete in U_j, and Lemma 4.12 is proved with $X_{c+\varepsilon}$ instead of X_d. In order to achieve the proof, we consider an increasing sequence $c = c_0 < c_1 < \ldots < c_N = d$ with $c_{k+1} - c_k \le \varepsilon_0$ and perform the same construction for each pair $X_{c_k} \subset X_{c_{k+1}}$, with c replaced by c_k and $\varepsilon = c_{k+1} - c_k$. □

(4.13) Proposition. *For every sublevel set $X_c \supset K$, the group $H^k(X_c, \mathcal{S})$ is Hausdorff and finite dimensional when $k \ge q$. Moreover, for $d > c$, the restriction map*

$$H^k(X_d, \mathcal{S}) \longrightarrow H^k(X_c, \mathcal{S})$$

is an isomorphism when $k \ge q$ and has a dense range when $k = q-1$.

Proof. Thanks to Lemma 4.12, we are led to consider the restriction maps

$$(4.14) \quad H^k(G_{j+1}, \mathcal{S}) \longrightarrow H^k(G_j, \mathcal{S}).$$

Let us apply the Mayer-Vietoris exact sequence IV-3.11 to $G_{j+1} = G_j \cup U_j$. For $k \ge q$ we have $H^k(U_j, \mathcal{S}) = H^k(G_j \cap U_j, \mathcal{S}) = 0$ by Lemma 4.4. Hence we get an exact sequence

$$\begin{array}{l} H^{q-1}(G_{j+1}, \mathcal{S}) \longrightarrow H^{q-1}(G_j, \mathcal{S}) \oplus H^{q-1}(U_j, \mathcal{S}) \longrightarrow H^{q-1}(G_j \cap U_j, \mathcal{S}) \longrightarrow \\ \quad H^k(G_{j+1}, \mathcal{S}) \longrightarrow \qquad H^k(G_j, \mathcal{S}) \qquad \longrightarrow \quad 0 \quad \longrightarrow \cdots, \qquad k \ge q. \end{array}$$

In this sequence, all the arrows are induced by restriction maps, so they define continuous linear operators. We already infer that the map (4.14) is bijective for $k > q$ and surjective for $k = q$. There exist a $\mathcal{S}$-acyclic covering $\mathcal{V} = (V_\alpha)$ of X_d and a finite family $\mathcal{V}' = (V'_{\alpha_1}, \ldots, V'_{\alpha_p})$ of open sets such that $V'_{\alpha_j} \subset\subset V_{\alpha_j}$ and $\bigcup V'_{\alpha_j} \supset \overline{X}_c$. Let $\mathcal{W}$ be a locally finite $\mathcal{S}$-acyclic covering of X_c which refines $\mathcal{V}' \cap X_c = (V'_{\alpha_j} \cap X_c)$. The refinement map

$$C^\bullet(\mathcal{V}, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{V}' \cap X_c, \mathcal{S}) \longrightarrow C^\bullet(\mathcal{W}, \mathcal{S})$$

is compact because the first arrow is, and it induces a surjective map

$$H^k(X_d, \mathcal{S}) \longrightarrow H^k(X_c, \mathcal{S}) \quad \text{for} \ \ k \ge q.$$

By Schwartz' theorem 1.9, we conclude that $H^k(X_c, \mathcal{S})$ is Hausdorff and finite dimensional for $k \geq q$. This is equally true for $H^q(G_j, \mathcal{S})$ because G_j is also a global sublevel set $\{x \in X\,;\, \psi_j(x) < c_j\}$ containing K. Now, the Mayer-Vietoris exact sequence implies that the composite

$$H^{q-1}(U_j, \mathcal{S}) \longrightarrow H^{q-1}(G_j \cap U_j, \mathcal{S}) \xrightarrow{\partial} H^q(G_{j+1}, \mathcal{S})$$

is equal to zero. However, the first arrow has a dense range by Lemma 4.9. As the target space is Hausdorff, the second arrow must be zero; we obtain therefore the injectivity of $H^q(G_{j+1}, \mathcal{S}) \longrightarrow H^q(G_j, \mathcal{S})$ and an exact sequence

$$\begin{array}{rcl} H^{q-1}(G_{j+1}, \mathcal{S}) \longrightarrow H^{q-1}(G_j, \mathcal{S}) \oplus H^{q-1}(U_j, \mathcal{S}) & \longrightarrow & H^{q-1}(G_j \cap U_j, \mathcal{S}) \longrightarrow 0 \\ g \oplus u & \longmapsto & u_{\restriction G_j \cap U_j} - g_{\restriction G_j \cap U_j}. \end{array}$$

The argument used in Rem. 1.10 shows that the surjective arrow is open. Let $g \in H^{q-1}(G_j, \mathcal{S})$ be given. By Lemma 4.9, we can approximate $g_{\restriction G_j \cap U_j}$ by a sequence $u_{\nu \restriction G_j \cap U_j}$, $u_\nu \in H^{q-1}(U_j, \mathcal{S})$. Then $w_\nu = u_{\nu \restriction G_j \cap U_j} - g_{\restriction G_j \cap U_j}$ tends to zero. As the second map in the exact sequence is open, we can find a sequence

$$g'_\nu \oplus u'_\nu \in H^{q-1}(G_j, \mathcal{S}) \oplus H^{q-1}(U_j, \mathcal{S})$$

converging to zero which is mapped on w_ν. Then $(g - g'_\nu) \oplus (u_\nu - u'_\nu)$ is mapped on zero, and there exists a sequence $f_\nu \in H^{q-1}(G_{j+1}, \mathcal{S})$ which coincides with $g - g'_\nu$ on G_j and with $u_\nu - u'_\nu$ on U_j. In particular $f_{\nu \restriction G_j}$ converges to g and we have shown that

$$H^{q-1}(G_{j+1}, \mathcal{S}) \longrightarrow H^{q-1}(G_j, \mathcal{S})$$

has a dense range. □

Proof of Andreotti-Grauert's Theorem 4.10. Let $\mathcal{W}$ be a countable basis of the topology of X consisting of strongly 1-convex open sets W_α contained in $\mathcal{S}$-distinguished patches of X. Let $L \subset U$ be an arbitrary compact subset. Select a smooth exhaustion function ψ on X such that ψ is strongly q-convex on $X \smallsetminus \overline{X}_b$ and $L \subset X_b \subset\subset U$ for some sublevel set X_b of ψ; choose $c > b$ such that $X_c \subset\subset U$. For every $d \in \mathbb{R}$, we denote by $\mathcal{W}_d \subset \mathcal{W}$ the collection of sets $W_\alpha \in \mathcal{W}$ such that $W_\alpha \subset X_d$. Then $\mathcal{W}_d$ is a $\mathcal{S}$-acyclic covering of X_d. We consider the sequence of Čech complexes

$$E^\bullet_\nu = C^\bullet(\mathcal{W}_{c+\nu}, \mathcal{S}), \qquad \nu \in \mathbb{N}$$

together with the surjective projection maps $E^\bullet_{\nu+1} \longrightarrow E^\bullet_\nu$, and their inverse limit $E^\bullet = C^\bullet(\mathcal{W}, \mathcal{S})$. Then we have $H^k(E^\bullet) = H^k(X, \mathcal{S})$ and $H^k(E^\bullet_\nu) = H^k(X_{c+\nu}, \mathcal{S})$. Propositions 1.11 (a,b,c) and 4.13 imply that $H^k(X, \mathcal{S}) \longrightarrow H^k(X_c, \mathcal{S})$ is bijective for $k \geq q$ and has a dense range for $k = q - 1$. It already follows that $H^k(X, \mathcal{S})$ is Hausdorff for $k \geq q$. Now, take an increasing sequence of open sets X_{c_ν} equal to sublevel sets of a sequence of exhaustions

ψ_ν, such that $U = \bigcup X_{c_\nu}$. Then all groups $H^k(X_{c_\nu}, \mathcal{S})$ are in bijection with $H^k(X, \mathcal{S})$ for $k \geq q$, and the image of $H^{q-1}(X_{c_{\nu+1}}, \mathcal{S})$ in $H^{q-1}(X_{c_\nu}, \mathcal{S})$ is dense because it contains the image of $H^{q-1}(X, \mathcal{S})$. Proposition 1.11 (a,b,c) again shows that $H^k(U, \mathcal{S}) \longrightarrow H^k(X_{c_0}, \mathcal{S})$ is bijective for $k \geq q$, and d) shows that $H^{q-1}(X, \mathcal{S}) \longrightarrow H^{q-1}(U, \mathcal{S})$ has a dense range. The theorem follows. □

A combination of Andreotti-Grauert's theorem with Th. 3.6 yields the following important consequence.

(4.15) Corollary. *Let $\mathcal{S}$ be a coherent sheaf over an analytic scheme $(X, \mathcal{O}_X)$ with* $\dim X \leq n$.

a) *We have* $H^k(X, \mathcal{S}) = 0$ *for all* $k \geq n+1$;

b) *If X has no compact irreducible component of dimension n, then we have $H^n(X, \mathcal{S}) = 0$.*

c) *If X has only finitely many n-dimensional compact irreducible components, then $H^n(X, \mathcal{S})$ is finite dimensional.* □

The special case of 4.15 b) when X is smooth and $\mathcal{S}$ locally free has been first proved by (Malgrange 1955), and the general case is due to (Siu 1969). Another consequence is the following approximation theorem for coherent sheaves over manifolds, which results from Prop. 3.8.

(4.16) Proposition. *Let $\mathcal{S}$ be a coherent sheaf over a non compact connected complex manifold M with* $\dim M = n$. *Let $U \subset M$ be an open subset such that the complement $M \smallsetminus U$ has no compact connected component. Then the restriction map $H^{n-1}(M, \mathcal{S}) \longrightarrow H^{n-1}(U, \mathcal{S})$ has a dense range.* □

5. Grauert's Direct Image Theorem

The goal of this section is to prove the following fundamental result on direct images of coherent analytic sheaves, due to (Grauert 1960).

(5.1) Direct image theorem. *Let X, Y be complex analytic schemes and let $F : X \to Y$ be a proper analytic morphism. If $\mathcal{S}$ is a coherent $\mathcal{O}_X$-module, the direct images $R^qF_\star\mathcal{S}$ are coherent $\mathcal{O}_Y$-modules.*

We give below a beautiful proof due to (Kiehl-Verdier 1971), which is much simpler than Grauert's original proof; this proof rests on rather deep properties of nuclear modules over nuclear Fréchet algebras. We first introduce the basic concept of topological tensor product. Our presentation owes much to the seminar lectures by (Douady-Verdier 1973).

5.A. Topological Tensor Products and Nuclear Spaces

The algebra of holomorphic functions on a product space $X \times Y$ is a completion $\mathcal{O}(X) \mathbin{\widehat{\otimes}} \mathcal{O}(Y)$ of the algebraic tensor product $\mathcal{O}(X) \otimes \mathcal{O}(Y)$. We are going to describe the construction and the basic properties of the required topological tensor products $\widehat{\otimes}$.

Let E, F be (real or complex) vector spaces equipped with semi-norms p and q, respectively. Then $E \otimes F$ can be equipped with any one of the two natural semi-norms $p \otimes_\pi q$, $p \otimes_\varepsilon q$ defined by

$$p \otimes_\pi q(t) = \inf \Big\{ \sum_{1 \le j \le N} p(x_j)\, q(y_j)\, ;\ t = \sum_{1 \le j \le N} x_j \otimes y_j\, ,\ x_j \in E\, ,\ y_j \in F \Big\},$$

$$p \otimes_\varepsilon q(t) = \sup_{||\xi||_p \le 1,\, ||\eta||_q \le 1} \big|\xi \otimes \eta(t)\big|, \qquad \xi \in E', \quad \eta \in F'\, ;$$

the inequalities in the last line mean that ξ, η satisfy $|\xi(x)| \le p(x)$ and $|\eta(y)| \le q(y)$ for all $x \in E$, $y \in F$. Then clearly $p \otimes_\varepsilon q \le p \otimes_\pi q$, for

$$p \otimes_\varepsilon q\Big(\sum x_j \otimes y_j\Big) \le \sum p \otimes_\varepsilon q(x_j \otimes y_j) \le \sum p(x_j)\, q(y_j).$$

Given $x \in E$, $y \in F$, the Hahn-Banach theorem implies that there exist ξ, η such that $||\xi||_p = ||\eta||_q = 1$ with $\xi(x) = p(x)$ and $\eta(y) = q(y)$, hence $p \otimes_\varepsilon q(x \otimes y) \ge p(x)\, q(y)$. On the other hand $p \otimes_\pi q(x \otimes y) \le p(x)\, q(y)$, thus

$$p \otimes_\varepsilon q(x \otimes y) = p \otimes_\pi q(x \otimes y) = p(x)\, q(y).$$

(5.2) Definition. *Let E, F be locally convex topological vector spaces. The topological tensor product $E \mathbin{\widehat{\otimes}_\pi} F$ (resp. $E \mathbin{\widehat{\otimes}_\varepsilon} F$) is the Hausdorff completion of $E \otimes F$, equipped with the family of semi-norms $p \otimes_\pi q$ (resp. $p \otimes_\varepsilon q$) associated to fundamental families of semi-norms on E and F.*

Since we may also write

$$p \otimes_\pi q(t) = \inf \Big\{ \sum |\lambda_j|\, ;\ t = \sum \lambda_j\, x_j \otimes y_j\, ,\ p(x_j) \le 1\, ,\ q(y_j) \le 1 \Big\}$$

where the λ_j's are scalars, we see that the closed unit ball $B(p \mathbin{\widehat{\otimes}_\pi} q)$ in $E \mathbin{\widehat{\otimes}_\pi} F$ is the closed convex hull of $B(p) \otimes B(q)$. From this, we easily infer that the topological dual space $(E \mathbin{\widehat{\otimes}_\pi} F)'$ is isomorphic to the space of continuous bilinear forms on $E \times F$. Another simple consequence of this interpretation of $B(p \mathbin{\widehat{\otimes}_\pi} q)$ is example a) below.

(5.3) Examples.

a) For all discrete spaces I, J, there is an isometry

$$\ell^1(I) \mathbin{\widehat{\otimes}_\pi} \ell^1(J) \simeq \ell^1(I \times J).$$

b) For Banach spaces (E, p), (F, q), the closed unit ball in $E \mathbin{\widehat{\otimes}_\varepsilon} F$ is dual to the unit ball $B(p' \mathbin{\widehat{\otimes}_\pi} q')$ of $E' \mathbin{\widehat{\otimes}_\pi} F'$ through the natural pairing extending the algebraic pairing of $E \otimes F$ and $E' \otimes F'$. If $c_0(I)$ denotes the space of bounded sequences on I converging to zero at infinity, we have $c_0(I)' = \ell^1(I)$, hence by duality $c_0(I) \mathbin{\widehat{\otimes}_\varepsilon} c_0(J)$ is isometric to $c_0(I \times J)$.

c) If X, Y are compact topological spaces and if $C(X)$, $C(Y)$ are their algebras of continuous functions with the sup norm, then

$$C(X) \mathbin{\widehat{\otimes}_\varepsilon} C(Y) \simeq C(X \times Y).$$

Indeed, $C(X)'$ is the space of finite Borel measures equipped with the mass norm. Thus for $f \in C(X) \otimes C(Y)$, the $\otimes_\varepsilon$-seminorm is given by

$$||f||_\varepsilon = \sup_{||\mu|| \le 1,\, ||\nu|| \le 1} \mu \otimes \nu(f) = \sup_{X \times Y} |f|\,;$$

the last equality is obtained by taking Dirac measures δ_x, δ_y for μ, ν (the inequality $\le$ is obvious). Now $C(X) \otimes C(Y)$ is dense in $C(X \times Y)$ by the Stone-Weierstrass theorem, hence its completion is $C(X \times Y)$, as desired. □

Let $f : E_1 \to E_2$ and $g : F_1 \to F_2$ be continuous morphisms. For all semi-norms p_2, q_2 on E_2, F_2, there exist semi-norms p_1, q_1 on E_1, F_1 and constants $||f|| = ||f||_{p_1,p_2}$, $||g|| = ||g||_{q_1,q_2}$ such that $p_2 \circ f \le ||f||\, p_1$ and $q_2 \circ g \le ||g||\, q_1$. Then we find

$$(p_2 \otimes_\pi q_2) \circ (f \otimes g) \le ||f||\, ||g||\, p_1 \otimes_\pi q_1$$

and a similar formula with $p_j \otimes_\varepsilon q_j$. It follows that there are well defined continuous maps

$$(5.4') \qquad f \mathbin{\widehat{\otimes}_\pi} g : E_1 \mathbin{\widehat{\otimes}_\pi} F_1 \longrightarrow E_2 \mathbin{\widehat{\otimes}_\pi} F_2,$$
$$(5.4'') \qquad f \mathbin{\widehat{\otimes}_\varepsilon} g : E_1 \mathbin{\widehat{\otimes}_\varepsilon} F_1 \longrightarrow E_2 \mathbin{\widehat{\otimes}_\varepsilon} F_2.$$

Another simple fact is that $\widehat{\otimes}_\pi$ preserves open morphisms:

(5.5) Proposition. *If $f : E_1 \to E_2$ and $g : F_1 \to F_2$ are epimorphisms, then $f \mathbin{\widehat{\otimes}_\pi} g : E_1 \mathbin{\widehat{\otimes}_\pi} F_1 \longrightarrow E_2 \mathbin{\widehat{\otimes}_\pi} F_2$ is an epimorphism.*

Proof. Recall that when E is locally convex complete and F Hausdorff, a morphism $u : E \to F$ is open if and only if $\overline{u(V)}$ is a neighborhood of 0 for every neighborhood of 0 (this can be checked essentially by the same proof as 1.8 b)). Here, for any semi-norms p, q on E_1, F_1 the closure of $f \mathbin{\widehat{\otimes}_\pi} g\big(B(p \mathbin{\widehat{\otimes}_\pi} q)\big)$ contains the closed convex hull of $f\big(B(p)\big) \otimes g\big(B(q)\big)$ in which $f\big(B(p)\big)$ and $g\big(B(q)\big)$ are neighborhoods of 0, so it is a neighborhood of 0 in $E \mathbin{\widehat{\otimes}_\pi} F$. □

If $E_1 \subset E_2$ is a closed subspace, every continuous semi-norm p_1 on E_1 is the restriction of a continuous semi-norm on E_2, and every linear form

$\xi_1 \in E_1'$ such that $||\xi_1||_{p_1} \le 1$ can be extended to a linear form $\xi_2 \in E_2$ such that $||\xi_2||_{p_2} = ||\xi_1||_{p_1}$ (Hahn-Banach theorem); similar properties hold for a closed subspace $F_1 \subset F_2$. We infer that

$$(p_2 \otimes_\varepsilon q_2)_{\restriction E_1 \otimes F_1} = p_1 \otimes_\varepsilon q_1 ,$$

thus $E_1 \mathbin{\widehat{\otimes}}_\varepsilon F_1$ is a closed subspace of $E_2 \mathbin{\widehat{\otimes}}_\varepsilon F_2$. In other words:

(5.6) Proposition. *If $f : E_1 \to E_2$ and $g : F_1 \to F_2$ are monomorphisms, then $f \mathbin{\widehat{\otimes}}_\varepsilon g : E_1 \mathbin{\widehat{\otimes}}_\varepsilon F_1 \longrightarrow E_2 \mathbin{\widehat{\otimes}}_\varepsilon F_2$ is a monomorphism.* □

Unfortunately, 5.5 fails for $\widehat{\otimes}_\varepsilon$ and 5.6 fails for $\widehat{\otimes}_\pi$, even with Fréchet or Banach spaces. It follows that neither $\widehat{\otimes}_\pi$ nor $\widehat{\otimes}_\varepsilon$ are exact functors in the category of Fréchet spaces. In order to circumvent this difficulty, it is necessary to work in a suitable subcategory.

(5.7) Definition. *A morphism $f : E \to F$ of complete locally convex spaces is said to be nuclear if f can be written as*

$$f(x) = \sum \lambda_j \, \xi_j(x) \, y_j$$

where (λ_j) is a sequence of scalars with $\sum |\lambda_j| < +\infty$, $\xi_j \in E'$ an equicontinuous sequence of linear forms and $y_j \in F$ a bounded sequence.

When E and F are Banach spaces, the space of nuclear morphisms is isomorphic to $E' \mathbin{\widehat{\otimes}}_\pi F$ and the nuclear norm $||f||_\nu$ is defined to be the norm in this space, namely

$$(5.8) \quad ||f||_\nu = \inf \Big\{ \sum |\lambda_j| \, ; \; f = \sum \lambda_j \, \xi_j \otimes y_j, \; ||\xi_j|| \le 1, \; ||y_j|| \le 1 \Big\}.$$

For general spaces E, F, the equicontinuity of (ξ_j) means that there is a seminorm p on E and a constant C such that $|\xi_j(x)| \le C \, p(x)$ for all j. Then the definition shows that $f : E \to F$ is nuclear if and only if f can be factorized as $E \to E_1 \to F_1 \to F$ where $E_1 \to F_1$ is a nuclear morphism of Banach spaces: indeed we need only take E_1 be equal to the Hausdorff completion $\widehat{E}_p$ of (E, p) and let F_1 be the subspace of F generated by the closed balanced convex hull of $\{y_j\}$ (= unit ball in F_1) ; moreover, if $u : S \to E$ and $v : F \to T$ are continuous, the nuclearity of f implies the nuclearity of $v \circ f \circ u$; its nuclear decomposition is then $v \circ f \circ u = \sum \lambda_j \, (\xi_j \circ u) \otimes v(y_j)$.

(5.9) Remark. Every nuclear morphism is compact: indeed, we may assume in Def. 5.7 that (y_j) converges to 0 and $\sum |\lambda_j| \le 1$, otherwise we replace y_j by $\varepsilon_j y_j$ with ε_j converging to zero such that $\sum |\lambda_j / \varepsilon_j| \le 1$; then, if $U \subset F$ is a neighborhood of 0 such that $|\xi_j(U)| \le 1$ for all j, the image $f(U)$ is contained in the closed convex hull of the compact set $\{y_j\} \cup \{0\}$, which is compact.

(5.10) Proposition. *If E, F, G are Banach spaces and if $f : E \to F$ is nuclear, there is a continuous morphism*

$$f \mathbin{\widehat{\otimes}} \mathrm{Id}_G : E \mathbin{\widehat{\otimes}}_\varepsilon G \longrightarrow F \mathbin{\widehat{\otimes}}_\pi G$$

extending $f \otimes \mathrm{Id}_G$, such that $||f \mathbin{\widehat{\otimes}} \mathrm{Id}_G|| \le ||f||_\nu$.

Proof. If $f = \sum \lambda_j \, \xi_j \otimes y_j$ as in (5.8), then for any $t \in E \otimes G$ we have

$$(f \otimes \mathrm{Id}_G)(t) = \sum \lambda_j \big(\xi_j \otimes \mathrm{Id}_G(t)\big) \otimes y_j$$

where $(\xi_j \otimes \mathrm{Id}_G)(t) \in G$ has norm

$$||(\xi_j \otimes \mathrm{Id}_G)(t)|| = \sup_{\eta \in G',\, ||\eta|| \le 1} \big|\eta\big(\xi_j \otimes \mathrm{Id}_G(t)\big)\big| = \sup_\eta \big|\xi_j \otimes \eta(t)\big| \le ||t||_\varepsilon.$$

Therefore $||f \otimes \mathrm{Id}_G(t)||_\pi \le \sum |\lambda_j| \, ||t||_\varepsilon$, and the infimum over all decompositions of f yields

$$||f \otimes \mathrm{Id}_G(t)||_\pi \le ||f||_\nu ||t||_\varepsilon.$$

Proposition 5.10 follows. □

If E is a Fréchet space and (p_j) an increasing sequence of semi-norms on E defining the topology of E, we have

$$E = \varprojlim \widehat{E}_{p_j},$$

where $\widehat{E}_{p_j}$ is the Hausdorff completion of (E, p_j) and $\widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j}$ the canonical morphism. Here $\widehat{E}_{p_j}$ is a Banach space for the induced norm $\widehat{p}_j$.

(5.11) Definition. *A Fréchet space E is said to be nuclear if the topology of E can be defined by an increasing sequence of semi-norms p_j such that each canonical morphism*

$$\widehat{E}_{p_{j+1}} \longrightarrow \widehat{E}_{p_j}$$

of Banach spaces is nuclear.

If E, F are arbitrary locally convex spaces, we always have a continuous morphism $E \mathbin{\widehat{\otimes}}_\pi F \to E \mathbin{\widehat{\otimes}}_\varepsilon F$, because $p \otimes_\varepsilon q \le p \otimes_\pi q$. If E, say, is nuclear, this morphism yields in fact an isomorphism $E \mathbin{\widehat{\otimes}}_\varepsilon F \simeq E \mathbin{\widehat{\otimes}}_\pi F$: indeed, by Prop. 5.10, we have $p_j \mathbin{\widehat{\otimes}}_\pi q \le C_j \, p_{j+1} \mathbin{\widehat{\otimes}}_\varepsilon q$ where C_j is the nuclear norm of $\widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j}$. Hence, when E or F is nuclear, we will identify $E \mathbin{\widehat{\otimes}}_\pi F$ and $E \mathbin{\widehat{\otimes}}_\varepsilon F$ and omit ε or π in the notation $E \mathbin{\widehat{\otimes}} F$.

(5.12) Example. Let $D = \prod D(0, R_j)$ be a polydisk in $\mathbb{C}^n$. For any $t \in \,]0, 1[$, we equip $\mathcal{O}(D)$ with the semi-norm

$$p_t(f) = \sup_{tD} |f|.$$

The completion of $\big(\mathcal{O}(D), p_t\big)$ is the Banach space E_t of holomorphic functions on tD which are continuous up to the boundary. We claim that for $t' < t < 1$ the restriction map

$$\rho_{t,t'} : E_{t'} \longrightarrow E_t$$

is nuclear. In fact, for $f \in \mathcal{O}(D)$, we have $f(z) = \sum a_\alpha z^\alpha$ where $a_\alpha = a_\alpha(f)$ satisfies the Cauchy inequalities $|a_\alpha(f)| \le p_{t'}(f)/(t'R)^\alpha$ for all $\alpha \in \mathbb{N}^n$. The formula $f = \sum a_\alpha(f)\, e_\alpha$ with $e_\alpha(z) = z^\alpha$ shows that

$$||\rho_{t,t'}||_\nu \le \sum ||a_\alpha||_{p_{t'}} ||e_\alpha||_{p_t} \le \sum (t'R)^{-\alpha}(tR)^\alpha = (1 - t/t')^{-n} < +\infty.$$

We infer that $\mathcal{O}(D)$ is a nuclear Fréchet space. It is also in a natural way a fully nuclear Fréchet algebra (see Def. 5.39 below). □

(5.13) Proposition. *Let E be a nuclear space. A morphism $f : E \to F$ is nuclear if and only if f admits a factorization $E \to M \to F$ through a Banach space M.*

Proof. By definition, a nuclear map $f : E \to F$ always has a factorization through a Banach space (even if E is not nuclear). Conversely, if E is nuclear, any continuous linear map $E \to M$ into a Banach space M is continuous for some semi-norm p_j on E, so this map has a factorization

$$E \to \widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j} \to M$$

in which the second arrow is nuclear. Hence any map $E \to M \to F$ is nuclear. □

(5.14) Proposition.

a) *If E, F are nuclear spaces, then $E \mathbin{\widehat{\otimes}} F$ is nuclear.*

b) *Any closed subspace or quotient space of a nuclear space is nuclear.*

c) *Any countable product of nuclear spaces is nuclear.*

d) *Any countable inverse limit of nuclear spaces is nuclear.*

Proof. a) If $f : E_1 \to F_1$ and $g : E_2 \to F_2$ are nuclear morphisms of Banach spaces, it is easy to check that $f \mathbin{\widehat{\otimes}_\pi} g$ and $f \mathbin{\widehat{\otimes}_\varepsilon} g$ are nuclear with $||f \mathbin{\widehat{\otimes}_?} g||_\nu \le ||f||_\nu ||g||_\nu$ in both cases. Property a) follows by applying this to the canonical morphisms $\widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j}$ and $\widehat{F}_{q_{j+1}} \to \widehat{F}_{q_j}$.

c) Let E_k, $k \in \mathbb{N}$, be nuclear spaces and $F = \prod E_k$. If (p_j^k) is an increasing family of semi-norms on E_k as in Def. 5.11, then the topology of F is defined by the family of semi-norms

$$q_j(x) = \max_{0\le k\le j} p_j^k(x_k), \qquad x = (x_k) \in F.$$

Then $\widehat{F}_{q_j} = \bigoplus_{0\le k\le j} \widehat{E}_{k,p_j^k}$ and

$$\big(\widehat{F}_{q_{j+1}} \to \widehat{F}_{q_j}\big) = \bigoplus_{0\le k\le j} \big(\widehat{E}_{k,p_{j+1}^k} \to \widehat{E}_{k,p_j^k}\big) \oplus \big(\widehat{E}_{j+1,p_{j+1}^{j+1}} \to \{0\}\big)$$

is easily seen to be nuclear.

b) If $F \subset E$ is closed, then $\widehat{F}_{p_j}$ can be identified to a closed subspace of $\widehat{E}_{p_j}$, the map $\widehat{F}_{p_{j+1}} \to \widehat{F}_{p_j}$ is the restriction of $\widehat{E}_{p_{j+1}} \to \widehat{E}_{p_j}$ and we have $\widehat{E/F}_{p_j} \simeq \widehat{E}_{p_j}/\widehat{F}_{p_j}$. It is not true in general that the restriction or quotient of a nuclear morphism is nuclear, but this is true for a binuclear $=$ (nuclear $\circ$ nuclear) morphism, as shown by Lemma 5.15 b) below. Hence $\widehat{F}_{p_{2j+2}} \to \widehat{E}_{p_{2j}}$ and $\widehat{E/F}_{p_{2j+2}} \to \widehat{E/F}_{p_{2j}}$ are nuclear, so (p_{2j}) is a fundamental family of semi-norms on F or E/F, as required in Def. 5.11.

d) follows immediately from b) and c), since $\varprojlim E_k$ is a closed subspace of $\prod E_k$. □

(5.15) Lemma. *Let E, F, G be Banach spaces.*

a) *If $f : E \to F$ is nuclear, then f can be factorized through a Hilbert space H as a morphism $E \to H \to F$.*

b) *Let $g : F \to G$ be another nuclear morphism. If $\mathrm{Im}(g \circ f)$ is contained in a closed subspace T of G, then $g \circ f : E \to T$ is nuclear. If $\ker(g \circ f)$ contains a closed subspace S of E, the induced map $(g \circ f)^\sim : E/S \to G$ is nuclear.*

Proof. a) Write $f = \sum_{j\in I} \xi_j \otimes y_j \in E' \mathbin{\widehat{\otimes}_\pi} F$ with $\sum ||\xi_j||\,||y_j|| < +\infty$. Without loss of generality, we may suppose $||\xi_j|| = ||y_j||$. Then f is the composition

$$E \longrightarrow \ell^2(I) \longrightarrow F, \qquad x \longmapsto \big(\xi_j(x)\big), \quad (\lambda_j) \longmapsto \sum \lambda_j y_j.$$

b) Decompose g into $g = v \circ u$ as in a) and write $g \circ f$ as the composition

$$E \xrightarrow{\ f\ } F \xrightarrow{\ u\ } H \xrightarrow{\ v\ } G$$

where H is a Hilbert space. If $\mathrm{Im}(g \circ f) \subset T$ and if $T \subset G$ is closed, then $H_1 = v^{-1}(T)$ is a closed subspace of H containing $\mathrm{Im}(u \circ f)$. Therefore $g \circ f : E \to T$ is the composition

$$E \xrightarrow{\ f\ } F \xrightarrow{\ u\ } H \xrightarrow{\ \mathrm{pr}^\perp\ } H_1 \xrightarrow{\ v\restriction H_1\ } T$$

where f is nuclear and $g \circ f : E \longrightarrow T$ is nuclear. Similar proof for $(g \circ f)^\sim : E/S \to G$ by using decompositions $f = v \circ u : E \to H \to F$ and

$$(g\circ f)^\sim : E/S \xrightarrow{\widetilde{u}} H/H_1 \simeq H_1^\perp \xrightarrow{v_{\restriction H_1^\perp}} F \xrightarrow{g} G$$

where $H_1 = \overline{u(S)}$ satisfies $H_1 \subset \ker(g\circ v) \subset H$. □

(5.16) Corollary. *Let E be a nuclear space and let $E \to F$ be a nuclear morphism.*

a) *If $f(E)$ is contained in a closed subspace T of F, then the morphism $f_1 : E \to T$ induced by f is nuclear.*

b) *If $\ker f$ contains a closed subspace S of E, then $\widetilde{f} : E/S \to F$ is nuclear.*

Proof. Let $E \xrightarrow{u} M \xrightarrow{v} F$ be a factorization of f through a Banach space M. In case a), resp. b), $M_1 = v^{-1}(T)$ is a closed subspace of M, resp. $M/\overline{u(S)}$ is a Banach space, and we have factorizations

$$f_1 : E \xrightarrow{u_1} M_1 \xrightarrow{v_1} T, \qquad \widetilde{f} : E/S \xrightarrow{\widetilde{u}} M/\overline{u(S)} \xrightarrow{\widetilde{v}} F$$

where u_1, $\widetilde{u}$ are induced by u and v_1, $\widetilde{v}$ by v. Hence f_1 and $\widetilde{f}$ are nuclear. □

(5.17) Proposition. *Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of Fréchet spaces and let F be a Fréchet space. If E_2 or F is nuclear, there is an exact sequence*

$$0 \longrightarrow E_1 \mathbin{\widehat{\otimes}} F \longrightarrow E_2 \mathbin{\widehat{\otimes}} F \longrightarrow E_3 \mathbin{\widehat{\otimes}} F \longrightarrow 0.$$

Proof. If E_2 is nuclear, then so are E_1 and E_3 by Prop. 5.14 b). Hence $E_1 \mathbin{\widehat{\otimes}} F \to E_2 \mathbin{\widehat{\otimes}} F$ is a monomorphism and $E_2 \mathbin{\widehat{\otimes}} F \to E_3 \mathbin{\widehat{\otimes}} F$ an epimorphism by Prop. 5.6 and 5.5. It only remains to show that

$$\mathrm{Im}\big(E_1 \mathbin{\widehat{\otimes}} F \longrightarrow E_2 \mathbin{\widehat{\otimes}} F\big) = \ker\big(E_2 \mathbin{\widehat{\otimes}} F \longrightarrow E_3 \mathbin{\widehat{\otimes}} F\big)$$

and for this, we need only show that the left hand side is dense in the right hand side (we already know it is closed). Let $\varphi \in (E_2 \mathbin{\widehat{\otimes}} F)'$ be a linear form, viewed as a continuous bilinear form on $E_2 \times F$. If φ vanishes on the image of $E_1 \mathbin{\widehat{\otimes}} F$, then φ induces a continuous bilinear form on $E_3 \times F$ by passing to the quotient. Hence φ must vanish on the kernel of $E_2 \mathbin{\widehat{\otimes}} F \to E_3 \mathbin{\widehat{\otimes}} F$, and our density statement follows by the Hahn-Banach theorem. □

5.B. Künneth Formula for Coherent Sheaves

As an application of the above general concepts, we now show how topological tensor products can be used to compute holomorphic functions and cohomology of coherent sheaves on product spaces.

(5.18) Proposition. *Let $\mathcal{F}$ be a coherent analytic sheaf on a complex analytic scheme $(X, \mathcal{O}_X)$. Then $\mathcal{F}(X)$ is a nuclear space.*

Proof. Let $A \subset \Omega \subset \mathbb{C}^N$ be an open patch of X such that the image sheaf $(i_A)_\star \mathcal{F}_{\restriction A}$ on Ω has a resolution

$$\mathcal{O}_\Omega^{p_1} \longrightarrow \mathcal{O}_\Omega^{p_0} \longrightarrow (i_A)_\star \mathcal{F}_{\restriction A} \longrightarrow 0$$

and let $D \subset\subset \Omega$ be a polydisk. As D is Stein, we get an exact sequence

$$(5.19) \quad \mathcal{O}^{p_1}(D) \longrightarrow \mathcal{O}^{p_0}(D) \longrightarrow \mathcal{F}(A \cap D) \longrightarrow 0.$$

Hence $\mathcal{F}(A \cap D)$ is a quotient of the nuclear space $\mathcal{O}^{p_0}(D)$ and so $\mathcal{F}(A \cap D)$ is nuclear by (5.14 b). Let (U_α) be a countable covering of X by open sets of the form $A \cap D$. Then $\mathcal{F}(X)$ is a closed subspace of $\prod \mathcal{F}(U_\alpha)$, thus $\mathcal{F}(X)$ is nuclear by (5.14 b,c). □

(5.20) Proposition. *Let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on complex analytic schemes X, Y respectively. Then there is a canonical isomorphism*

$$\mathcal{F} \boxtimes \mathcal{G}(X \times Y) \simeq \mathcal{F}(X) \mathbin{\widehat{\otimes}} \mathcal{G}(Y).$$

Proof. We show the proposition in several steps of increasing generality.

a) *$X = D \subset \mathbb{C}^n$, $Y = D' \subset \mathbb{C}^p$ are polydisks, $\mathcal{F} = \mathcal{O}_X$, $\mathcal{G} = \mathcal{O}_Y$.*

Let $p_t(f) = \sup_{tD} |f|$, $p'_t(f) = \sup_{tD'} |f|$ and $q_t(f) = \sup_{t(D\times D')} |f|$ be the semi-norms defining the topology of $\mathcal{O}(D)$, $\mathcal{O}(D')$ and $\mathcal{O}(D \times D')$, respectively. Then $\widehat{E}_{p_t}$ is a closed subspace of the space $C(t\overline{D})$ of continuous functions on $t\overline{D}$ with the sup norm, and we have $p_t \otimes_\varepsilon p'_t = q_t$ by example (5.3 c). Now, $\mathcal{O}(D) \otimes \mathcal{O}(D')$ is dense in $\mathcal{O}(D \times D')$, hence its completion with respect to the family (q_t) is $\mathcal{O}(D) \mathbin{\widehat{\otimes}}_\varepsilon \mathcal{O}(D') = \mathcal{O}(D \times D')$.

b) *X is embedded in a polydisk $D \subset \mathbb{C}^n$, $X = A \cap D \overset{i}{\hookrightarrow} D$,*
$i_\star \mathcal{F}$ is the cokernel of a morphism $\mathcal{O}_D^{p_1} \longrightarrow \mathcal{O}_D^{p_0}$,
$Y = D' \subset \mathbb{C}^p$ is a polydisk and $\mathcal{G} = \mathcal{O}_Y$.

By taking the external tensor product with $\mathcal{O}_Y$, we get an exact sequence

$$(5.21) \quad \mathcal{O}_{D\times Y}^{p_1} \longrightarrow \mathcal{O}_{D\times Y}^{p_0} \longrightarrow i_\star \mathcal{F} \boxtimes \mathcal{O}_Y \longrightarrow 0.$$

Then we find a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}^{p_1}(D) \mathbin{\widehat{\otimes}} \mathcal{O}(Y) & \longrightarrow & \mathcal{O}^{p_0}(D) \mathbin{\widehat{\otimes}} \mathcal{O}(Y) & \longrightarrow & \mathcal{F}(X) \mathbin{\widehat{\otimes}} \mathcal{O}(Y) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
\mathcal{O}^{p_1}(D \times Y) & \longrightarrow & \mathcal{O}^{p_0}(D \times Y) & \longrightarrow & \mathcal{F} \boxtimes \mathcal{O}_Y(X \times Y) & \longrightarrow & 0
\end{array}$$

in which the first line is exact as the image of (5.19) by the exact functor $\bullet \mathbin{\widehat{\otimes}} \mathcal{O}(Y)$, and the second line is exact because the exact sequence of sheaves (5.21) gives an exact sequence of spaces of sections on the Stein space $D \times Y$; note that $i_\star \mathcal{F} \boxtimes \mathcal{O}_Y(D \times Y) = \mathcal{F} \boxtimes \mathcal{O}_Y(X \times Y)$. As the first two vertical arrows are isomorphisms by a), the third one is also an isomorphism.

c) *X, $\mathcal{F}$ are as in* b),
Y is embedded in a polydisk $D' \subset \mathbb{C}^p$, $Y = A' \cap D' \overset{j}{\hookrightarrow} D'$
and $j_\star \mathcal{G}$ is the cokernel of $\mathcal{O}_{D'}^{q_1} \longrightarrow \mathcal{O}_{D'}^{q_0}$.

Taking the external tensor product with $\mathcal{F}$, we get an exact sequence

$$\mathcal{F} \boxtimes \mathcal{O}_{D'}^{q_1} \longrightarrow \mathcal{F} \boxtimes \mathcal{O}_{D'}^{q_0} \longrightarrow \mathcal{F} \boxtimes j_\star \mathcal{G} \longrightarrow 0$$

and with the same arguments as above we obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{F}(X) \widehat{\otimes}\, \mathcal{O}^{q_1}(D') & \longrightarrow & \mathcal{F}(X) \widehat{\otimes}\, \mathcal{O}^{q_0}(D') & \longrightarrow & \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(Y) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
\mathcal{F} \boxtimes \mathcal{O}_{D'}^{q_1}(X \times D') & \longrightarrow & \mathcal{F} \boxtimes \mathcal{O}_{D'}^{q_0}(X \times D') & \longrightarrow & \mathcal{F} \boxtimes \mathcal{G}(X \times Y) & \longrightarrow & 0.
\end{array}$$

d) *X, $\mathcal{F}$ are as in* b),c) *and Y, $\mathcal{G}$ are arbitrary.*

Then Y can be covered by open sets $U_\alpha = A_\alpha \cap D_\alpha$ embedded in polydisks D_α, on which the image of $\mathcal{G}$ admits a two-step resolution. We have $\mathcal{F} \boxtimes \mathcal{G}(X \times U_\alpha) \simeq \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(U_\alpha)$ by c), and the same is true over the intersections $X \times U_{\alpha\beta}$ because $U_{\alpha\beta} = U_\alpha \cap U_\beta$ can be embedded by the cross product embedding $j_\alpha \times j_\beta : U_{\alpha\beta} \to D_\alpha \times D_\beta$. We have an exact sequence

$$0 \longrightarrow \mathcal{G}(Y) \longrightarrow \prod_\alpha \mathcal{G}(U_\alpha) \longrightarrow \prod_{\alpha,\beta} \mathcal{G}(U_{\alpha\beta})$$

where the last arrow is $(c_\alpha) \mapsto (c_\beta - c_\alpha)$, and a commutative diagram with exact lines

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(Y) & \longrightarrow & \prod \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(U_\alpha) & \longrightarrow & \prod\ \mathcal{F}(X) \widehat{\otimes}\, \mathcal{G}(U_{\alpha\beta}) \\
& \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
0 \longrightarrow & \mathcal{F} \boxtimes \mathcal{G}(X \times Y) & \longrightarrow & \prod \mathcal{F} \boxtimes \mathcal{G}(X \times U_\alpha) & \longrightarrow & \prod \mathcal{F} \boxtimes \mathcal{G}(X \times U_{\alpha\beta}).
\end{array}$$

Therefore the first vertical arrow is an isomorphism.

e) *X, $\mathcal{F}$, Y, $\mathcal{G}$ are arbitrary.*

This case is treated exactly in the same way as d) by reversing the roles of $\mathcal{F}$, $\mathcal{G}$ and by using d) to get the isomorphism in the last two vertical arrows. □

(5.22) Corollary. *Let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves over complex analytic schemes X, Y and let $\pi : X \times Y \to X$ be the projection. Suppose that $H^\bullet(Y, \mathcal{G})$ is Hausdorff.*

a) *If X is Stein, then $H^q(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \simeq \mathcal{F}(X) \widehat{\otimes}\, H^q(Y, \mathcal{G})$.*

b) *In general, for every open set $U \subset X$,*

$$\big(R^q \pi_\star(\mathcal{F} \boxtimes \mathcal{G})\big)(U) = \mathcal{F}(U) \widehat{\otimes}\, H^q(Y, \mathcal{G}).$$

c) *If $H^q(Y, \mathcal{G})$ is finite dimensional, then*

$$R^q \pi_\star(\mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} \otimes H^q(Y, \mathcal{G}).$$

Proof. a) Let $\mathcal{V} = (V_\alpha)$ be a countable Stein covering of Y. By the Leray theorem, $H^\bullet(Y, \mathcal{G})$ is equal to the cohomology of the Čech complex $C^\bullet(\mathcal{V}, \mathcal{G})$. Similarly $X \times \mathcal{V} = (X \times V_\alpha)$ is a Stein covering of $X \times Y$ and we have

$$H^q(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = H^q\big(C^\bullet(X \times \mathcal{V}, \mathcal{F} \boxtimes \mathcal{G})\big).$$

However, Prop. 5.20 shows that $C^\bullet(X \times \mathcal{V}, \mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F}(X) \mathbin{\widehat{\otimes}} C^\bullet(\mathcal{V}, \mathcal{G})$. Our assumption that $C^\bullet(\mathcal{V}, \mathcal{G})$ has Hausdorff cohomology implies that the cocycle and coboundary groups are (nuclear) Fréchet spaces, and that each cohomology group can be computed by means of short exact sequences in this category. By Prop. 5.17, we thus get the desired equality

$$H^q\big(C^\bullet(X \times \mathcal{V}, \mathcal{F} \boxtimes \mathcal{G})\big) = \mathcal{F}(X) \mathbin{\widehat{\otimes}} H^q\big(C^\bullet(\mathcal{V}, \mathcal{G})\big).$$

b) The presheaf $U \mapsto \mathcal{F}(U) \mathbin{\widehat{\otimes}} H^q(Y, \mathcal{G})$ is in fact a sheaf, because the tensor product with the nuclear space $H^q(Y, \mathcal{G})$ preserves the exactness of all sequences

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod \mathcal{F}(U_\alpha) \longrightarrow \prod \mathcal{F}(U_{\alpha\beta})$$

associated to arbitrary coverings (U_α) of U. Property b) thus follows from a) and from the fact that $R^q\pi_\star(\mathcal{F} \boxtimes \mathcal{G})$ is the sheaf associated to the presheaf $U \mapsto H^q(U \times Y, \mathcal{F} \boxtimes \mathcal{G})$.

c) is an immediate consequence of b), since the finite dimensionality of $H^q(Y, \mathcal{G})$ implies that this space is Hausdorff. $\square$

(5.23) Künneth formula. *Let $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves over complex analytic schemes X, Y and suppose that the cohomology spaces $H^\bullet(X, \mathcal{F})$ and $H^\bullet(Y, \mathcal{G})$ are Hausdorff. Then there is an isomorphism*

$$\bigoplus_{p+q=k} H^p(X, \mathcal{F}) \mathbin{\widehat{\otimes}} H^q(Y, \mathcal{G}) \xrightarrow{\ \simeq\ } H^k(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$$

$$\bigoplus \alpha_p \otimes \beta_q \longmapsto \sum \alpha_p \smile \beta_q.$$

Proof. Consider the Leray spectral sequence associated to the coherent sheaf $\mathcal{S} = \mathcal{F} \boxtimes \mathcal{G}$ and to the projection $\pi : X \times Y \to X$. By Cor. 5.22 b) and a use of Čech cohomology, we find

$$E_2^{p,q} = H^p(X, R^q\pi_\star \mathcal{F} \boxtimes \mathcal{G}) = H^p(X, \mathcal{F}) \mathbin{\widehat{\otimes}} H^q(Y, \mathcal{G}).$$

It remains to show that the Leray spectral sequence degenerates in E_2. For this, we argue as in the proof of Th. IV-15.9. In that proof, we defined a morphism of the double complex $C^{p,q} = \mathcal{F}^{[p]}(X) \otimes \mathcal{G}^{[q]}(Y)$ into the double complex that defines the Leray spectral sequence (in IV-15.9, we only considered the sheaf theoretic external tensor product $\mathcal{F} \boxtimes \mathcal{G}$, but there is an obvious morphism of that one into the analytic tensor product). We get a morphism of spectral sequences which induces at the E_2-level the obvious morphism

$$H^p(X, \mathcal{F}) \otimes H^q(Y, \mathcal{G}) \longrightarrow H^p(X, \mathcal{F}) \mathbin{\widehat{\otimes}} H^q(Y, \mathcal{G}).$$

It follows that the Leray spectral sequence $E_r^{p,q}$ is obtained for $r \geq 2$ by taking the completion of the spectral sequence of $C^{\bullet,\bullet}$. Since this spectral sequence degenerates in E_2 by the algebraic Künneth theorem, the Leray spectral sequence also satisfies $d_r = 0$ for $r \geq 2$. □

(5.24) Remark. If X or Y is compact, the Künneth formula holds with $\otimes$ instead of $\widehat{\otimes}$, and the assumption that both cohomology spaces are Hausdorff is unnecessary. The proof is exactly the same, except that we use (5.22 c) instead of (5.22 b).

5.C. Modules over Nuclear Fréchet Algebras

Throughout this subsection, we work in the category of nuclear Fréchet spaces. Recall that a topological algebra (commutative, with unit element 1) is an algebra A together with a topological vector space structure such that the multiplication $A \times A \to A$ is continuous. A is said to be a Fréchet (resp. nuclear) algebra if it is Fréchet (resp. nuclear) as a topological vector space.

(5.25) Definition. *A (Fréchet, resp. nuclear) A-module E is a (Fréchet, resp. nuclear) space E with a A-module structure such that the multiplication $A \times E \to E$ is continuous. The module E is said to be nuclearly free if E is of the form $A \mathbin{\widehat{\otimes}} V$ where V is a nuclear Fréchet space.*

Assume that A is nuclear and let E be a nuclear A-module. A *nuclearly free resolution* $L_\bullet$ of E is an exact sequence of A-modules and continuous A-linear morphisms

$$\cdots \longrightarrow L_q \xrightarrow{d_q} L_{q-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow E \longrightarrow 0 \tag{5.26}$$

in which each L_q is a nuclearly free A-module. Such a resolution is said to be *direct* if each map d_q is direct, i.e. if $\operatorname{Im} d_q$ has a topological supplementary space in L_{q-1} (as a vector space over $\mathbb{R}$ or $\mathbb{C}$, not necessarily as a A-module).

(5.27) Proposition. *Every nuclear A-module E admits a direct nuclearly free resolution.*

Proof. We define the "standard resolution" of E to be

$$L_q = A \mathbin{\widehat{\otimes}} \ldots \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} E$$

where A is repeated $(q+1)$ times; the A-module structure of L_q is chosen to be the one given by the first factor and we set $d_0(a_0 \otimes x) = a_0 x$,

$$d_q(a_0 \otimes \ldots \otimes a_q \otimes x) = \sum_{0 \le i < q} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_q \otimes x$$
$$+ (-1)^q a_0 \otimes \ldots \otimes a_{q-1} \otimes a_q x.$$

Then there is a homotopy operator $h_q : L_q \to L_{q+1}$ given by $h_q(t) = 1 \otimes t$ for all q (h_q, however, is not A-linear). This implies easily that $L_\bullet$ is a direct nuclearly free resolution. □

If E and F are two nuclear A-modules, we define $E \mathbin{\widehat{\otimes}}_A F$ to be

(5.28) $$E \mathbin{\widehat{\otimes}}_A F = \operatorname{coker}\big(E \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} F \xrightarrow{\ d\ } E \mathbin{\widehat{\otimes}} F\big) \quad \text{where}$$
$$d(x \otimes a \otimes y) = ax \otimes y - x \otimes ay.$$

Then $E \mathbin{\widehat{\otimes}}_A F$ is a A-module which it is not necessarily Hausdorff. If $E \mathbin{\widehat{\otimes}}_A F$ is Hausdorff, it is in fact a nuclear A-module by Prop. 5.14. If E is nuclearly free, say $E = A \mathbin{\widehat{\otimes}} V \simeq V \mathbin{\widehat{\otimes}} A$, we have $E \mathbin{\widehat{\otimes}}_A F = V \mathbin{\widehat{\otimes}} F$ (which is thus Hausdorff): indeed, there is an exact sequence

$$V \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} F \longrightarrow V \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} F \longrightarrow V \mathbin{\widehat{\otimes}} F \longrightarrow 0,$$
$$v \otimes a_0 \otimes a_1 \otimes x \longmapsto v \otimes a_0 a_1 \otimes x - v \otimes a_0 \otimes a_1 x, \qquad v \otimes a \otimes x \longmapsto v \otimes ax,$$

obtained by tensoring the standard resolution of F with $V \mathbin{\widehat{\otimes}}$; observe that the tensor product $\widehat{\otimes}$ with a nuclear space preserves exact sequences thanks to Prop. 5.17. We further define $\widehat{\operatorname{Tor}}{}_q^A(E,F)$ to be

(5.29) $$\widehat{\operatorname{Tor}}{}_q^A(E,F) = H_q(E \mathbin{\widehat{\otimes}}_A L_\bullet),$$

where $L_\bullet$ is the standard resolution of F. There is in fact an isomorphism

$$E \mathbin{\widehat{\otimes}}_A L_\bullet \xrightarrow{\ \simeq\ } E \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} F$$
$$x \otimes_A (a_0 \otimes a_1 \otimes \ldots \otimes a_q \otimes y) \longmapsto a_0 x \otimes a_1 \otimes \ldots \otimes a_q \otimes y$$

where A is repeated q times in the target space. In this isomorphism, the differential becomes

$$d_q(x \otimes a_1 \otimes \ldots \otimes a_q \otimes y) = a_1 x \otimes a_2 \otimes \ldots \otimes a_q \otimes y$$
$$+ \sum_{1 \le i < q} (-1)^i x \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_q \otimes y$$
$$+ (-1)^q x \otimes a_1 \otimes \ldots \otimes a_{q-1} \otimes a_q y.$$

In particular, we get $\widehat{\operatorname{Tor}}{}_0^A(E,F) = E \mathbin{\widehat{\otimes}}_A F$. Moreover, if we exchange the roles of E and F, we obtain a complex which is isomorphic to the above one up to the sign of d_q, hence $\widehat{\operatorname{Tor}}{}_q^A(E,F) \simeq \widehat{\operatorname{Tor}}{}_q^A(F,E)$. If $E = A \mathbin{\widehat{\otimes}} V$ is nuclearly free, the complex $E \mathbin{\widehat{\otimes}}_A L_\bullet = V \mathbin{\widehat{\otimes}} L_\bullet$ is exact, thus

$$E \text{ or } F \text{ nuclearly free} \Longrightarrow \widehat{\operatorname{Tor}}{}_q^A(E,F) = 0 \quad \text{for } q \ge 1.$$

(5.30) Proposition. *For any exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ of nuclear A-modules and any nuclear A-module F, there is an (algebraic) exact sequence*

$$\begin{array}{l} \cdots \hat{\mathrm{Tor}}_q^A(E_1, F) \longrightarrow \hat{\mathrm{Tor}}_q^A(E_2, F) \longrightarrow \hat{\mathrm{Tor}}_q^A(E_3, F) \longrightarrow \hat{\mathrm{Tor}}_{q-1}^A(E_1, F) \cdots \\ \longrightarrow \quad E_1 \mathbin{\widehat{\otimes}_A} F \quad \longrightarrow \quad E_2 \mathbin{\widehat{\otimes}_A} F \quad \longrightarrow \quad E_3 \mathbin{\widehat{\otimes}_A} F \quad \longrightarrow 0. \end{array}$$

Proof. As the standard resolution $L_\bullet \to F$ is nuclearly free, $L_q = A \mathbin{\widehat{\otimes}} V_q$ say, then $E_j \mathbin{\widehat{\otimes}_A} L_\bullet = E_j \mathbin{\widehat{\otimes}} V_\bullet$ for $j = 1, 2, 3$, so we have a short exact sequence of complexes

$$0 \longrightarrow E_1 \mathbin{\widehat{\otimes}_A} L_\bullet \longrightarrow E_2 \mathbin{\widehat{\otimes}_A} L_\bullet \longrightarrow E_3 \mathbin{\widehat{\otimes}_A} L_\bullet \longrightarrow 0. \qquad \square$$

(5.31) Corollary. *For any nuclearly free (possibly non direct) resolution $L_\bullet$ of F, there is a canonical isomorphism*

$$\hat{\mathrm{Tor}}_q^A(E, F) \simeq H_q(E \mathbin{\widehat{\otimes}_A} L_\bullet).$$

Proof. Set $B_q = \mathrm{Im}(L_{q+1} \to L_q)$ for all $q \geq 0$ and $B_{-1} = F$. Then apply (5.30) to the short exact sequences $0 \to B_q \to L_q \to B_{q-1} \to 0$ and the fact that L_q is nuclearly free to get

$$\hat{\mathrm{Tor}}_k^A(E, B_{q-1}) \simeq \begin{cases} \hat{\mathrm{Tor}}_{k-1}^A(E, B_q) & \text{for } k > 1, \\ \ker(E \mathbin{\widehat{\otimes}_A} B_q \to E \mathbin{\widehat{\otimes}_A} L_q) & \text{for } k = 1. \end{cases}$$

Hence we obtain inductively

$$\begin{aligned} \hat{\mathrm{Tor}}_q^A(E, F) &= \hat{\mathrm{Tor}}_q^A(E, B_{-1}) \simeq \ldots \simeq \hat{\mathrm{Tor}}_1^A(E, B_{q-2}) \\ &\simeq \ker(E \mathbin{\widehat{\otimes}_A} B_{q-1} \to E \mathbin{\widehat{\otimes}_A} L_{q-1}) \end{aligned}$$

and a commutative diagram

$$\begin{tikzcd} E \mathbin{\widehat{\otimes}_A} L_{q+1} \arrow[r] \arrow[dr] & E \mathbin{\widehat{\otimes}_A} L_q \arrow[r] & E \mathbin{\widehat{\otimes}_A} B_{q-1} \arrow[r] & 0 \\ & E \mathbin{\widehat{\otimes}_A} B_q \arrow[u] & & \end{tikzcd}$$

in which the horizontal line is exact (thanks to the surjectivity of the left oblique arrow and the exactness of the sequence with $E \mathbin{\widehat{\otimes}_A} B_q$ as first term). Therefore $\ker(E \mathbin{\widehat{\otimes}_A} B_{q-1} \to E \mathbin{\widehat{\otimes}_A} L_{q-1})$ can be interpreted as the kernel of $E \mathbin{\widehat{\otimes}_A} L_q \to E \mathbin{\widehat{\otimes}_A} L_{q-1}$ modulo the image of $E \mathbin{\widehat{\otimes}_A} L_{q+1} \to E \mathbin{\widehat{\otimes}_A} L_q$, and this is is precisely the definition of $H_q(E \mathbin{\widehat{\otimes}_A} L_\bullet)$. $\square$

Now, we are ready to introduce the crucial concept of transversality.

(5.32) Definition. *We say that two nuclear A-modules E, F are transverse if $E \mathbin{\widehat{\otimes}}_A F$ is Hausdorff and if* $\widehat{\mathrm{Tor}}^A_q(E,F) = 0$ *for $q \geq 1$.*

For example, a nuclearly free A-module $E = A \mathbin{\widehat{\otimes}} V$ is transverse to any nuclear A-module F. Before proving further general properties, we give a fundamental example.

(5.33) Proposition. *Let X, Y be Stein spaces and let $U' \subset U \subset\subset X$, $V \subset\subset Y$ be Stein open subsets. If $\mathcal{F}$ is a coherent sheaf over $X \times Y$, then $\mathcal{O}(U')$ and $\mathcal{F}(U \times V)$ are transverse over $\mathcal{O}(U)$. Moreover*

$$\mathcal{O}(U') \mathbin{\widehat{\otimes}}_{\mathcal{O}(U)} \mathcal{F}(U \times V) = \mathcal{F}(U' \times V).$$

Proof. Let $\mathcal{L}_\bullet \to \mathcal{F}$ be a free resolution of $\mathcal{F}$ over $U \times V$; such a resolution exists by Cartan's theorem A. Then $\mathcal{L}_\bullet(U \times V)$ is a resolution of $\mathcal{F}(U \times V)$ which is nuclearly free over $\mathcal{O}(U)$, for $\mathcal{O}(U \times V) = \mathcal{O}(U) \mathbin{\widehat{\otimes}} \mathcal{O}(V)$; in particular, we get

$$\begin{aligned} \mathcal{O}(U') \mathbin{\widehat{\otimes}}_{\mathcal{O}(U)} \mathcal{O}(U \times V) &= \mathcal{O}(U') \mathbin{\widehat{\otimes}} \mathcal{O}(V) = \mathcal{O}(U' \times V), \\ \mathcal{O}(U') \mathbin{\widehat{\otimes}}_{\mathcal{O}(U)} \mathcal{L}_\bullet(U \times V) &= \mathcal{L}_\bullet(U' \times V). \end{aligned}$$

But $\mathcal{L}_\bullet(U' \times V)$ is a resolution of $\mathcal{F}(U' \times V)$, so

$$\widehat{\mathrm{Tor}}^{\mathcal{O}(U)}_q\big(\mathcal{O}(U'), \mathcal{F}(U \times V)\big) = \begin{cases} \mathcal{F}(U' \times V) & \text{for } q = 0, \\ 0 & \text{for } q \geq 1. \end{cases} \qquad \square$$

(5.34) Properties.

a) *If $0 \to E_1 \to E_2 \to E_3 \to 0$ is an exact sequence of nuclear A-modules and if E_2, E_3 are transverse to F, then E_1 is transverse to F.*

b) *Let $A \to A_1 \to A_2$ be homomorphisms of nuclear algebras and let E be a nuclear A-module. if A_1 and A_2 are transverse to E over A, then A_2 is tranverse to $A_1 \mathbin{\widehat{\otimes}}_A E$ over A_1.*

c) *Let $E^\bullet$ be a complex of nuclear A-modules, bounded on the right side, and let M be a nuclear A-module which is transverse to all E^n. If $E^\bullet$ is acyclic in degrees $\geq k$, then $M \mathbin{\widehat{\otimes}}_A E^\bullet$ is also acyclic in degrees $\geq k$.*

d) *Let $E^\bullet$, $F^\bullet$ be complexes of nuclear A-modules, bounded on the right side. Let $f^\bullet : E^\bullet \to F^\bullet$ be a A-linear morphism and let M be a nuclear A-module which is transverse to all E^q and F^q. If $f^\bullet$ induces an isomorphism $H^q(f^\bullet) : H^q(E^\bullet) \to H^q(F^\bullet)$ in degrees $q \geq k$ and an epimorphism in degree $q = k-1$, then*

$$\mathrm{Id}_M \mathbin{\widehat{\otimes}}_A f^\bullet : M \mathbin{\widehat{\otimes}}_A E^\bullet \to M \mathbin{\widehat{\otimes}}_A F^\bullet$$

has the same property.

Proof. a) is an immediate consequence of the Tôr exact sequence.

To prove b), we need only check that if A_1 is transverse to E over A, then

$$\mathrm{T\hat{o}r}_q^{A_1}(A_2, A_1 \mathbin{\widehat{\otimes}}_A E) = \mathrm{T\hat{o}r}_q^{A}(A_2, E), \qquad \forall n \geq 0.$$

Indeed, if $L_\bullet = A \mathbin{\widehat{\otimes}} V_\bullet$ is a nuclearly free resolution of E over A, then $A_1 \mathbin{\widehat{\otimes}}_A L_\bullet = A_1 \mathbin{\widehat{\otimes}} V_\bullet$ is a nuclearly free resolution of $A_1 \mathbin{\widehat{\otimes}}_A E$ over A_1, since $H_q(A_1 \mathbin{\widehat{\otimes}}_A L_\bullet) = \mathrm{T\hat{o}r}_q^A(A_1, E) = 0$ for $q \geq 1$. Hence

$$\begin{aligned}\mathrm{T\hat{o}r}_q^{A_1}(A_2, A_1 \mathbin{\widehat{\otimes}}_A E) &= H_q\big(A_2 \mathbin{\widehat{\otimes}}_{A_1} (A_1 \mathbin{\widehat{\otimes}}_A L_\bullet)\big) = H_q\big(A_2 \mathbin{\widehat{\otimes}}_{A_1} (A_1 \mathbin{\widehat{\otimes}} V_\bullet)\big)\\ &= H_q(A_2 \mathbin{\widehat{\otimes}} V_\bullet) = H_q(A_2 \mathbin{\widehat{\otimes}}_A L_\bullet) = \mathrm{T\hat{o}r}_q^A(A_2, E).\end{aligned}$$

c) The short exact sequences $0 \to Z^q(E^\bullet) \hookrightarrow E^q \xrightarrow{d^q} Z^{q+1}(E^\bullet) \to 0$ show by backward induction on q that M is transverse to $Z^q(E^\bullet)$ for $q \geq k-1$. Hence for $q \geq k-1$ we obtain an exact sequence

$$0 \longrightarrow M \mathbin{\widehat{\otimes}}_A Z^q(E^\bullet) \hookrightarrow M \mathbin{\widehat{\otimes}}_A E^q \xrightarrow{d^q} M \mathbin{\widehat{\otimes}}_A Z^{q+1}(E^\bullet) \longrightarrow 0,$$

which gives in particular $Z^q(M \mathbin{\widehat{\otimes}}_A E^\bullet) = B^q(M \mathbin{\widehat{\otimes}}_A E^\bullet) = M \mathbin{\widehat{\otimes}}_A Z^q(E^\bullet)$ for $q \geq k$, as desired.

d) is obtained by applying c) to the *mapping cylinder* $C(f^\bullet)$, as defined in the following lemma (the proof is straightforward and left to the reader). □

(5.35) Lemma. *If $f^\bullet : E^\bullet \to F^\bullet$ is a morphism of complexes, the mapping cylinder $C^\bullet = C(f^\bullet)$ is the complex defined by $C^q = E^q \oplus F^{q-1}$ with differential*

$$\begin{pmatrix} d_E^q & 0 \\ -f^q & d_F^{q-1}\end{pmatrix} : E^q \oplus F^{q-1} \longrightarrow E^{q+1} \oplus F^q.$$

Then there is a short exact sequence $0 \to F^{\bullet-1} \to C^\bullet \to E^\bullet \to 0$ and the associated connecting homomorphism $\partial^q : H^q(E^\bullet) \to H^q(F^\bullet)$ is equal to $H^q(f^\bullet)$; in particular, $C^\bullet$ is acyclic in degree q if and only if $H^q(f^\bullet)$ is injective and $H^{q-1}(f^\bullet)$ is surjective. □

5.D. *A*-Subnuclear Morphisms and Perturbations

We now introduce a notion of nuclearity relatively to an algebra A. This notion is needed for example to describe the properties of the $\mathcal{O}(S)$-linear restriction map $\mathcal{O}(S \times U) \to \mathcal{O}(S \times U')$ when $U' \subset\subset U$.

(5.36) Definition. *Let E and F be Fréchet A-modules over a Fréchet algebra A and let $f : E \to F$ be a A-linear map. We say that*

a) *f is A-nuclear if there exist a scalar sequence (λ_j) with $\sum |\lambda_j| < +\infty$, an equicontinuous family of A-linears maps $\xi_j : E \to A$ and a bounded sequence y_j in F such that for all $x \in E$*

$$f(x) = \sum \lambda_j \, \xi_j(x) y_j.$$

b) *f is A-subnuclear if there exists a Fréchet A-module M and an epimorphism $p : M \to E$ such that $f \circ p$ is A-nuclear; if E is nuclear, we also require M to be nuclear.*

If $f : E \to F$ is A-nuclear and if $u : S \to E$ and $v : F \to T$ are continuous A-linear maps then $v \circ f \circ u$ is A-nuclear; the same is true for A-subnuclear maps. If V and W are nuclear spaces and if $u : V \to W$ is $\mathbb{C}$-nuclear, then $\mathrm{Id}_A \widehat{\otimes} u : A \widehat{\otimes} V \to A \widehat{\otimes} W$ is A-nuclear. From this we infer:

(5.37) Proposition. *Let S, Z be Stein spaces and let $U' \subset\subset U \subset\subset Z$ be Stein open subsets. Then the restriction $\rho : \mathcal{O}(S \times U) \to \mathcal{O}(S \times U')$ is $\mathcal{O}(S)$-nuclear. If $\mathcal{F}$ is a coherent sheaf over $Y \times Z$ with Y Stein and $S \subset\subset Y$, then the restriction map $\rho : \mathcal{F}(S \times U) \to \mathcal{F}(S \times U')$ is $\mathcal{O}(S)$-subnuclear.*

Proof. As $\mathcal{O}(S \times U) = \mathcal{O}(S) \widehat{\otimes} \mathcal{O}(U)$ and $\mathcal{O}(U) \to \mathcal{O}(U')$ is $\mathbb{C}$-nuclear, only the second statement needs a proof. By Cartan's theorem A, there exists a free resolution $\mathcal{L}_\bullet \to \mathcal{F}$ over $S \times U$. Then there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{L}_0(S{\times}U) & \longrightarrow & \mathcal{F}(S{\times}U) \\
\rho\downarrow & & \downarrow\rho \\
\mathcal{L}_0(S{\times}U') & \longrightarrow & \mathcal{F}(S{\times}U')
\end{array}$$

in which the top horizontal arrow is an $\mathcal{O}(S)$-epimorphism and the left vertical arrow is an $\mathcal{O}(S)$-nuclear map; its composition with the bottom horizontal arrow is thus also $\mathcal{O}(S)$-nuclear. □

Let $f : E \to F$ be a A-linear morphism of Fréchet A-modules. Suppose that $f(E) \subset F_1$ where F_1 is a *closed* A-submodule of F and let $f_1 : E \to F_1$ be the map induced by f. If f is A-nuclear, it is not true in general that f_1 is A-nuclear or A-subnuclear, even if A, E, F are nuclear. However:

(5.38) Proposition. *With the above notations, suppose A, E, F nuclear. Let B be a nuclear Fréchet algebra and let $\rho : A \to B$ be a $\mathbb{C}$-nuclear homomorphism. Suppose that B is transverse to E, F and F/F_1 over A. If $f : E \to F$ is A-subnuclear, then $\mathrm{Id}_B \widehat{\otimes}_A f_1 : B \widehat{\otimes}_A E \to B \widehat{\otimes}_A F_1$ is B-subnuclear.*

Proof. We first show that $\rho \widehat{\otimes}_A f_1 : E = A \widehat{\otimes}_A E \to B \widehat{\otimes}_A F_1$ is $\mathbb{C}$-nuclear. Since a quotient of a $\mathbb{C}$-nuclear map is $\mathbb{C}$-nuclear by Cor. 5.16 b), we may suppose for this that f is A-nuclear. Write

$$f(x) = \sum \lambda_j \, \xi_j(x) y_j, \quad \xi_j : E \to A, \quad \sum |\lambda_j| < +\infty, \quad y_j \in F,$$
$$\rho(t) = \sum \mu_k \, \eta_k(t) b_k, \quad \eta_k : A \to \mathbb{C}, \quad \sum |\mu_k| < +\infty, \quad b_k \in B$$

as in the definition of (A-)nuclearity. Then $\rho \mathbin{\widehat{\otimes}_A} f : E \longrightarrow B \mathbin{\widehat{\otimes}_A} F$ is $\mathbb{C}$-nuclear: for any $x \in E$, we have $\rho(\xi_j(x)) = \xi_j(x)\rho(1)$ in the A-module structure of B, hence

$$\begin{aligned} \rho \mathbin{\widehat{\otimes}_A} f(x) = \rho \otimes f(1 \otimes x) &= \sum \lambda_j \, \rho(\xi_j(x)) \mathbin{\widehat{\otimes}_A} y_j \\ &= \sum \lambda_j \mu_k \, (\eta_k \circ \xi_j)(x) \, b_k \mathbin{\widehat{\otimes}_A} y_j. \end{aligned}$$

By our transversality assumptions, $B \mathbin{\widehat{\otimes}_A} F_1$ is a closed subspace of $B \mathbin{\widehat{\otimes}_A} F$. As $\mathrm{Im}(\rho \mathbin{\widehat{\otimes}_A} f) \subset B \mathbin{\widehat{\otimes}_A} F_1$, the induced map $\rho \mathbin{\widehat{\otimes}_A} f_1 : E \to B \mathbin{\widehat{\otimes}_A} F_1$ is $\mathbb{C}$-nuclear by Cor. 5.16 a). Finally, there is a commutative diagram

$$\begin{array}{ccc} B \mathbin{\widehat{\otimes}} E & \xrightarrow{\mathrm{Id}_B \widehat{\otimes} (\rho \widehat{\otimes}_A f_1)} & B \mathbin{\widehat{\otimes}} (B \mathbin{\widehat{\otimes}_A} F_1) \\ \downarrow & & \downarrow \\ B \mathbin{\widehat{\otimes}_A} E & \xrightarrow{\mathrm{Id}_B \widehat{\otimes}_A f_1} & B \mathbin{\widehat{\otimes}_A} F_1 \end{array}$$

in which the vertical arrows are B-linear epimorphisms. The top horizontal arrow is B-nuclear by the $\mathbb{C}$-nuclearity of $\rho \mathbin{\widehat{\otimes}_A} f_1$, hence $\mathrm{Id}_B \mathbin{\widehat{\otimes}_A} f_1$ is B-subnuclear. □

Example 5.12 suggests the following definition (which is somewhat less general than some other in current use, but sufficient for our purposes).

(5.39) Definition. *We say that a Fréchet algebra A is fully nuclear if the topology of A is defined by an increasing family $(p_t)_{t \in]0,1[}$ of multiplicative semi-norms (that is, $p_t(xy) \le p_t(x)\, p_t(y)$), such that the Banach algebra homomorphism $\widehat{A}_{p_{t'}} \to \widehat{A}_{p_t}$ is nuclear for all $t < t' < 1$.*

If A is fully nuclear and $t \in]0,1]$, we define A_t to be the completion of A equipped with the family of semi-norms $p_{\lambda t}$, $\lambda \in]0,1[$. Then A_t is again a fully nuclear algebra, and for all $t < t' < 1$ the canonical map $A_{t'} \to A_t$ is nuclear: indeed, for $t \le u < u' < t'$, there is a factorization

$$A_{t'} \longrightarrow \widehat{A}_{p_{u'}} \longrightarrow \widehat{A}_{p_u} \longrightarrow A_t.$$

If E is a nuclear A-module, we say that E is fully A-transverse if E is transverse to all A_t over A. Then by 5.34 b), each nuclear space

$$(5.40) \quad E_t = A_t \mathbin{\widehat{\otimes}_A} E$$

is a fully A_t-transverse A_t-module. If $f : E \to F$ is a morphism of fully A-transverse nuclear modules, there is an induced map

$$(5.40') \quad f_t = \mathrm{Id}_{A_t} \widehat{\otimes}_A f : E_t \longrightarrow F_t, \qquad \forall t \in]0,1].$$

(5.41) Example. Let X be a closed analytic subscheme of an open set $\Omega \subset \mathbb{C}^N$, $D = D(a,R) \subset\subset \Omega$ a polydisk and $U = D \cap X$. We have an epimorphism $\mathcal{O}(D) \to \mathcal{O}(U)$. Denote by $\widetilde{p}_t$ the quotient semi-norm of $p_t(f) = \sup_{D(a,tR)} |f|$ on $\mathcal{O}(U)$. Then $\mathcal{O}(U)$ equipped with $(\widetilde{p}_t)_{t\in]0,1[}$ is a fully nuclear algebra, and $\mathcal{O}(U)_t = \mathcal{O}\big(D(a,tR) \cap X\big)$.

Now, let Y be a Stein space, $V \subset\subset Y$ a Stein open subset and $\mathcal{F}$ a coherent sheaf over $X \times Y$. Then Prop. 5.33 shows that $\mathcal{F}(U \times V)$ is a fully transverse nuclear $\mathcal{O}(U)$-module.

(5.42) Subnuclear perturbation theorem. *Let A be a fully nuclear algebra, let E and F be two fully A-transverse nuclear A-modules and let $f, u : E \to F$ be A-linear maps. Suppose that u is A-subnuclear and that f is an epimorphism. Then for every $t < 1$, the cokernel of*

$$f_t - u_t : E_t \longrightarrow F_t$$

is a finitely generated A_t-module (as an algebraic module; we do not assert that the cokernel is Hausdorff).

Proof. We argue in several steps. The first step is the following special case.

(5.43) Lemma. *Let B be a Banach algebra, S a Fréchet B-module and $v : S \to S$ a B-nuclear morphism. Then* $\mathrm{Coker}(\mathrm{Id}_S - v)$ *is a finitely generated B-module.*

Proof. Let $v(x) = \sum \lambda_j\, \xi_j(x) y_j$ be a B-nuclear decomposition of v. We have a factorization

$$v = \beta \circ \alpha : S \xrightarrow{\ \alpha\ } \ell^1(B) \xrightarrow{\ \beta\ } S$$

where $\alpha(x) = \big(\lambda_j \xi_j(x)\big)$ and $\beta(t_j) = \sum t_j y_j$. Set $w = \alpha \circ \beta : \ell^1(B) \to \ell^1(B)$. As α is B-nuclear, so is w, and α, β induce isomorphisms

$$\mathrm{Coker}(\mathrm{Id}_S - v) \ \underset{\widetilde{\beta}}{\overset{\widetilde{\alpha}}{\rightleftarrows}}\ \mathrm{Coker}\big(\mathrm{Id}_{\ell^1(B)} - w\big).$$

We are thus reduced to the case when S is a Banach module. Then we write $v = v' + v''$ with

$$v'(x) = \sum_{1 \le j \le N} \lambda_j\, \xi_j(x) y_j, \qquad v''(x) = \sum_{j > N} \lambda_j\, \xi_j(x) y_j.$$

For N large enough, we have $||v''|| < 1$, hence $\mathrm{Id}_S - v''$ is an automorphism and $\mathrm{Coker}(\mathrm{Id}_S - v' - v'')$ is generated by the classes of $y_1, \ldots, y_N$. $\square$

Proof of Theorem 5.42. a) We may suppose that E is nuclearly free and that u is A-nuclear, otherwise we replace f, u by their composition with $A \widehat{\otimes} M \longrightarrow M \xrightarrow{p} E$, where M is nuclear and $p : M \to E$ is an epimorphism such that $u \circ p$ is A-nuclear.

b) As in (5.9), there is a A-nuclear decomposition $u(x) = \sum \lambda_j \, \xi_j(x) y_j$ where (y_j) converges to 0 in F. Since f is an epimorphism, we can find a sequence (x_j) converging to 0 in E such that $f(x_j) = y_j$. Hence we have $u = f \circ v$ where $v(x) = \sum \lambda_j \, \xi_j(x) x_j$ is a A-nuclear endomorphism of E, and the cokernel of $f - u$ is the image by f of the cokernel of $\mathrm{Id}_E - v$.

c) By a), b) we may suppose that $F = E = A \widehat{\otimes} M$, $f = \mathrm{Id}_E$ and that u is A-nuclear. Let B be the Banach algebra $B = \widehat{A}_{p_t}$. Then $B \widehat{\otimes}_A E = B \widehat{\otimes} M$ is a Fréchet B-module and $\mathrm{Id}_B \ \widehat{\otimes}_A \ u$ is B-nuclear. By Lemma 5.42, $\mathrm{Id}_B \widehat{\otimes}_A \mathrm{Id}_E - \mathrm{Id}_B \widehat{\otimes}_A u$ has a finitely generated cokernel over B. Now, there is an obvious morphism $B \to A_t$, hence by taking the tensor product with $A_t \widehat{\otimes}_B \bullet$ we get

$$A_t \widehat{\otimes}_B (B \widehat{\otimes}_A E) = A_t \widehat{\otimes}_B (B \widehat{\otimes} M) = A_t \widehat{\otimes} M = A_t \widehat{\otimes}_A E = E_t$$

and we see that

$$\mathrm{Id}_{E_t} - u_t = \mathrm{Id}_{A_t} \widehat{\otimes}_A \mathrm{Id}_E - \mathrm{Id}_{A_t} \widehat{\otimes}_A u$$

has a finitely generated cokernel over A_t. □

5.E. Proof of the Direct Image Theorem

We first prove a functional analytic version of the result, which appears as a relative version of Schwartz' theorem 1.9.

(5.44) Theorem. *Let A be a fully nuclear algebra, $E^\bullet$ and $F^\bullet$ complexes of fully A-transverse nuclear A-modules. Let $f^\bullet : E^\bullet \to F^\bullet$ be a morphism of complexes such that each f^q is A-subnuclear. Suppose that $E^\bullet$ and $F^\bullet$ are bounded on the right and that $H^q(f^\bullet)$ is an isomorphism for each q. Then for every $t < 1$, there is a complex $L^\bullet$ of finitely generated free A_t-modules and a complex morphism $h^\bullet : L^\bullet \to E_t^\bullet$ which induces an isomorphism on cohomology.*

Proof. a) We first show the following statement:

> *Suppose that $E_t^\bullet$ and $F_t^\bullet$ are acyclic in degrees $> q$. Then for every $t' < t$, the cohomology space $H^q(E_{t'}^\bullet) \simeq H^q(F_{t'}^\bullet)$ is a finitely generated $A_{t'}$-module.*

Indeed, the exact sequences $0 \to Z^k(E_t^\bullet) \to E_t^k \to Z^{k+1}(E_t^\bullet) \to 0$ show by backward induction on k that $Z^k(E_t^\bullet)$ is fully A_t-transverse for $k \geq q$. The same is true for $Z^k(F_t^\bullet)$. Then f_t^q is a A_t-subnuclear map from

$Z^q(E_t^\bullet)$ into F_t^q, and its image is contained in the closed subspace $Z^q(F_t^\bullet)$. By Prop. 5.38, for all $t'' < t$, the map $f_{t''}^q = \mathrm{Id}_{A_{t''}} \widehat{\otimes}_{A_t} f_t^q$ is a $A_{t''}$-subnuclear map $Z^q(E_{t''}^\bullet) \to Z^q(F_{t''}^\bullet)$. By Prop. 5.34 d), $H^\bullet(f_{t''}^\bullet)$ is an isomorphism in all degrees, hence

$$d_{t''}^q \oplus f_{t''}^q : F_{t''}^{q-1} \oplus Z^q(E_{t''}^\bullet) \longrightarrow Z^q(F_{t''}^\bullet)$$

is surjective. By the subnuclear perturbation theorem, the map

$$d_{t'}^q \oplus 0 = \mathrm{Id}_{A_{t'}} \widehat{\otimes}_{A_{t''}} \big((d_{t''}^q \oplus f_{t''}^q) - (0 \oplus f_{t''}^q)\big)$$

has a finitely generated $A_{t'}$-cokernel for $t' < t'' < t$, as desired.

b) Let N be an index such that $E^k = F^k = 0$ for $k > N$. Fix a sequence $t < \ldots < t_q < t_{q+1} < \ldots < t_N < 1$. To prove the theorem, we construct by backward induction on q a finitely generated free module L^q over A_{t_q} and morphisms $d^q : L^q \to L_{t_q}^{q+1}$, $h^q : L^q \to E_{t_q}^q$ such that

i) $L_{\geq q,\, t_q}^\bullet : 0 \to L^q \to L_{t_q}^{q+1} \to \cdots \to L_{t_q}^N \to 0$ *is a complex and* $h_{\geq q,\, t_q}^\bullet : L_{\geq q,\, t_q}^\bullet \to E_{t_q}^\bullet$ *is a complex morphism.*

ii) *The mapping cylinder* $M_q^\bullet = C(h_{\geq q,\, t_q}^\bullet)$ *defined by* $M_q^k = \bigoplus_{k\in\mathbb{Z}} \big(L_{\geq q,\, t_q}^k \oplus E_{t_q}^{k-1}\big)$ *is acyclic in degrees* $k > q$.

Suppose that L^k has been constructed for $k \geq q$. Consider the mapping cylinder $N_q^\bullet = C(f_{t_q}^\bullet \circ h_{\geq q,\, t_q}^\bullet)$ and the complex morphism

$$M_q^\bullet \longrightarrow N_q^\bullet, \qquad L_{\geq q,\, t_q}^k \oplus E_{t_q}^{k-1} \longrightarrow L_{\geq q,\, t_q}^k \oplus F_{t_q}^{k-1}$$

given by $\mathrm{Id} \oplus f_{t_q}^{k-1}$. This morphism is A_{t_q}-subnuclear in each degree and induces an isomorphism in cohomology (compare the cohomology of the short exact sequences associated to each mapping cylinder, with the obvious morphism between them). Moreover, $M_q^\bullet$ and $N_q^\bullet$ are acyclic in degrees $k > q$. By step a), the cohomology space $H^q(M_{q,\, t_{q-1}}^\bullet)$ is a finitely generated $A_{t_{q-1}}$-module. Therefore, we can find a finitely generated free $A_{t_{q-1}}$-module L^{q-1} and a morphism

$$d^{q-1} \oplus h^{q-1} : L^{q-1} \to M_{q,\, t_{q-1}}^q = L_{t_{q-1}}^q \oplus E_{t_{q-1}}^{q-1}$$

such that the image is contained in $Z^q(M_{q,\, t_{q-1}}^\bullet)$ and generates the cohomology space $H^q(M_{q,\, t_{q-1}}^\bullet)$. As $M_{q,\, t_{q-1}}^{q-1} = E_{t_{q-1}}^{q-2}$, this means that $M_{q-1}^\bullet$ is also acyclic in degree q. Thus L^{q-1}, together with the maps (d^{q-1}, h^{q-1}) satisfies the induction hypotheses for $q-1$, and $L_t^\bullet$ together with the induced map $h_t^\bullet : L_t^\bullet \to E_t^\bullet$ is the required morphism of complexes. □

Proof of theorem 5.1. Let X, Y be complex analytic schemes, let $F : X \to Y$ be a proper analytic morphism and let $\mathcal{S}$ be a coherent sheaf over X. Fix a point $y_0 \in Y$, a neighborhood of y_0 which is isomorphic to a closed analytic subscheme of a Stein open set $W \subset \mathbb{C}^n$ and a polydisk $D^0 = D(y_0, R_0) \subset\subset W$.

The compact set $K = F^{-1}(\overline{D}^0 \cap Y)$ can be covered by finitely many open subsets $U_\alpha^0 \subset\subset X$ which possess embeddings as closed analytic subschemes of Stein open sets $\Omega_\alpha^0 \subset \mathbb{C}^{N_\alpha}$. Let $\Omega'_\alpha \subset\subset \Omega_\alpha \subset\subset \Omega_\alpha^0$ be Stein open subsets such that $U_\alpha = U_\alpha^0 \cap \Omega_\alpha$ and $U'_\alpha = U_\alpha^0 \cap \Omega'_\alpha$ still cover K. Let $i_\alpha : U_\alpha^0 \to \Omega_\alpha^0$ and $j : Y \cap D^0 \to D^0$ be the embeddings and $\mathcal{S}_\alpha = \big(i_\alpha \times (j \circ F)\big)_\star \mathcal{S}$ the image sheaf of $\mathcal{S}$ on $\Omega_\alpha^0 \times D^0$. Let $D \subset\subset D^0$ be a concentric polydisk. Then $\mathcal{S}\big(U_\alpha \cap F^{-1}(D)\big) = \mathcal{S}_\alpha(\Omega_\alpha \times D)$ is a fully transverse $\mathcal{O}(D)$-module by Ex. 5.41, and so is $\mathcal{S}\big(U'_\alpha \cap F^{-1}(D)\big) = \mathcal{S}_\alpha(\Omega'_\alpha \times D)$. Moreover, the restriction map

$$\mathcal{S}\big(U_\alpha \cap F^{-1}(D)\big) \longrightarrow \mathcal{S}\big(U'_\alpha \cap F^{-1}(D)\big)$$

is $\mathcal{O}(D)$-subnuclear by Prop. 5.37 applied to $\mathcal{F} = \mathcal{S}_\alpha$. For every Stein open set $V \subset D$, Prop. 5.33 shows that

$$\mathcal{O}(V)\, \widehat{\otimes}_{\mathcal{O}(D)}\, \mathcal{S}\big(U_\alpha \cap F^{-1}(D)\big) = \mathcal{S}\big(U_\alpha \cap F^{-1}(V)\big).$$

Denote by $\mathcal{U} \cap F^{-1}(D)$ the collection $\big(U_\alpha \cap F^{-1}(D)\big)$ and use a similar notation with $\mathcal{U}' = (U'_\alpha)$. As $\mathcal{U} \cap F^{-1}(D)$, $\mathcal{U}' \cap F^{-1}(D)$ are Stein coverings of $F^{-1}(D)$, the Leray theorem applied to the alternate Čech complex of $\mathcal{S}$ over $\mathcal{U} \cap F^{-1}(D)$ and $\mathcal{U}' \cap F^{-1}(D)$ gives an isomorphism

$$H^\bullet\big(AC^\bullet(\mathcal{U} \cap F^{-1}(D), \mathcal{S})\big) = H^\bullet\big(AC^\bullet(\mathcal{U}' \cap F^{-1}(D), \mathcal{S})\big) = H^\bullet\big(F^{-1}(D), \mathcal{S}\big).$$

By the above discussion, $AC^\bullet(\mathcal{U} \cap F^{-1}(D), \mathcal{S})$ and $AC^\bullet(\mathcal{U}' \cap F^{-1}(D), \mathcal{S})$ are finite complexes of fully transverse nuclear $\mathcal{O}(D)$-modules, the restriction map

$$AC^\bullet(\mathcal{U} \cap F^{-1}(D), \mathcal{S}) \longrightarrow AC^\bullet(\mathcal{U}' \cap F^{-1}(D), \mathcal{S})$$

is $\mathcal{O}(D)$-subnuclear and induces an isomorphism on cohomology groups. Set $D = D(y_0, R)$ and $D_t = D(y_0, tR)$. Theorem 5.44 shows that for every $t < 1$ there is a complex of finitely generated free $\mathcal{O}$-modules $\mathcal{L}^\bullet$ and a $\mathcal{O}(D_t)$-linear morphism of complexes

$$\mathcal{L}^\bullet(D_t) \to AC^\bullet(\mathcal{U} \cap F^{-1}(D_t), \mathcal{S})$$

which induces an isomorphism on cohomology. Let $V \subset D_t$ be an arbitrary Stein open set. By Prop. 5.34 d) applied with $M = \mathcal{O}(V)$, we conclude that $\mathcal{L}^\bullet(V) \to AC^\bullet(\mathcal{U} \cap F^{-1}(V), \mathcal{S})$ induces an isomorphism on cohomology. If we take the direct limit as V runs over all Stein neighborhoods of a point $y \in Y \cap D_t$, we see that $\mathcal{H}^q(\mathcal{L}^\bullet) \simeq R^q F_\star \mathcal{S}$ over $Y \cap D_t$, hence $R^q F_\star \mathcal{S}$ is $\mathcal{O}_Y$-coherent near y_0. □

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